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The arithmetic of Euler's integrals (**)

Contents

1 - Historical introduction

1.1 - *Classical results*

In 1730, with the aim of interpolating in a natural way the sequence $n!$, Euler introduced the gamma-function

$$\Gamma(z+1)=\prod_{k=1}^{\infty}\frac{k^{1-z}(k+1)^z}{k+z}=\int_0^1(-\log t)^z\,\mathrm{d}t=\int_0^{\infty}e^{-x}x^z\,\mathrm{d}x$$

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([Eu], vol. XIV, pp. 1-24). In connection with this, he later considered some definite integrals over the interval $(0, 1)$ which have since been called Euler's integrals, in particular the beta-function

$$B(u, v) = \int_0^1 x^{u-1}(1-x)^{v-1}\,\mathrm{d}x = \frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)} \tag{1.1}$$

([Eu], vol. XVII, pp. 268-288, 343-344, 355) and the hypergeometric integral

$$\int_0^1 \frac{x^{\beta-1}(1-x)^{\gamma-\beta-1}}{(1-xy)^{\alpha}}\,\mathrm{d}x \tag{1.2}$$

([Eu], vol. XII, pp. 254-256). In the special case $\alpha = \gamma$, the integral (1.2) is related to the beta-function as follows:

$$\int_0^1 \frac{x^{\beta-1}(1-x)^{\gamma-\beta-1}}{(1-xy)^{\gamma}}\,\mathrm{d}x = \frac{B(\beta, \gamma-\beta)}{(1-y)^{\beta}} \qquad (\mathrm{Re}\,\gamma > \mathrm{Re}\,\beta > 0).$$

More generally, for any α and for $\mathrm{Re}\,\gamma > \mathrm{Re}\,\beta > 0$ we have

$$\int_0^1 \frac{x^{\beta-1}(1-x)^{\gamma-\beta-1}}{(1-xy)^{\alpha}}\,\mathrm{d}x = B(\beta, \gamma-\beta)\;{}_2F_1(\alpha, \beta; \gamma; y), \tag{1.3}$$

where ${}_2F_1$ denotes the hypergeometric function, defined (with modern notation) by the power series

$${}_2F_1(\alpha, \beta; \gamma; y) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\,\frac{y^n}{n!} \tag{1.4}$$

for $|y| < 1$. Here the parameters α, β, γ are any complex numbers with $\gamma \neq 0, -1, -2, \ldots$, and the Pochhammer symbols $(\alpha)_n$, $(\beta)_n$ and $(\gamma)_n$ are defined by

$$(\xi)_0 = 1, \qquad (\xi)_n = \xi(\xi+1)\ldots(\xi+n-1) \quad (n = 1, 2, \ldots).$$

Euler introduced the hypergeometric series (1.4) as a solution to the second order linear differential equation

$$y(1-y)\,\frac{\mathrm{d}^2 z}{\mathrm{d}y^2} + (\gamma - (\alpha+\beta+1)\,y)\,\frac{\mathrm{d}z}{\mathrm{d}y} - \alpha\beta z = 0 \tag{1.5}$$

([Eu], vol. XVI_2, pp. 41-55). The formula (1.3) can be easily proved either by expanding $(1-xy)^{-\alpha}$ in a binomial series and by term-by-term integration, or by showing that the integral (1.2) is also a solution, regular at $y = 0$, of the hypergeometric differential equation (1.5) ([Eu], vol. XII, p. 256). Since the integral (1.2) is plainly a holomorphic function of y in the cut plane $\mathbb{C}\setminus[1, +\infty)$, (1.3) yields the analytic continuation of ${}_2F_1(\alpha, \beta; \gamma; y)$ outside the unit disc $|y| < 1$.

Suitable choices of the parameters α, β, γ transform the hypergeometric series (1.4) into the Taylor expansions of some elementary functions (see [Er] pp. 101-102, formulae (4)-(17)). Therefore, general methods to investigate arithmetical properties of the values of ${}_2F_1$ at special points yield interesting consequences about the arithmetic of several «natural» constants. This has been traditionally pursued by viewing ${}_2F_1$ as a solution of the differential equation (1.5), and by employing (1.5) for the construction of Padé or Padé-type approximations to ${}_2F_1$ (see, e.g., [C]). However, in recent years new attention was directed to the Euler integral representation (1.3) of ${}_2F_1$, as well as to its generalizations to the higher dimensional cases.

We incidentally recall that natural generalizations of ${}_2F_1$ are the functions

$$ {}_pF_q(\alpha_1, \dots, \alpha_p; \gamma_1, \dots, \gamma_q; y) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\gamma_1)_n \dots (\gamma_q)_n} \frac{y^n}{n!}, $$

the most interesting cases in number theory being obtained for $p = q + 1$.

By (1.1) and by the invariance of the series (1.4) under the interchange of α and β, the integral representation (1.3) of ${}_2F_1$ yields

$$ (1.6) \qquad {}_2F_1(\alpha, \beta; \gamma; y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)} \int_0^1 \frac{x^{\beta-1}(1-x)^{\gamma-\beta-1}}{(1-xy)^{\alpha}}\, \mathrm{d}x $$

$$ = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)} \int_0^1 \frac{x^{\alpha-1}(1-x)^{\gamma-\alpha-1}}{(1-xy)^{\beta}}\, \mathrm{d}x $$

for $\operatorname{Re}\gamma > \max\{\operatorname{Re}\alpha, \operatorname{Re}\beta\}$ and $\min\{\operatorname{Re}\alpha, \operatorname{Re}\beta\} > 0$, and can be easily generalized to ${}_{q+1}F_q$ as follows:

