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Some remarks on integral points close to hypersurfaces (**)

1 - The one-dimensional case

We begin by discussing rational approximation to algebraic points on the line. In this context, we have the celebrated theorem of Roth which states the following:

Theorem of Roth 1.1. *Let α be a real irrational algebraic number, $\varepsilon > 0$ a positive real number. Then for all but finitely many rational numbers p/q (p, q are coprime integers, $q > 0$),*

$$\left| \alpha - \frac{p}{q} \right| > \left(\frac{1}{q} \right)^{2+\varepsilon}.$$

It is a milestone in the theory of diophantine approximation and a key tool in attaking many problems on integral solutions to diophantine equations.

Observe at once that Roth theorem can be viewed as a lower bound for the distance from the rational point $(q : p) \in \mathbb{P}_1(\mathbb{Q})$ to the algebraic point (i.e. hypersurface) $(1 : \alpha)$. Also, if we let $f(X) \in \mathbb{Q}[X]$ be the minimal polynomial of α and $\tilde{f}(X_0, X_1) \in \mathbb{Q}[X_0, X_1]$ be the corresponding homogeneous form, i.e. a homogenous polynomial of degree $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ with $\tilde{f}(1, \alpha) = 0$, we can restate Roth's theorem as follows:

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Given $\varepsilon > 0$, *all but finitely many rational points* $(q : p) \in \mathbb{P}_1(\mathbb{Q})$ *satisfy*

$$\frac{\tilde{f}(q, p)}{\max\{|q|, |p|\}^d} > \max\{|q|, |p|\}^{-2-\varepsilon}.$$

In other words, it consists of a lower bound for the distance of the rational points $(q: p) \in \mathbb{P}_1(\mathbb{Q})$ to the *geometrically reducible* algebraic hypersurface defined by the vanishing of $\tilde{f}(X_0, X_1)$.

While in dimension one all hypersurfaces of degree > 1 are geometrically completely reducible, in higher dimension this is certainly not the case. It will turn out that all known methods to study the rational approximation to algebraic hypersurfaces give good bounds only in the case of (sufficiently) reducible hypersurfaces.

Before passing to higher dimension, let us see the most known general result in the one-dimensional case. The form given below is due to Lang, after previous work by Mahler and Ridout.

We first introduce some notation, which will be used in the rest of the paper. For a number field K and a place v, we say that a corresponding absolute value $|\cdot|_v$ is normalized with respect to K if for all rational numbers $x \in \mathbb{Q}$,

$$|x|_v = |x|_w^{[K_v:\,\mathbb{Q}_w]/[K:\,\mathbb{Q}]}, \tag{1.1}$$

where w stands for the only place of $\mathbb{Q}$ below v, the absolute value $|\cdot|_w$ of $\mathbb{Q}$ is normalized in the usual way and K_v, $\mathbb{Q}_w$ are the completions of K and $\mathbb{Q}$ respectively. With this convention, the product formula holds (without weights) and the Weil height reads

$$H(x) = \prod_v \max\{1, |x|_v\},$$

the product being taken over all the valuations of the number field K. With this notation we have

Generalized Roth's Theorem 1.2. *Let K be a number field, S be a finite set of absolute values of K; for each $v \in S$, let $\alpha_v \in K$ be an element of K. Let $\varepsilon > 0$ be a positive real number. Then for all but finitely many elements $\beta \in K$,*

$$\prod_{v \in S} |\alpha_v - \beta|_v > H(\beta)^{-2-\varepsilon}.$$

Again, the statement can be viewed as a lower bound for the distance (with respect to several absolute values) from the approximating point β to the (zero dimensional) hypersurface defined by the points $\{\alpha_v \mid v \in S\}$. Such a bound is an easy consequence of the properties of heights (the so called Liouville inequality) in the case the set $\{\alpha_v \mid v \in S\}$ has cardinality $\leqslant 2$. On the contrary, the Generalized Roth's theorem is deep and best possible in the case the above mentioned hypersurface has degree at least three, which amounts to having at least three components.

As for Roth's theorem, one can reformulate the generalized version given above in terms of lower bounds for the product of linear forms with algebraic coefficients. This point of view will be taken up in the next paragraph.

