Riv. Mat. Univ. Parma (7) **3*** (2004), 223-231

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The exceptional set in short intervals for two additive problems with primes: a survey (**)

1 - Goldbach problem

The first problem we examine here is the well-known Goldbach's conjecture: is it true that every even number $n > 2$ can be written as a sum of two primes? At present it is not known if this statement is true or false. In the following an even number which is a sum of two primes will be called a G-number. A possible approach to this problem is trying to estimate the number of its exceptions: so denote by E the set of even integers larger than two which are not G-numbers. It is clear that a positive answer to the Goldbach problem is equivalent to proving that $E = \emptyset$ and hence $|E| = 0$. As we said before, we are unfortunately not able to prove such a strong result and, in fact, we are very far from it. To explain why, after letting X be a sufficiently large parameter and $E(X) = E \cap [1, X]$, we recall the best known result on $|E(X)|$ (the cardinality of the set $E(X)$):

Theorem 1.1 (Montgomery-Vaughan [**15**], 1975). *There exists an effectively computable positive constant δ such that*

$$|E(X)| \ll X^{1-\delta}.$$

In this statement, and in the following, we denote with the I.M. Vinogradov's

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(**) Received February 2nd 2004 and in revised form September 30th 2004. AMS classification 11 P 32, 11 P 55.

notation $f(x) \ll g(x)$ the existence of a positive constant $C > 0$ such that $|f(x)| \leq C|g(x)|$.

Montgomery-Vaughan's result means, from an asymptotic point of view, that there are few exceptions to the Goldbach conjecture, but the quality of the estimation is very far from $|E(X)| \ll 1$ (which means that every sufficiently large even integer is a G-number). Concerning the order of magnitude of δ in Theorem 1.1, we remark that, in 1999, H. Li [**12**] was able to prove that $\delta = 0.079$ is admissible while, recently, J. Pintz [**21**] announced that Theorem 1.1 holds with $\delta = 1/3$.

We recall that Theorem 1.1 is proved using the famous circle method which was developed, around 1920, by G.H. Hardy, J.E. Littlewood and S. Ramanujan. Roughly speaking, the circle method is an analytic technique which connects the number of representations of an integer as a sum of other integers with an integral of trigonometric functions of period 1; then the interval of integration is split in two subsets: the first one is a union of few subsets of $(0, 1)$ (in which the integrands are «big»: the «major arcs») and furnishes the expected main term of the additive problem considered and the second one is a union of a large numbers of other subsets of $(0, 1)$ (in which the integrands are «small»: the «minor arcs») and furnishes the expected error term. Hardy, Littlewood and Ramanujan used their method to prove results on the partition function (Hardy-Ramanujan; problem completely solved by H. Rademacher in 1937) and on several additive problems with primes (Hardy-Littlewood; see their series of articles named «Partitio Numerorum»).

We also recall that, relaxing the condition on one of the two summands, we obtain better results. For example we have the following:

Theorem 1.2 (J. Chen [**3**], [**4**], 1966). *Every sufficiently large number n can be written as a sum of a prime and of an integer which has at most two prime factors.*

The proof of Chen's theorem is based on sieve techniques (see, *e.g.*, Halberstam-Richert [**5**], ch. 11). By relaxing the condition on the number of summands, we have other important results:

Theorem 1.3 (I.M. Vinogradov [**25**], 1937). *Every sufficiently large odd number n can be written as a sum three primes.*

Theorem 1.4 (O. Ramaré [**23**], 1995). *Every even integer n can be written as a sum of at most six primes. Every integer $n > 1$ can be written as a sum of at most seven primes.*

Vinogradov's theorem is proved by using the circle method, while Ramaré's result follows from a combination of sieve techniques and of effective results on the distribution of primes in arithmetic progressions.

Now we turn back to the Goldbach problem. Another kind of situation one might study is the number of exceptions belonging to a «short» interval. This means that we consider the exceptions to the Goldbach problem in the interval $[X, X+H]$, where $H = o(X)$ as $X \to +\infty$. From now on we will write such a set as $E(X, H) = E \cap [X, X+H]$. It is clear that Theorem 1.1 cannot give us any information on such a short interval. It is also clear that H cannot be too small; for example if we were able to prove $E(X, H) = \emptyset$ for $H = 2$ and X sufficiently large, we would have that every sufficiently large even number n is a sum of two primes, *i.e.*, a proof of a weak version of the Goldbach problem!