$$ {}_{q+1}F_q(\alpha_1, \dots, \alpha_{q+1}; \gamma_1, \dots, \gamma_q; y) = \frac{\Gamma(\gamma_1) \dots \Gamma(\gamma_q)}{\Gamma(\alpha_1)\,\Gamma(\gamma_1-\alpha_1) \dots \Gamma(\alpha_q)\,\Gamma(\gamma_q-\alpha_q)} $$

$$ \times \int_0^1 \dots \int_0^1 \frac{x_1^{\alpha_1-1}(1-x_1)^{\gamma_1-\alpha_1-1} \dots x_q^{\alpha_q-1}(1-x_q)^{\gamma_q-\alpha_q-1}}{(1-x_1 \dots x_q y)^{\alpha_{q+1}}}\, \mathrm{d}x_1 \dots \mathrm{d}x_q $$

for $\operatorname{Re}\gamma_h > \operatorname{Re}\alpha_h > 0$ $(h = 1, \dots, q)$.

1.2 - *Further developments*

In 1979, a few months after the appearance of Apéry's celebrated proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$, Beukers [B] employed double and triple Euler-type integrals to produce through a new and more natural method the same sequences of rational approximations to $\zeta(2) = \sum_{n=1}^{\infty} n^{-2} = \pi^2/6$ and to $\zeta(3)$ already found by Apéry, and hence the same irrationality measures of these constants obtained by Apéry (see (1.11) and (1.13) below).

We recall that μ is said to be an *irrationality measure* of an irrational number α if for any $\varepsilon > 0$ there exists a constant $q_0 = q_0(\varepsilon) > 0$ such that

$$\left| \alpha - \frac{p}{q} \right| > q^{-\mu-\varepsilon}$$

for all integers p and q with $q > q_0$. As usual, we denote by $\mu(\alpha)$ the least irrationality measure of α. Also, an irrationality measure of a number α is usually obtained by means of a sequence (r_n/s_n) of rational approximations to α, by applying the following well-known

Proposition. *Let $\alpha \in \mathbb{R}$, and let (r_n), (s_n) be sequences of integers satisfying*

$$\lim_{n\to\infty} \frac{1}{n} \log |r_n - s_n \alpha| = -R$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log |s_n| \leqslant S$$

for some positive numbers R and S. Then $\alpha \notin \mathbb{Q}$, and

$$\mu(\alpha) \leqslant \frac{S}{R} + 1 .$$

Beukers [B] considered the integrals

$$\int_0^1 \int_0^1 \left(\frac{x(1-x)\, y(1-y)}{1-xy} \right)^n \frac{\mathrm{d}x\, \mathrm{d}y}{1-xy} \tag{1.7}$$

for $\zeta(2)$, and

$$(1.8)\qquad \int_0^1\int_0^1\int_0^1 \left(\frac{x(1-x)\,y(1-y)\,z(1-z)}{1-(1-xy)\,z}\right)^n \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)\,z}$$

for $\zeta(3)$, and employed two different representations for each of the integrals (1.7) or (1.8). By n-fold partial integration we have

$$\int_0^1 x^n(1-x)^n \frac{\mathrm{d}x}{(1-xy)^{n+1}} = -\frac{1}{ny}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}(x^n(1-x)^n)\frac{\mathrm{d}x}{(1-xy)^n} = \ldots$$

$$= \frac{(-1)^n}{n!\,y^n}\int_0^1 \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n(1-x)^n)\frac{\mathrm{d}x}{1-xy}\,.$$

Hence, defining

$$(1.9)\qquad L_n(x) := \frac{1}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n(1-x)^n) = \sum_{k=0}^n (-1)^k \binom{n}{k}\binom{n+k}{k} x^k \in \mathbb{Z}[x]\,,$$

the double integral (1.7) can be written as

$$(1.10)\qquad \int_0^1\int_0^1 \left(\frac{x(1-x)\,y(1-y)}{1-xy}\right)^n \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy} = (-1)^n \int_0^1\int_0^1 L_n(x)(1-y)^n\,\frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}\,.$$

From (1.9) we get

$$L_n\left(\frac{1-t}{2}\right) = \frac{1}{2^n n!}\,\frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^2-1)^n,$$

and this is the n-th Legendre polynomial ([Er] p. 151, formula (17)).

By expanding $(1-xy)^{-1} = \sum_{k=0}^{\infty} x^k y^k$ and integrating term by term, one easily sees that the right side of (1.10) has the arithmetic form $a_n - b_n\,\zeta(2)$ for suitable $a_n \in \mathbb{Q}$ and $b_n \in \mathbb{Z}$. Moreover we have $d_n^2 a_n \in \mathbb{Z}$, where $d_0 = 1$ and $d_n = \text{l.c.m.}\{1, \ldots, n\}$ for $n = 1, 2, \ldots$. Thus one can apply the Proposition above with $\alpha = \zeta(2)$, $r_n = d_n^2 a_n$ and $s_n = d_n^2 b_n$. Standard asymptotic estimates for the integral (1.7) as $n \to \infty$, together with the asymptotic formula $d_n = \exp(n + o(n))$

given by the prime number theorem, show that $\zeta(2) \notin \mathbb{Q}$ and

$$\mu(\zeta(2)) \leqslant \frac{5\ \log \dfrac{\sqrt{5}+1}{2} + 2}{5\ \log \dfrac{\sqrt{5}+1}{2} - 2} + 1 = 11.85078\ldots. \tag{1.11}$$