2 - The subspace theorem

A major advance in the theory of diophantine approximations was performed by W. Schmidt in the seventies, when he proved a best possible lower bound for linear forms with algebraic coefficients in several variables. His first result can be stated as follows:

Schmidt's Theorem 2.1. *Let $f(x_0, \dots, x_n)$ be a linear form with (real) algebraic coefficients, and let $\varepsilon > 0$. Then there are only finitely many points $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ with*

$$0 < |f(\mathbf{x})| < \max\{|x_0|, \dots, |x_n|\}^{-n-\varepsilon}.$$

Of course, one could restate the above theorem in «projective form». We introduce the necessary notation, which extends the one used in §1.

For a vector $\mathbf{x} = (x_0, \dots, x_n) \in K^{n+1}$, where K is a number field, and a valuation v of K, let us denote by $\|\mathbf{x}\|_v$ the v-adic norm of $\mathbf{x}$:

$$\|\mathbf{x}\|_v = \|(x_0, \dots, x_n)\|_v = \max\{|x_0|_v \dots, |x_n|_v\}.$$

The projective height of the point $\mathbf{x} \in K^{n+1} \setminus \{0\}$ is $H(\mathbf{x}) = \prod_v \|\mathbf{x}\|_v$, the product being taken over all the normalized absolute values of K. By the product formula, it only depends on the projective class of $\mathbf{x}$ in $\mathbb{P}_n(K)$. Note that for a vector $\mathbf{x} \in \mathbb{Z}^{n+1}$ with coprime entries, its height is just the ordinary norm $\|\mathbf{x}\|$. With this notation, Theorem 2.1 can be stated as a lower bound

$$\frac{|f(\mathbf{x})|}{\|\mathbf{x}\|} > H(\mathbf{x})^{-n-1-\varepsilon} \tag{2.1}$$

valid for all rational points $\mathbf{x} \in \mathbb{P}_n(\mathbb{Q})$, with possibly the exception of a finite set and the hyperplane of equation $f(\mathbf{x}) = 0$ (in case the latter has any rational point).

Schmidt's Theorem was soon generalized by Schmidt himself to treat the approximations to several hyperplanes, in the spirit of the Generalized Roth's Theorem. Then Schlickewei managed to extend Schmidt's theorem to cover several distinct absolute values, as well as several hyperplanes for each absolute value. The most general result is now called the Subspace Theorem. We give here its formulation in the projective version:

Subspace Theorem 2.2. *Let K be a number field, S be a finite set of absolute values of K, $n \geq 1$ a positive integer. For each $v \in S$, let $L_{v0}, \ldots, L_{vn}$ be a set of linearly independent linear forms in $n+1$ variables $\mathbf{x} = (x_0, \ldots, x_n)$. Let $\varepsilon > 0$ be a positive real number. Then the solutions $\mathbf{x} \in \mathbb{P}_n(K)$ to the inequality*

$$\prod_{v \in S} \prod_{i=0}^{n} \frac{|L_{vi}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} < H(\mathbf{x})^{-n-1-\varepsilon} \tag{2.2}$$

are contained in a finite set of hyperplanes of $\mathbb{P}_n$ defined over K.

Among the first and most spectacular applications of the Subspace Theorem, let us mention Schmidt's classification of the so called *norm form equations* admitting infinitely many solutions ([8], chap. 4, §3) and the unit-equation theorem (proved by Evertse and van der Poorten-Schlickewei) concerning the linear equation $x_0 + \ldots + x_n = 0$ in elements of a finitely generated multiplicative group ([8], chap. 4, §2).