Several results the exceptional set in short intervals were proved during the last twenty years but we recall now just the following ones since they are deeply connected with ours. In 1996 C.H. Jia, combining analytic method with sieve techniques, proved that

Theorem 1.5 (Jia [8], 1996). *Let $A > 0$, $\varepsilon > 0$ be arbitrary constants and $H \geqslant X^{\frac{7}{108}+\varepsilon}$; then*

$$|E(X, H)| \ll H \log^{-A} X .$$

Using only circle method techniques, the best result on $E(X, H)$ was proved in 1993 by Perelli-Pintz:

Theorem 1.6 (Perelli-Pintz [18], 1993). *Let $A > 0$, $0 < \varepsilon < 5/6$ be arbitrary constants and $H \geqslant X^{\frac{7}{36}+\varepsilon}$; then*

$$|E(X, H)| \ll H \log^{-A} X .$$

As can be seen from the previous statements, there are two important parameters: the quality of the estimate on $|E(X, H)|$ and the uniformity on H.

So a natural question is: is it possible to obtain a short interval analogue of Theorem 1.1, *i.e.*, to save a power of H in the estimate of $|E(X, H)|$?

Near 1980 an extension of Montgomery-Vaughan's result to short intervals was given by S.T. Luo-Q. Yao and Yao. They stated that there exists an effectively computable positive constant δ such that, for every $\varepsilon > 0$, $|E(X, H)| \ll H^{1-\delta}$ for $H \geqslant X^{\frac{7}{12}+\varepsilon}$. But they made an oversight in the proof (in fact the mistake is in the application of zero-density estimates for Dirichlet L-functions) and so, after correction, their theorem becomes:

Theorem 1.7 (Luo-Yao [**13**] and Yao [**26**]). *There exists an effectively computable positive constant* δ *such that for* $H \geqslant X^{\frac{7}{12}+2\delta}$ *we have*

$$|E(X, H)| \ll H^{1-\delta}.$$

In 2001 T.P. Peneva [**16**], see also the Corrigendum [**17**], obtained that the same estimate on $|E(X, H)|$ holds in the wider range $H \geqslant X^{\frac{1}{3}+\delta}$. In fact an oversight in Peneva's estimate of the minor arcs let her state the result in the range $H > X^{\frac{1}{3}}$; unfortunately this uniformity on H is not reached by her proof.

In 2003 we proved the following

Theorem 1.8 (L. [**10**]). *There exists an effectively computable positive constant* δ *such that for* $H \geqslant X^{\frac{7}{24}+7\delta}$ *we have*

$$|E(X, H)| \ll H^{1-\delta/600}.$$

Here we just give an outline of the proof.

i) First we introduce a localization parameter Y for the primes and so we write an even integer $n \in [X, X+H]$ as $p_1 + p_2$ with $X - Y < p_1 \leqslant X + Y$ and $Y/2 < p_2 \leqslant Y$. Our result hence is obtained by using $Y = X^{\frac{7}{8}+7\delta+\varepsilon}$ and $H = Y^{\frac{1}{3}+6\delta+\varepsilon}$.

ii) Essentially, we follow the Montgomery-Vaughan argument to treat the contribution of the major arcs and the Mikawa-Peneva technique to estimate the mean squares of minor arcs. For technical reasons the main term estimate is performed only at the centre of the major arcs. In the remaining part («periphery» of the major arcs), we study the individual contributions of the non-exceptional zeros of Dirichlet L-functions located in a thin constant strip near $\Re(s) = 1$ («excluded zeros»). The mean-square estimation of the non-excluded zeros in the periphery of the major arcs is performed using a slightly modified version of Perelli-Pintz's minor arcs technique.

iii) In the body of the proof we will use the zero-density estimate

$$\sum_{q \leqslant P} \sum_{\chi}{}^{*} N(\sigma, T, \chi) \ll (P^2 T)^{\frac{12}{5}(1-\sigma)} (\log PT)^{22}, \tag{1}$$

for $\sigma \in [1/2, 1]$, see Ramachandra [**22**], and the log-free zero-density estimate

$$\sum_{q \leqslant P} \sum_{\chi}{}^{*} N(\sigma, T, \chi) \ll (P^4 T)^{\frac{3}{2}(1-\sigma)}, \tag{2}$$

for $\sigma \in [27/28, 1]$, see Peneva [**16**] and the Corrigendum [**17**], where * means that the summation is over primitive characters and $N(\sigma, T, \chi) = |\{\varrho = \beta + i\gamma : L(\varrho, \chi) = 0, \beta \geq \sigma \text{ and } |\gamma| \leq T\}|$ is the density function for the zeros of the Dirichlet L-functions.