A similar but more involved computation, again based on repeated partial integration, yields the analogue of (1.10) for the triple integral (1.8), i.e.,

$$\int_0^1\int_0^1\int_0^1 \left(\frac{x(1-x)\,y(1-y)\,z(1-z)}{1-(1-xy)\,z}\right)^n \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)\,z} = \int_0^1\int_0^1\int_0^1 L_n(x)\,L_n(y)\,\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)\,z}\,, \tag{1.12}$$

where L_n is again the polynomial (1.9). The right side of (1.12) is easily seen to be $a_n' - 2b_n'\,\zeta(3)$ with $b_n' \in \mathbb{Z}$ and $d_n^3\,a_n' \in \mathbb{Z}$. Then standard estimates for (1.8) yield $\zeta(3) \notin \mathbb{Q}$ and

$$\mu(\zeta(3)) \leqslant \frac{4\ \log(\sqrt{2}+1)+3}{4\ \log(\sqrt{2}+1)-3} + 1 = 13.41782\ldots. \tag{1.13}$$

The remarkable importance and originality of Beukers' paper [B] generated a diffuse – and, in a sense, misleading – opinion that the key instruments for fruitful applications of Euler-type integrals to irrationality problems should be the Legendre (or Legendre-type) polynomials and the related partial integration method. For instance, in 1980 Alladi and Robinson [AR] applied to simple Euler-type integrals the one-dimensional analogue of Beukers' method, again based upon Legendre polynomials and partial integration. Thus they obtained irrationality measures, subsequently improved by other authors, for $\log 2$, $\pi/\sqrt{3}$, and some other constants.

The next step forward, again in the main stream of Beukers' method involving Legendre polynomials and partial integration, was made by Hata [H] who considered one-, two-, and three-dimensional integrals containing suitable Legendre-type polynomials whose coefficients are products of binomial coefficients of a special type. Such polynomials are relevant because their coefficients possess a large common divisor. In [H] as well as in a series of subsequent papers, Hata showed how to eliminate common prime factors of the above-mentioned binomial coefficients occurring in Legendre-type polynomials. Thus he succeeded in reducing the size of the rational coefficients of linear forms involving constants such as $\log 2$, $\pi/\sqrt{3}$, $\zeta(2)$, $\zeta(3)$, etc., given by suitable Euler-type integrals, and obtained improvements on the irrationality measures of the constants involved.

As with (1.10) and (1.12), Hata's highly technical method ultimately relies on a twofold representation, obtained by repeated partial integration, for each of the Euler-type integrals considered. In the representation similar to (1.7)-(1.8), i.e. where the Legendre-type polynomials do not explicitly appear, Hata's integrals for $\zeta(2)$ and $\zeta(3)$ essentially correspond to integrals (1.7) or (1.8) where the exponents of the five factors in the rational function $x(1-x)y(1-y)(1-xy)^{-1}$, or of the seven factors in $x(1-x)y(1-y)z(1-z)(1-(1-xy)z)^{-1}$, are not all equal.

2 - Birational transformations and permutation groups

The state of the art changed in 1996 with the publication of the paper [RV1] by Rhin and the author, where we introduced new ideas to deal with suitable families of double integrals of Euler-Beukers' type and we proved the best irrationality measure of $\zeta(2)$ obtained so far, namely

$$\mu(\zeta(2)) < 5.441242\ldots. \tag{2.1}$$

The method of [RV1] was subsequently adapted to one-dimensional integrals for the diophantine study of logarithms of rational numbers [V2] and of logarithms of algebraic numbers [AV], and, overcoming considerable new difficulties, to triple integrals of Euler-Beukers' type in the paper [RV2], where we obtained the best known irrationality measure of $\zeta(3)$, namely

$$\mu(\zeta(3)) < 5.513890\ldots. \tag{2.2}$$

We refer the reader to [V3], Section 1, for the successive improvements on the irrationality measures of $\zeta(2)$ and $\zeta(3)$ obtained since Apéry's upper bounds (1.11) and (1.13), until our results (2.1) and (2.2).

Very recently, in [RV3], Rhin and the author have also applied their method to the diophantine study of double integrals of Euler's type related to the dilogarithm $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} x^n/n^2$. Thus we have obtained qualitative and quantitative improvements on all the best previously known irrationality results for the values of the dilogarithm at positive rational numbers.

The core of Rhin and Viola's method consists in the construction of a permutation group acting on a family of Euler-type integrals related to the constant studied ($\zeta(2)$, or $\zeta(3)$, or the dilogarithm of a rational number, etc.). Such a permutation group is obtained by combining the hypergeometric integral transformation expressed by the relation (1.6) with the action of a suitable birational transformation. The latter, and in particular its dimension, depends on the constant studied, whereas the hypergeometric transformation (1.6) is the same in all cases. The algebraic structure of the above-mentioned permutation group yields strong information on the factorization of the rational coefficients of the linear forms involving the constant considered, and hence on the diophantine properties of that constant. Presumably, a better knowledge of the geometry of the birational transformations coming into play would yield further consequences on the arithmetic of the constants involved (see [F] for a first step towards the study of varieties related to the Rhin-Viola permutation groups).

2.1 - *Double Euler-type integrals*

We outline here the main results of the paper [RV1]. For integers $h, i, j, k, l \geqslant 0$ let

$$I(h, i, j, k, l) = \int_0^1 \int_0^1 \frac{x^h (1-x)^i y^k (1-y)^j}{(1-xy)^{i+j-l}} \frac{\mathrm{d}x\, \mathrm{d}y}{1-xy}, \tag{2.3}$$

and let $\tau : (x, y) \mapsto (X, Y)$ be the birational transformation defined by

$$\tau : \begin{cases} X = \dfrac{1-x}{1-xy} \\ Y = 1 - xy\,. \end{cases} \tag{2.4}$$