3 - The non linear case

Quoting Wolfgang Schmidt from [7], §11.3: «A better question perhaps is how close rational points can come to a given algebraic variety...». Suppose now $V \subset \mathbb{P}_n$ is an algebraic hypersurface containing no rational point $\mathbf{x} \in \mathbb{P}_n(\mathbb{Q})$, defined by the equation $f(\mathbf{x}) = 0$, where f is a form of degree d with rational integer coefficients. (This means that the equation $f(\mathbf{x}) = 0$ has $(0, \ldots, 0)$ as its only integral solution). For every integer point $\mathbf{x} \neq 0$ in $\mathbb{Z}^{n+1}$ we have $|f(\mathbf{x})| \geq 1$. Such an inequality, which can also take the form

$$\frac{|f(\mathbf{x})|}{\|\mathbf{x}\|^d} \geq H(\mathbf{x})^{-d}, \tag{3.1}$$

can be interpreted as a generalization of Liouville's inequality. Note that the left side term can be taken by definition to be the distance from $\mathbf{x}$ to $V(\mathbb{C})$ in $\mathbb{P}_n(\mathbb{C})$, with respect to the ordinary absolute value. Still quoting from [7], §11.3: «Any improvement on this inequality, even though perhaps it may apply only to special cases of non-linear hypersurfaces, would be of great interest and would shed light on certain diophantine equations».

In [1] and [2] Zannier and the author proved the following

Theorem 3.1. *Let* $f(\mathbf{x}) \in \overline{\mathbb{Q}}[\mathbf{x}]$ *be a polynomial in* n *variables with algebraic coefficients of degree* $\delta \geq 1$. *Then for every* $\varepsilon > 0$ *there exists a number* $c > 0$ *such that for all* $\mathbf{x} \in \mathbb{Z}^n$ *with* $f(\mathbf{x}) \neq 0$,

$$|f(\mathbf{x})| > c \cdot H(\mathbf{x})^{-\delta(n-1)-\varepsilon}. \tag{3.2}$$

Some remarks are in order. First, the polynomial f is not assumed to be homogeneous. Second, $f(\mathbf{x})$ has algebraic, not necessarily rational, coefficients. Hence, the analogue of Liouville's bound (3.1) would be now

$$|f(\mathbf{x})| \geq c(f) \cdot H(\mathbf{x})^{-\delta(r-1)}$$

where r is the degree of the number field generated by the coefficients of f, and $c(f)$ is an effectively computable constant.

On the other hand, in the particular case of a polynomial with integral coefficients, the lower bound provided by Theorem 3.1 is *weaker* then Liouville's bound $|f(\mathbf{x})| \geq 1$, valid for every $\mathbf{x} \in \mathbb{Z}^n$ with $f(\mathbf{x}) \neq 0$. However, we shall see (Corollary 3.4) some improvement on Liouville's bound even for polynomials with rational coefficients, although only in some special cases, namely when the polynomials in question are geometrically reducible.

Theorem 3.1 was obtained as a corollary of Theorem 3.2 below. We first need a definition: for a finite set S of absolute values of K, containing the archimedean ones, we let $\mathcal{O}_S \subset K$ be the ring of S-integers of K, i.e. the set of elements $x \in K$ with $|x|_v \leq 1$ for all $v \notin S$.

Theorem 3.2. *Let* $\mathcal{W} \subset \mathbb{A}^n$ *be an affine algebraic variety of dimension* r, *defined over a number field* K *and irreducible over* K. *Let for each place* $v \in S$, $f_v(\mathbf{x}) \in K[\mathbf{x}]$ *be a polynomial of degree* $\delta \geq 1$. *For every positive* ε, *there are only finitely many points* $\mathbf{x} \in \mathcal{W}(\mathcal{O}_S)$ *such that*

$$0 < \prod_{v \in S} |f_v(\mathbf{x})|_v < H(\mathbf{x})^{-\delta(r-1)-\varepsilon}. \tag{3.3}$$

Hence the exponent can be chosen to depend on the dimension of any algebraic variety $\mathcal{W}$ containing the approximations in question. (If we have no information on the distribution of the approximating points, i.e. they are Zariski dense in $\mathbb{A}^n$, we reobtain the exponent of Theorem 3.1 by taking $\mathcal{W} = \mathbb{A}^n$).