iv) The meaning of the previously mentioned constants 7/8 and 1/3 can be explained as follows. In the centre of the major arcs, our treatment requires (1) and so Y has to be greater than $X^{\frac{7}{12} + 3\delta + \varepsilon}$. In the periphery of major arcs unfortunately we are not able to reach the level $X^{\frac{7}{12} + 3\delta + \varepsilon}$ but only $X^{\frac{7}{8} + 7\delta + \varepsilon}$. This loss of uniformity is due to the use of the partial summation formula in the estimate of

$$\sum_{Y/2 < m \leq Y} \Lambda(n-m)\, \chi(n-m)\, m^{\varrho - 1},$$

where ϱ is an excluded zero of a Dirichlet L-function of a primitive character χ and Λ is the von Mangoldt function. Moreover, in the mean-square estimates of the minor arcs, we will have to choose H equal to $Y^{\frac{1}{3} + 6\delta + \varepsilon}$.

Theorem 1.8 is, at present, the Montgomery-Vaughan's type estimate which has the widest uniformity on H. It won, in 2003, the Distinguished Award of Hardy-Ramanujan Society.

It is an open problem to obtain the same estimate on $|E(X, H)|$ with a better uniformity on H, *e.g.*, for $H \geq X^{\frac{7}{36} + \delta}$. We think that it should be true, but, so far, we have no proof of it. We also think that, to prove a further better uniformity on H, some type of sieve method has to be inserted into our proof.

2 - Hardy-Littlewood problem

The second additive question we discuss here is the Hardy-Littlewood problem. In 1923 Hardy and Littlewood [**6**], [**7**] conjectured that every sufficiently large integer is either a k^{th}-power of an integer or a sum of a prime and a k^{th}-power of an integer, for $k = 2, 3$. In the following we will call Hardy-Littlewood number (HL-number) an integer which is a sum of a prime and of a k^{th}-power of an integer, $k \in \mathbb{N}$, $k \geq 2$.

As we said for the Goldbach problem, a strategy to prove results on the HL-problem is to study the number of its exceptions. So denote by E_k the set of integers which are neither an HL-number nor a k^{th}-power of an integer. We will study the cardinality of the sets $E_k(X) = E_k \cap [1, X]$ and $E_k(X, H) = E_k \cap [X, X + H]$, where X is a sufficiently large parameter and $H = o(X)$ as $X \to +\infty$. It is clear that the Hardy-Littlewood conjectures are equivalent to $|E_k(X)| \ll 1$ for $k = 2, 3$. At first sight such a problem seems to be deeply connected with Goldbach's conjecture but the integer powers are sparser and have a more regular distribution than primes. The regular distribution of powers suggests that the HL-problem should be easier than Goldbach's one but their sparsity suggests it should be harder. By now, in fact, several results on HL-numbers, which are similar to the ones known on Goldbach problem, can be proved. For example, it is possible to obtain that the exceptions of the HL-problem are asymptotically few. In fact, about fifteen years ago, Brünner-Perelli-Pintz proved the following result:

Theorem 2.1 (Brünner-Perelli-Pintz [**2**], 1989). *There exists an effectively computable positive constant* δ *such that*

$$|E_2(X)| \ll X^{1-\delta}.$$

The same estimate was independently proved in the same years by A.I. Vinogradov [**24**]. Theorem 2.1 can be considered an analogue of Theorem 1.1 but in its proof several technical difficulties concerning the contributions to the main term of the non-exceptional zeros of Dirichlet L-functions located in a strip near the line $\Re(s) = 1$ (such contributions are not present in the proof of Theorem 1.1) have to be avoided. A similar result was proved for the general case $k \geq 2$ by A. Zaccagnini [**27**] in 1992. To generalize Brünner-Perelli-Pintz's proof to the case $k \geq 2$, Zaccagnini had to develop a more sophisticated treatment of the arithmetic part (the singular series).

As for Theorem 1.1 on the Goldbach problem, Theorem 2.1 does not have any consequences on the estimate of $|E_k(X, H)|$. In this case, several results were proved in the latest ten years; as before, we cite only the ones which are strictly connected with ours. In the first part of the nineties of the last century, Perelli-Pintz and H. Mikawa proved independently:

Theorem 2.2 (Perelli-Pintz [**19**] and Mikawa [**14**]). *Let* $A > 0$, $\varepsilon > 0$ *be arbitrary constants and* $H \geq X^{7/24+\varepsilon}$; *then*

$$|E_2(X, H)| \ll H \log^{-A} X.$$

Moreover, in 1995, Perelli-Zaccagnini were able to generalize this result to the following:

Theorem 2.3 (Perelli-Zaccagnini [**20**]). *Let* $A > 0$, $\varepsilon > 0$ *be arbitrary constants and* $H \geq X^{7/12(1-1/k)+\varepsilon}$; *then*

$$|E_k(X, H)| \ll H \log^{-A} X.$$

Recently the author proved the following Montgomery-Vaughan type estimate for the exceptional set of the Hardy-Littlewood problem ($k \geq 2$) in short intervals:

Theorem 2.4 (L. [**11**], 2003). *Let $k \geqslant 2$ be a fixed integer and $K = 2^{k-2}$. There exists an effectively computable positive constant δ such that for $H \geqslant X^{7/12\left(1-\frac{1}{k}\right)+\delta}$*

$$|E_k(X, H)| \ll H^{1-\delta/(5K)}.$$

Again, we give just an outline of the proof.

i) We insert a localization parameter Y for the primes and write an HL-number $n \in [X, X+H]$ as $p + m^k$ with $X - Y \leqslant p \leqslant X + Y$ and $Y/2 \leqslant m^k \leqslant Y$. Theorem 2.4 is obtained using $Y = X^{7/12+10\delta+\varepsilon}$ and $H = Y^{\left(1-\frac{1}{k}\right)+\delta}$.

ii) To treat the centre of the major arcs we adopt the circle method setting used by Brünner, Perelli, Pintz and Zaccagnini. So we estimate the contribution of the zeros of Dirichlet L-functions located in a suitable thin strip near $\Re(s) = 1$ («excluded zeros») as «secondary» main terms.

iii) In the body of the proof we will use the zero-density estimate (1) and the following log-free zero-density estimate: let $\varepsilon > 0$, then

$$\sum_{q \leqslant P} \sum_{\chi}{}^{*} N(\sigma, T, \chi) \ll (P^2 T)^{(2+\varepsilon)(1-\sigma)}, \tag{3}$$

for $\sigma \in [4/5, 1]$, see M. Jutila [**9**]. In this case we have no need for the sharper log-free density estimate (2) since the level of minor arcs essentially implies $Y > X^{1/2}$, see iv).

iv) The meaning of the previously mentioned constants 7/12 and $\left(1 - \dfrac{1}{k}\right)$ can be explained as follows. In the error term of the explicit formula for the function $\psi(x, \chi) = \sum_{m \leqslant x} \Lambda(m)\chi(m)$ we have to choose the vertical level T of the zeros as $T \geqslant X^{1+7\delta} Y^{-1} \log^2 X$ and, to estimate the contribution of the secondary main terms using (3), we have to choose $T \leqslant X^{1/2-\varepsilon-2\delta}$. Combining such relations we get $Y \geqslant X^{1/2+\varepsilon+9\delta}$ which is already satisfied since in the centre of the major arcs our treatment requires (1) and hence $Y \geqslant X^{7/12+\varepsilon+10\delta}$. Moreover, in the mean-square estimates of the minor arcs and of the periphery of major arcs, we will choose $H \geqslant Y^{\left(1-\frac{1}{k}\right)+\delta}$. Here, unlike the Goldbach case, we can use the regular distribution of powers to give a careful estimate the contributions of the periphery of the'major arcs without any loss of uniformity on Y.

v) Finally, other differences with the proof of Theorem 1.8 are that a stronger zero-free region for Dirichlet L-functions and a stronger result on Deuring-Heilbronn phenomenon are needed. Moreover, the arithmetic part (the singular series) is more difficult to treat with respect to the Goldbach case.

Finally, we remark that the uniformity on H in Theorem 2.4 seems to be the best possible for this problem while the quality of the estimate can be improved to $|E_k(X, H)| \ll H^{1-\delta/(ck)}$, where $c > 0$ is an absolute constant, using the Brüdern-

Perelli [**1**] approach (we did not use it in the proof of Theorem 2.4 to avoid the technical difficulties involved).

Since it seems that Theorem 2.4 is essentially optimal, a natural and interesting question is which result can be proved on $|E_k(X, H)|$ assuming some hypothesis on the distribution of the zeros of the Dirichlet L-functions. To be more explicit: is it possible, assuming the Generalized Riemann Hypothesis, to prove an estimate of the type $|E_k(X, H)| \ll H^{1-\frac{1}{ck}}$, where $c > 0$ is an absolute constant, uniformly for $H \gg X^{\frac{1}{2}(1-1/k)+\varepsilon}$? We have a preliminary result of this type but, for now, it holds only with a weaker uniformity on H.

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Abstract

We give a brief account about the exceptional sets in short intervals for the Goldbach and the Hardy-Littlewood problems. In particular, we present two recent results about Montgomery-Vaughan's type estimates for such exceptional sets.

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