It is easy to check that τ has period 5 and maps the open unit square $(0, 1)^2$ onto itself. Moreover, the jacobian determinant of (2.4) is

$$\frac{\mathrm{d}(X, Y)}{\mathrm{d}(x, y)} = \frac{x}{1-xy},$$

and by (2.4) we have $x = 1 - XY$. Hence

$$\frac{\mathrm{d}X\, \mathrm{d}Y}{1-XY} = \frac{\mathrm{d}x\, \mathrm{d}y}{1-xy},$$

so that both the integration domain $(0,1)^2$ and the measure $\mathrm{d}x\,\mathrm{d}y/(1-xy)$ in the integral (2.3) are invariant under the action of τ. Therefore, if we apply the birational transformation τ to $I(h,i,j,k,l)$, i.e., if we make the change of variables

$$\tau^{-1}:\begin{cases} x = 1 - XY \\ y = \dfrac{1-Y}{1-XY} \end{cases}$$

and then replace X, Y with x, y respectively, we easily obtain

$$I(h,i,j,k,l) = I(i,j,k,l,h),$$

whence the value of the integral (2.3) is invariant under the action of the cyclic permutation $\boldsymbol{\tau}$ defined by

$$\boldsymbol{\tau} = (h\ i\ j\ k\ l).$$

Similarly, let $\sigma:(x,y)\mapsto(X,Y)$ be defined by

$$\sigma:\begin{cases} X = y \\ Y = x\,. \end{cases}$$

If we apply the transformation σ to $I(h,i,j,k,l)$, i.e., if we interchange the variables x, y in the integral (2.3), we get

$$I(h,i,j,k,l) = I(k,j,i,h,l),$$

so that the value of (2.3) is also invariant under the action of the permutation $\boldsymbol{\sigma}$ defined by

$$\boldsymbol{\sigma} = (h\ k)(i\ j).$$

Thus the value of (2.3) is invariant under the action of the permutation group

$$\boldsymbol{T} = \langle \boldsymbol{\tau}, \boldsymbol{\sigma} \rangle \tag{2.5}$$

generated by $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, which is plainly isomorphic to the dihedral group $\mathfrak{D}_5$ of order 10.

Besides the integers

$$h,\ i,\ j,\ k,\ l, \tag{2.6}$$

a crucial role is played by the five integers

$$j+k-h, \quad k+l-i, \quad l+h-j, \quad h+i-k, \quad i+j-l. \tag{2.7}$$

We extend the actions of the permutations $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ on any linear combination of the integers (2.6) by linearity. Therefore $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ act on the ten integers (2.6)-(2.7) as follows:

$$\boldsymbol{\tau} = (h\ \ i\ \ j\ \ k\ \ l)(j+k-h\ \ k+l-i\ \ l+h-j\ \ h+i-k\ \ i+j-l) \tag{2.8}$$

and

$$\boldsymbol{\sigma} = (h\ \ k)(i\ \ j)(j+k-h\ \ h+i-k)(k+l-i\ \ l+h-j). \tag{2.9}$$

Let $d_0 = 1$ and $d_n = $ l.c.m. $\{1, \dots, n\}$ for any integer $n \geqslant 1$. Also, we denote by max, max′, max″, ... the successive maxima in a finite sequence of real numbers: if $\mathcal{A} = (a_1, \dots, a_n)$ is any finite sequence of real numbers (with $n \geqslant 3$) and $i_1, \dots, i_n$ is a reordering of $1, \dots, n$ such that

$$a_{i_1} \geqslant a_{i_2} \geqslant a_{i_3} \geqslant \dots \geqslant a_{i_n},$$

we define

$$\max \mathcal{A} = a_{i_1}, \qquad \max{}' \mathcal{A} = a_{i_2}, \qquad \max{}'' \mathcal{A} = a_{i_3}. \tag{2.10}$$

Let $\mathcal{S}$ be the sequence of the integers (2.7):

$$\mathcal{S} = (j+k-h,\ k+l-i,\ l+h-j,\ h+i-k,\ i+j-l),$$

and let

$$M = \max \mathcal{S}, \qquad N = \max{}' \mathcal{S}. \tag{2.11}$$

In [RV1], Theorem 2.2, we prove that

$$I(h, i, j, k, l) = a - b\zeta(2) \quad \text{with } b \in \mathbb{Z} \text{ and } d_M d_N a \in \mathbb{Z}, \tag{2.12}$$

where M and N are defined by (2.11).

As far as I know, the proof of (2.12) given in Theorem 2.2 of [RV1] is the first example in the literature where the arithmetical structure (i.e., in the present case, the expression $a - b\zeta(2)$ with $a \in \mathbb{Q}$ and $b \in \mathbb{Z}$) of an Euler-type integral of dimension $\geqslant 2$ is obtained without using Legendre or Legendre-type polynomials and partial integration. The proof of (2.12) in [RV1] is based on the invariance of the value of $I(h, i, j, k, l)$ under the action of the group (2.5), and on a method of descent obtained by suitable linear decompositions of the rational function

$$f(x,\, y) := \frac{x^h (1-x)^i y^k (1-y)^j}{(1-xy)^{i+j-l}}\,. \tag{2.13}$$

Moreover, the same method of descent shows that the integer b in (2.12) is given by a double contour integral:

$$b = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \oint_C \oint_{C_x} \frac{x^h (1-x)^i y^k (1-y)^j}{(1-xy)^{i+j-l}}\, \frac{\mathrm{d}x\, \mathrm{d}y}{1-xy}\,, \tag{2.14}$$

where $C = \{x \in \mathbb{C} : |x| = \varrho_1\}$ and $C_x = \{y \in \mathbb{C} : |y - 1/x| = \varrho_2\}$ for any ϱ_1, $\varrho_2 > 0$.