In analogy with the Subspace Theorem, one can consider approximating several hypersurfaces with respect to the same place v, i.e. one might be interested in bounding from below the «distance» from a rational point to the intersection of several hypersurfaces (here the «distance», however, is measured as the product of the values of the forms defininig the subvariety). In [**1**], Theorem 3 we proved the following:

Theorem 3.3. *For* $v \in S$, *let* f_{iv}, $i = 1, \ldots, n-1$, *be polynomials in* $K[X_1, \ldots, X_n]$ *of degrees* $\delta_{iv} > 0$. *Put* $\delta_v = \max_i \delta_{iv}$ *and* $\mu := \min_{v \in S} \sum_{i=1}^{n-1} \frac{\delta_{iv}}{\delta_v}$. *Fix* $\varepsilon > 0$ *and consider the Zariski closure* $\mathcal{H}$ *in* $\mathbb{P}_n$ *of the set of solutions* $\mathbf{x} \in \mathcal{O}_S^n$ *of*

$$\prod_{v \in S} \prod_{i=1}^{n-1} |f_{iv}(\mathbf{x})|_v^{\frac{1}{\delta_v}} \leq H(\mathbf{x})^{\mu - n - \varepsilon}. \tag{3.4}$$

Suppose that, for $v \in S$, *the* $\tilde{f}_{iv}$, $i = 1, \ldots, n-1$, *define a variety of dimension* 1. *Then* $\dim \mathcal{H} \leq n-1$. *Moreover, if* $\mathcal{H}'$ *is a component of* $\mathcal{H}$ *of dimension* $n-1$, *there exists* $v \in S$ *such that the* $\tilde{f}_{iv}$ *determine in* $\mathcal{H}'$ *a variety of dimension* 1.

We shall derive from the above statement the following Corollary, which gives some improvement on Liouville's bound for values at integral points of polynomials with integral coefficients, at least in special cases.

Corollary 3.4. *Let* $f(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$ *be a polynomial with integral coefficients, irreducible in* $\mathbb{Z}[\mathbf{x}]$. *Suppose it splits over* $\overline{\mathbb{Q}}$ *in the product of* r *factors* $f_1, \ldots, f_r$ *(each) of degree* δ. *Suppose moreover that the* r *hypersurfaces of the hyperplane at infinity defined by the equation* $\tilde{f}_i(\mathbf{x}) = x_0 = 0$ $(i = 1, \ldots, r)$ *are in general position. Then*

(1) *The solutions* $\mathbf{x} \in \mathbb{Z}^n$ *of the inequality*

$$|f(\mathbf{x})| < H(\mathbf{x})^{(r-n)\delta - \varepsilon} \tag{3.5}$$

are not Zariski dense in $\mathbb{A}^n$.

(2) *If* $r > n$, *for every* $k \neq 0$ *the integral points in the affine hypersurface*

$$f(\mathbf{x}) = k$$

are not Zariski dense in the hypersurface defined by the above equation.

Note that the above equation $f(\mathbf{x}) = k$ is a so-called *norm form equation* in the particular case when f is a homogeneous polynomial and $\delta = 1$. This particular case was treated already by Schmidt in the seventies (see [8], chap. 4).

We sketch the deduction of Corollary 3.4 from Theorem 3.3. Note that the first statement is trivial if $n-r\leqslant 0$, so we may assume $n-r\geqslant 1$.

Let us denote by K the splitting field of $f(\mathbf{x})$, so that K is a Galois extension of $\mathbb{Q}$ and the factors $f_i(\mathbf{x})$ $(i=1,\dots,r)$ are defined over K and are conjugates over $\mathbb{Q}$. The idea is to apply Theorem 3.2 to such factors. We recall at once that the absolute value used in (3.5) is the standard (real) absolute value, while those appearing in (3.4) are normalized with respect to the number field K, i.e. according to the formula (1.1).

Now, let us suppose to have a sequence $\mathbf{x}_1, \mathbf{x}_2, \dots$ of solutions to the inequality (3.5). Put for each $j=1,2,\dots$

$$P_j=(1:x_{j1}:\dots:x_{jn})\in\mathbb{P}_n(\mathbb{Q})$$

where $\mathbf{x}_j=(x_{j1},\dots,x_{jn})$. In other terms we embed $\mathbb{A}^n$ into $\mathbb{P}_n$ in the usual way and denote by P_j the corresponding image of $\mathbf{x}_j$. After partitioning the sequence $\{P_j\}_{j=1,\dots}$ into finitely many subsequences, we can suppose that for each archimedean absolute value v, the sequence P_j converges in $\mathbb{P}_n(\mathbb{Q}_v)$ to a point P_v. Let us denote by S' the set of the archimedean absolute values v of K for which the point P_v is at infinity, i.e. has a vanishing 0-th coordinate; then denote by S'' the set of archimedean absolute values v for which P_v is in $\mathbb{A}^n(\mathbb{Q}_v)$. Finally set $S:=S'\cup S''$, the set of all archimedean absolute values of K. Note that unless the set $\{P_1,P_2,\dots\}$ is finite, the set S' is non-empty.