It is easy to see that if $i + j - l > \min\{h,\, i,\, j,\, k\}$, which is in fact the case with the numerical choice (2.34) below yielding the best known irrationality measure (2.1) of $\zeta(2)$, the integral $I(h,\, i,\, j,\, k,\, l)$ cannot be transformed by partial integration into an integral containing Legendre-type polynomials to which Hata's method applies. Therefore it is essential to dispense with the partial integration method and with Legendre-type polynomials, and to use instead the method of descent introduced in [RV1].

From (2.8) and (2.9) we see that the permutation group (2.5) is intransitive over the integers (2.6)-(2.7), because each of the two sets (2.6) and (2.7) is mapped onto itself by both $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$. As we anticipated above, under the further assumption that the integers (2.7) are all $\geqslant 0$ we can enlarge the permutation group (2.5) by introducing a «hypergeometric» permutation $\boldsymbol{\varphi}$, induced by the relation (1.6), with the effect that the new permutation group $\boldsymbol{\Phi} = \langle \boldsymbol{\varphi},\, \boldsymbol{\tau},\, \boldsymbol{\sigma} \rangle$ is transitive over (2.6)-(2.7). Moreover, the hypergeometric permutation $\boldsymbol{\varphi}$ brings into play the factorials of the integers (2.6) and (2.7) through the gamma-factors appearing in (1.6). The p-adic valuation of such factorials replaces – with the double advantage of being simpler and more general – Hata's p-adic valuation of binomial coefficients occurring in Legendre-type polynomials.

We choose in (1.6)

$$\alpha = i + j - l + 1, \quad \beta = h + 1, \quad \gamma = h + i + 2\,.$$

Then (1.6) yields

$$\int_0^1 \frac{x^h (1-x)^i}{(1-xy)^{i+j-l+1}}\, \mathrm{d}x = \frac{h!\, i!}{(i+j-l)!\, (l+h-j)!} \int_0^1 \frac{x^{i+j-l} (1-x)^{l+h-j}}{(1-xy)^{h+1}}\, \mathrm{d}x\,.$$

Multiplying by $y^k(1-y)^j$ and integrating in $0 \leqslant y \leqslant 1$ we obtain

$$I(h,\, i,\, j,\, k,\, l) = \frac{h!\, i!}{(i+j-l)!\,(l+h-j)!}\, I(i+j-l,\ l+h-j,\ j,\ k,\ l), \tag{2.15}$$

whence

$$\frac{I(h,\, i,\, j,\, k,\, l)}{h!\ i!\, j!\ k!\ l!} = \frac{I(i+j-l,\ l+h-j,\ j,\ k,\ l)}{(i+j-l)!\ (l+h-j)!\, j!\ k!\ l!}\,. \tag{2.16}$$

Let φ be the integral transformation acting on the quotient

$$\frac{I(h,\, i,\, j,\, k,\, l)}{h!\ i!\, j!\ k!\ l!} \tag{2.17}$$

as is given by (2.16), and let $\boldsymbol{\varphi}$ be the corresponding permutation, mapping h, i, j, k, l respectively to $i+j-l$, $l+h-j$, j, k, l and extended to any linear combination of h, i, j, k, l by linearity. Then the action of $\boldsymbol{\varphi}$ on (2.6)-(2.7) is

$$\boldsymbol{\varphi} = (h\ \ i+j-l)(i\ \ l+h-j)(j+k-h\ \ k+l-i)$$

and, by (2.16), the value of the quotient (2.17) is clearly invariant under the action of the permutation group

$$\boldsymbol{\Phi} = \langle \boldsymbol{\varphi},\ \boldsymbol{\tau},\ \boldsymbol{\sigma} \rangle$$

generated by $\boldsymbol{\varphi}$, $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$. In [RV1], p. 38, we show that the group $\boldsymbol{\Phi}$ is isomorphic to the symmetric group $\mathfrak{S}_5$ of permutations of five elements. In particular, $\boldsymbol{\Phi}$ has order

$$|\boldsymbol{\Phi}| = 5! = 120\,.$$

Since the value of (2.17) is invariant under the action of $\boldsymbol{\Phi}$, for any permutation $\boldsymbol{\varrho} \in \boldsymbol{\Phi}$ we have

$$\frac{I(h,\, i,\, j,\, k,\, l)}{h!\ i!\, j!\ k!\ l!} = \frac{I(\boldsymbol{\varrho}(h),\, \boldsymbol{\varrho}(i),\, \boldsymbol{\varrho}(j),\, \boldsymbol{\varrho}(k),\, \boldsymbol{\varrho}(l))}{\boldsymbol{\varrho}(h)!\, \boldsymbol{\varrho}(i)!\, \boldsymbol{\varrho}(j)!\, \boldsymbol{\varrho}(k)!\, \boldsymbol{\varrho}(l)!}\,. \tag{2.18}$$

Thus we associate with $\boldsymbol{\varrho}$ the quotient

$$\frac{h!\ i!\, j!\ k!\ l!}{\boldsymbol{\varrho}(h)!\, \boldsymbol{\varrho}(i)!\, \boldsymbol{\varrho}(j)!\, \boldsymbol{\varrho}(k)!\, \boldsymbol{\varrho}(l)!} \tag{2.19}$$

resulting from the transformation formula (2.18) for $I(h, i, j, k, l)$. If $\boldsymbol{\varrho}, \boldsymbol{\varrho}' \in \boldsymbol{\Phi}$

lie in the same left coset of the subgroup $\boldsymbol{T} = \langle \boldsymbol{\tau}, \boldsymbol{\sigma} \rangle$ in $\boldsymbol{\Phi}$, the quotient (2.19) equals the analogous quotient for $\boldsymbol{\varrho}'$. Thus with each left coset of $\boldsymbol{T}$ in $\boldsymbol{\Phi}$ we associate the corresponding quotient (2.19), where $\boldsymbol{\varrho}$ is any representative of the coset considered.