Fix now a place $v\in S'$. Denote by $\{i_1(v),\dots,i_{n-1}(v)\}$ the set of indices minimizing the absolute value of $|\tilde{f}_i(P_v)|_v$.

By our assumption, the point at infinity P_v can belong to at most $n-1$ hypersurfaces $\tilde{f}_i(\mathbf{x})=0$. Then for $i\notin\{i_1(v),\dots,i_{n-1}(v)\}$, we have $\tilde{f}_i(P_v)\neq 0$. Then for such indices i, $|\tilde{f}_i(P_v)|_v/\|P_v\|^\delta\geqslant c_v$, for a positive constant c_v; so, by continuity of the function $\mathbb{P}_n(\mathbb{Q}_v)\ni P\mapsto\tilde{f}_iP)/\|P\|_v^\delta$ in the v-adic topology, for $i\notin\{i_1(v),\dots,i_{n-1}(v)\}$,

$$|f_i(\mathbf{x}_j)|_v=|\tilde{f}_i(P_j)|_v\gg\|P_j\|_v^\delta=\|\mathbf{x}_j\|_v^\delta.$$

Taking the product over all $i\in\{1,\dots,r\}$ we get, for $v\in S'$,

$$|f(\mathbf{x}_j)|_v\gg\left(\prod_{h=1}^{n-1}|f_{i_h(v)}(\mathbf{x}_j)|_v\right)\cdot\|P_j\|_v^{(r-n+1)\delta}. \tag{3.6}$$

Now, recall that the polynomials $f_1,\dots,f_r$ are conjugates over $\mathbb{Q}$; by this we mean that for each $i\in\{1,\dots,r\}$ and each automorphism $\sigma\in\mathrm{Gal}(K/\mathbb{Q})$, there exists an index σ_i such that $f_i^\sigma=f_{\sigma_i}$. (We have denoted by f_i^σ the polynomial obtained by applying the automorphism σ to each of its coefficients.) Also, the places of S are conjugates over $\mathbb{Q}$: if $v, w \in S$, then there exists a $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that for all $x \in K$, $|x|_v = |\sigma(x)|_w$. Then, using the fact that the algebraic number $f(\mathbf{x}_j)$ is in fact a rational integer, we know that $|f(\mathbf{x}_j)|_v = |f(\mathbf{x}_j)|_w$ for all $w \in S$ (and all $j = 1, \ldots$). Hence the relation (3.6) must hold for every $v \in S$, so

$$|f(\mathbf{x}_j)| = \prod_{v \in S} |f(\mathbf{x}_j)|_v \gg \left(\prod_{v \in S} \prod_{h=1}^{n-1} |f_{i_h(v)}(\mathbf{x}_j)|_v\right) \cdot \left(\prod_{v \in S} \|P_j\|_v^{\delta(r-n+1)}\right).$$

Since $\prod_{v \in S} \|P_j\|_v = H(P_j) = H(\mathbf{x}_j)$ we have

$$|f(\mathbf{x}_j)| \gg \left(\prod_{v \in S} \prod_{h=1}^{n-1} |f_{i_h(v)}(\mathbf{x}_j)|_v\right) \cdot H(\mathbf{x}_j)^{\delta(r-n+1)}.$$

On the other hand, if $\mathbf{x}_j$ is a solution of (3.5), then the above inequality implies

$$\prod_{v \in S} \prod_{h=1}^{n-1} |f_{i_h(v)}(\mathbf{x}_j)|_v \ll H(\mathbf{x}_j)^{-\delta-\varepsilon}. \tag{3.7}$$

We can now apply Theorem 3.3, with S the set of archimedean absolute values of K, $f_{hv} = f_{i_h(v)}$, $\delta_{hv} = \delta_v = \delta$ (for $h = 1, \ldots, n-1$ and $v \in S$).

Then $\mu = n - 1$. Choose $\varepsilon/2\delta$ for ε. Then our inequality (3.7) implies that the inequality (3.4) of Theorem 3.3 holds for large j (note that we assumed $n - r \geqslant 1$ otherwise the statement (1) of Corollary 3.4 is trivial). Theorem 3.3 gives at once the conclusion (1) of Corollary 3.4, i.e. the degeneracy (in the Zariski topology) of the solution to inequality (3.5). To prove (2) we have to use the further conclusion of Theorem 3.3; we omit the details (see also Theorems 1 and 2 in [**1**]).

We end this section by remarking that some results similar to those of [**1**] and [**2**] have been independently obtained by Evertse and Ferretti (see for instance [**4**] and [**5**]). Both our and their methods are ultimately based on the Subspace Theorem of Schmidt, albeit in a different way. Also the final results are different in some respect.

4 - Final remarks

We conclude with some results in the opposite direction. It is easy to see, in the linear case, that the exponent in the Subspace Theorem is best possible. In fact we have the following theorem of Dirichlet:

Dirichlet's Theorem 4.1. *Let $\alpha_0, \ldots, \alpha_n$ be linearly independent real numbers. Then there exists a constant $C = C(\alpha_0, \ldots, \alpha_n)$ such that for infinitely* *many integer vectors* $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$,

$$|x_0 \alpha_0 + \ldots + x_n \alpha_n| < C \cdot \|\mathbf{x}\|^{-n}.$$

The proof is a standard application of Dirichlet's box principle, or of the Minkowski convex body theorem (actually in the easy case of parallelepipeds).

The above result implies that the exponent $-n - \varepsilon$ in (2.1) of Schmidt's Theorem cannot be improved by taking a negative ε.

In the non-linear case, however, it seems very difficult to guess what the right result may be. Some estimations in the direction of Dirichlet's bound are known in the case of *diagonal* homogeneous forms, and were obtained by the circle method.

As a «companion theorem» for 3.1, we have just the following result of Davenport and Heilbronn [**9**], Theorem 11.1 (see also [**3**], chap. 20):

Theorem 4.2. *Let* n, δ *be positive integers with* $n \geq 2^\delta + 1$. *Let* $\lambda_1, \ldots, \lambda_n$ *be non-zero real numbers, not all rational, and not all of the same sign if* δ *is even. Then for every* $\varepsilon > 0$ *there exist integers* $x_1, \ldots, x_n$ *such that*

$$|\lambda_1 x_1^\delta + \ldots + \lambda_n x_n^\delta| < \varepsilon .$$

Hence Theorem 4.2 shows that the exponential bound $H(\mathbf{x})^{-\delta(n-1)-\varepsilon}$ of Theorem 3.1 cannot be replaced by a $O(1)$ bound, at least for diagonal homogeous forms in sufficiently many variables. Of course, our knowledge in the non-linear case is very far from being satisfactory, but it seems that any improvement in both directions needs substantially new ideas.

For non-diagonal inequalities, to the author's knowledge the only general result is Margulis' proof of the Oppenheim conjecture (see [**6**] for a recent survey on it), stating that every non-degenerate indefinite quadratic form in three variables with non-proportional coefficients takes arbitrarily small values at integral points. Hence the condition $n \geq 5$ for $\delta = 2$ in Theorem 4.1, arising from the circle method, can be relaxed to $n \geq 3$, even for non-diagonal forms. Also, Margulis' Theorem enables to treat the case of non-homogeneous quadratic polynomials. Again, it seems that no such general result is known for higher degree non-homogeneous polynomials.

The author is grateful to professors A. Perelli and U. Zannier for drawing his attention to the books of Davenport [**3**] *and Vaughan* [**9**], *which are connected with the results of this last section.*

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Abstract

The purpose of this note is to discuss the state of the art in the theory of (non) linear forms in algebraic numbers. In particular, we shall present a new result obtained in collaboration with U. Zannier, which is the object of the paper [**1**]. *Then we shall show a very particular but significant corollary.*

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