For any $\boldsymbol{\varrho} \in \boldsymbol{\Phi}$ we simplify the quotient (2.19) by removing the factorials of the integers appearing both in the numerator and in the denominator. If, after this simplification, the resulting quotient has v factorials in the numerator and v in the denominator, we say that $\boldsymbol{\varrho}$ is a permutation of level v, or that the left coset $\boldsymbol{\varrho T}$ has level v.

Since $|\boldsymbol{\Phi}| = 120$ and $|\boldsymbol{T}| = 10$, there are 12 left cosets of $\boldsymbol{T}$ in $\boldsymbol{\Phi}$, yielding 12 distinct quotients of factorials (2.19), which can be classified as follows (see [RV1], pp. 39-40):

$$\begin{gathered} 1 \text{ coset of level } 0, \\ 5 \text{ cosets of level } 2, \\ 5 \text{ cosets of level } 3, \\ 1 \text{ coset of level } 5. \end{gathered}$$

We remark that the integers M and N defined by (2.11) are invariant under the actions of the permutations $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, but not under the action of $\boldsymbol{\varphi}$. However, in order to get arithmetical consequences for $\zeta(2)$ from the group-theoretic arguments outlined above, we require M and N to be invariant under the action of the whole permutation group $\boldsymbol{\Phi}$. Therefore, at this point we must change the definitions of M and N as follows. Let

$$\mathcal{C} = (h,\ i,\ j,\ k,\ l,\ j+k-h,\ k+l-i,\ l+h-j,\ h+i-k,\ i+j-l)$$

be the sequence of the integers (2.6)-(2.7). We define

$$M = \max \mathcal{C}, \qquad N = \max{}' \mathcal{C}\,. \tag{2.20}$$

Plainly the M and N defined by (2.20) are invariant under the action of $\boldsymbol{\Phi}$. Also, since the former M and N defined by (2.11) do not exceed the new ones, (2.12) holds *a fortiori* with M and N given by (2.20).

We replace in (2.12) h, i, j, k, l by hn, in, jn, kn, ln respectively, where h, i, j, k, l are fixed and $n = 1, 2, \dots$, so that M and N are replaced by Mn and Nn respectively. Then (2.12) yields

$$I(hn,\ in,\ jn,\ kn,\ ln) = a_n - b_n\,\zeta(2) \tag{2.21}$$

with $b_n \in \mathbb{Z}$ and $d_{Mn}\, d_{Nn}\, a_n \in \mathbb{Z}$.

Let

$$A_n = d_{Mn} d_{Nn} a_n, \qquad B_n = d_{Mn} d_{Nn} b_n.$$

From (2.21) we have

$$d_{Mn} d_{Nn} I(hn, in, jn, kn, ln) = A_n - B_n \zeta(2) \tag{2.22}$$

with $A_n, B_n \in \mathbb{Z}$. In order to get a good irrationality measure of $\zeta(2)$ through the Proposition of Section 1.2, one (essentially) requires two sequences of integers, say the (A_n) and (B_n) in (2.22), such that $A_n - B_n \zeta(2) \neq 0$ and, for $n \to \infty$,

$$|A_n - B_n \zeta(2)| \to 0 \tag{2.23}$$

as rapidly as possible, with

$$|B_n| \to \infty \tag{2.24}$$

as slowly as possible. For suitable h, i, j, k, l the permutation group method allows one to improve considerably the sequences (A_n) and (B_n) in (2.22). One finds a large common divisor $\varDelta_n \varDelta'_n$ of A_n and B_n (see (2.28) below), so that, defining

$$D_n = \frac{d_{Mn} d_{Nn}}{\varDelta_n \varDelta'_n}, \tag{2.25}$$

the integers $D_n a_n = A_n / (\varDelta_n \varDelta'_n)$ and $D_n b_n = B_n / (\varDelta_n \varDelta'_n)$ yield a linear form $|D_n a_n - D_n b_n \zeta(2)|$ tending to 0 more rapidly than (2.23), with a growth of $|D_n b_n|$ slower than (2.24).

The construction of the common divisor $\varDelta_n \varDelta'_n$ of A_n and B_n is as follows. By the invariance of M and N under the action of the permutation $\boldsymbol{\varphi}$ we get, as in (2.22),

$$d_{Mn} d_{Nn} I((i+j-l)n, (l+h-j)n, jn, kn, ln) = A'_n - B'_n \zeta(2)$$

with $A'_n, B'_n \in \mathbb{Z}$. Therefore, if we apply in (2.22) the transformation formula (2.15) corresponding to $\boldsymbol{\varphi}$ we obtain

$$A_n - B_n \zeta(2) = \frac{(hn)!\,(in)!}{((i+j-l)n)!\,((l+h-j)n)!} (A'_n - B'_n \zeta(2)),$$

whence, by the irrationality of $\zeta(2)$,

$$
\begin{aligned}
&((i+j-l)\,n)!\,((l+h-j)n)!\,A_n = (hn)!\,(in)!\,A_n', \\
&((i+j-l)\,n)!\,((l+h-j)n)!\,B_n = (hn)!\,(in)!\,B_n'.
\end{aligned}
\tag{2.26}
$$

For a prime number p, let

$$\omega = \{n/p\} = n/p - [n/p]$$

denote the fractional part of n/p. Using the p-adic valuation of the factorials appearing in (2.26), it is easy to see ([RV1], pp. 44-45) that any prime $p > \sqrt{Mn}$ for which

$$[(i+j-l)\,\omega] + [(l+h-j)\,\omega] < [h\omega] + [i\omega] \tag{2.27}$$

divides A_n and B_n.

The above discussion applies to each transformation formula (2.18) corresponding to a left coset of $\boldsymbol{T}$ in $\boldsymbol{\Phi}$ of level 2. The five quotients of factorials (2.19) associated with the five left cosets of level 2 are easily seen to yield the five inequalities for ω obtained by applying the powers of the permutation $\boldsymbol{\tau}$ to $(i+j-l,\ l+h-j,\ h,\ i)$ in (2.27). Let Ω be the set of real numbers $\omega \in [0,1)$ satisfying at least one of such five inequalities. We infer that any prime $p > \sqrt{Mn}$ for which $\{n/p\} \in \Omega$ divides A_n and B_n.

A similar analysis applies to the five transformation formulae (2.18) corresponding to the five left cosets of $\boldsymbol{T}$ in $\boldsymbol{\Phi}$ of level 3, and yields a subset $\Omega' \subset \Omega$ such that p^2 divides A_n and B_n for any prime $p > \sqrt{Mn}$ satisfying $\{n/p\} \in \Omega'$.

Let

$$\Delta_n = \prod_{\substack{p > \sqrt{Mn} \\ \{n/p\} \in \Omega}} p, \qquad \Delta'_n = \prod_{\substack{p > \sqrt{Mn} \\ \{n/p\} \in \Omega'}} p \qquad (n = 1, 2, \ldots), \tag{2.28}$$

where p denotes a prime. From the above discussion we get $\Delta_n \Delta'_n \,|\, A_n$ and $\Delta_n \Delta'_n \,|\, B_n$. Also, by standard arguments ([V1], pp. 463-464) we have

$$\lim_{n \to \infty} \frac{1}{n} \log \Delta_n = \int_{\Omega} \mathrm{d}\psi(x), \qquad \lim_{n \to \infty} \frac{1}{n} \log \Delta'_n = \int_{\Omega'} \mathrm{d}\psi(x),$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the Euler gamma-function. Since $d_{Mn} d_{Nn} = \exp((M+N)\,n + o(n))$ by the prime number theorem, the definition (2.25) yields

$$\lim_{n\to\infty} \frac{1}{n} \log D_n = M + N - \left(\int_{\Omega} \mathrm{d}\,\psi(x) + \int_{\Omega'} \mathrm{d}\,\psi(x) \right). \tag{2.29}$$

Dividing (2.22) by $\Delta_n \Delta'_n$ we get

$$D_n I(hn, in, jn, kn, ln) = D_n a_n - D_n b_n \zeta(2) \in \mathbb{Z} + \mathbb{Z}\zeta(2), \tag{2.30}$$

and we obtain an irrationality measure of $\zeta(2)$ by applying to (2.30) the Proposition in Section 1.2. For this purpose we use (2.29) together with

$$\lim_{n\to\infty} \frac{1}{n} \log I(hn, in, jn, kn, ln) \tag{2.31}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log |b_n|. \tag{2.32}$$

The advantage of using (2.30) instead of (2.22) is quantified by the arithmetical correction

$$\int_{\Omega} \mathrm{d}\psi(x) + \int_{\Omega'} \mathrm{d}\psi(x)$$

in (2.29).

Assuming the integers (2.6)-(2.7) to be all strictly positive, a straightforward computation shows that the function $f(x, y)$ defined by (2.13) has exactly two stationary points (x_0, y_0) and (x_1, y_1) for which $x(1-x)y(1-y) \neq 0$, and that these points satisfy the inequalities $0 < x_0 < 1$, $0 < y_0 < 1$, $x_1 < 0$, $y_1 < 0$, $x_1 y_1 > 1$. Thus the limit (2.31) is given by $\log f(x_0, y_0)$, and using the double contour integral representation for b_n given by (2.14) one easily proves that (2.32) does not exceed $\log |f(x_1, y_1)|$.

If we denote

$$c_0 = -\log f(x_0, y_0), \qquad c_1 = \log |f(x_1, y_1)|,$$

$$c_2 = M + N - \left(\int_{\Omega} \mathrm{d}\psi(x) + \int_{\Omega'} \mathrm{d}\psi(x) \right),$$

from (2.30) and the Proposition in Section 1.2 we obtain the irrationality measure

$$\mu(\zeta(2)) \leqslant \frac{c_1 + c_2}{c_0 - c_2} + 1 = \frac{c_0 + c_1}{c_0 - c_2}, \tag{2.33}$$

provided that $c_0 > c_2$.

The choice

$$h = i = 12, \qquad j = k = 14, \qquad l = 13 \tag{2.34}$$

gives $c_0 = 31.27178857\ldots$, $c_1 = 30.41828189\ldots$, $c_2 = 19.93429159\ldots$. From (2.33) we obtain

$$\mu(\zeta(2)) < 5.441242\ldots.$$

2.2 - *Triple Euler-type integrals*

We summarize here the chief points of the arithmetic theory for triple integrals of Euler-Beukers' type related to $\zeta(3)$, given in the paper [RV2]. We consider the integral

$$\int_0^1\int_0^1\int_0^1 \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)\,z)^{q+h-r}}\,\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)\,z} \tag{2.35}$$

for integer parameters $h, j, k, l, q, r, s \geqslant 0$ such that $h \leqslant k + r$ (these inequalities are easily seen to be necessary and sufficient for the integral (2.35) to be finite). Let $\vartheta : (x, y, z) \mapsto (X, Y, Z)$ be the birational transformation defined by

$$\vartheta : \begin{cases} X = (1-y)\,z \\[2mm] Y = \dfrac{(1-x)(1-z)}{1-(1-xy)\,z} \\[4mm] Z = \dfrac{y}{1-(1-y)\,z}\,. \end{cases} \tag{2.36}$$

It is easy to check that ϑ has period 8 and maps the open unit cube $(0, 1)^3$ onto itself. Moreover, under the action of ϑ we have

$$\frac{\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z}{1-(1-XY)\,Z} = -\,\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)\,z}\,. \tag{2.37}$$

Thus, if we apply the transformation ϑ to the integral (2.35), i.e., if we make the change of variables

$$\vartheta^{-1}\colon \begin{cases} x = \dfrac{(1-Y)(1-Z)}{1-(1-XY)\,Z} \\ y = (1-X)\,Z \\ z = \dfrac{X}{1-(1-X)\,Z} \end{cases}$$

and then replace X, Y, Z with x, y, z respectively, the integral becomes

$$\int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{x^j(1-x)^{k+r-h}\,y^l(1-y)^h z^k(1-z)^r}{(1-(1-x)\,z)^{j+q-l-s}(1-(1-xy)\,z)^{r+l-q}}\,\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)\,z}\,. \tag{2.38}$$

We define $m = k+r-h$, whence $m \geqslant 0$ and

$$h+m = k+r\,, \tag{2.39}$$

and we make the further assumption

$$j+q = l+s\,, \tag{2.40}$$

which eliminates from (2.38) the «parasite» factor $1-(1-x)\,z$. By (2.40) we have $r+l-q = r+j-s$, so that the integral (2.38) can be written as

$$\int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{x^j(1-x)^m y^l(1-y)^h z^k(1-z)^r}{(1-(1-xy)\,z)^{r+j-s}}\,\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)\,z}\,.$$

This integral is obtained from (2.35) by applying to the parameters the cyclic permutation $(h\ j\ k\ l\ m\ q\ r\ s)$. Therefore, if for any integers $h, j, k, l, m, q, r,$

$s \geqslant 0$ satisfying (2.39) and (2.40) we define

$$I(h, j, k, l, m, q, r, s) \tag{2.41}$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{x^h (1-x)^l y^k (1-y)^s z^j (1-z)^q}{(1-(1-xy)\,z)^{q+h-r}} \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)\,z},$$

where m is a «hidden» parameter, the transformation ϑ given by (2.36) changes (2.41) into $I(j, k, l, m, q, r, s, h)$. Therefore the value of the integral (2.41) is invariant under the action of the cyclic permutation $\boldsymbol{\vartheta}$ defined by

$$\boldsymbol{\vartheta} = (h\ j\ k\ l\ m\ q\ r\ s)\,. \tag{2.42}$$

Let now $\sigma : (x, y, z) \mapsto (X, Y, Z)$ be defined by

$$\sigma : \begin{cases} X = y \\ Y = x \\ Z = z\,. \end{cases} \tag{2.43}$$

If we apply the transformation σ to (2.41), i.e., if we interchange the variables x, y, by virtue of (2.39) we get the integral $I(k, j, h, s, r, q, m, l)$. Hence the value of (2.41) is also invariant under the action of the permutation $\boldsymbol{\sigma}$ defined by

$$\boldsymbol{\sigma} = (h\ k)(l\ s)(m\ r)\,.$$

Thus the value of (2.41) is invariant under the action of the permutation group

$$\boldsymbol{\Theta} = \langle \boldsymbol{\vartheta}, \boldsymbol{\sigma} \rangle \tag{2.44}$$

generated by $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$, which is isomorphic to the dihedral group $\mathfrak{D}_8$ of order 16.

Besides the integers

$$h, j, k, l, m, q, r, s, \tag{2.45}$$

we consider the eight integers

$$
\begin{aligned}
h' &:= h + l - j = h + q - s,\\
j' &:= j + m - k = j + r - h,\\
k' &:= k + q - l = k + s - j,\\
l' &:= l + r - m = l + h - k,\\
m' &:= m + s - q = m + j - l,\\
q' &:= q + h - r = q + k - m,\\
r' &:= r + j - s = r + l - q,\\
s' &:= s + k - h = s + m - r,
\end{aligned}
\tag{2.46}
$$

where the double expression for each of (2.46) is obtained by applying (2.39) or (2.40).

We extend the actions of the permutations $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$ on any linear combination of the integers (2.45) by linearity. Note that this is possible because $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$ preserve the linear conditions (2.39) and (2.40). In fact we have

$$\boldsymbol{\vartheta}(h) + \boldsymbol{\vartheta}(m) = j + q = l + s = \boldsymbol{\vartheta}(k) + \boldsymbol{\vartheta}(r),$$

$$\boldsymbol{\vartheta}(j) + \boldsymbol{\vartheta}(q) = k + r = m + h = \boldsymbol{\vartheta}(l) + \boldsymbol{\vartheta}(s),$$

$$\boldsymbol{\sigma}(h) + \boldsymbol{\sigma}(m) = k + r = h + m = \boldsymbol{\sigma}(k) + \boldsymbol{\sigma}(r),$$

$$\boldsymbol{\sigma}(j) + \boldsymbol{\sigma}(q) = j + q = s + l = \boldsymbol{\sigma}(l) + \boldsymbol{\sigma}(s).$$

Therefore $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$ act on the sixteen integers (2.45)-(2.46) as follows:

$$\boldsymbol{\vartheta} = (h\ j\ k\ l\ m\ q\ r\ s)(h'\ j'\ k'\ l'\ m'\ q'\ r'\ s')$$

and

$$\boldsymbol{\sigma} = (h\ k)(l\ s)(m\ r)(h'\ k')(l'\ s')(m'\ r').$$

Let

$$\mathcal{U} = (h', j', k', l', m', q', r', s')$$

be the sequence of the integers (2.46), and let

$$M = \max \mathcal{U}, \qquad N = \max{}' \mathcal{U}, \qquad Q = \max{}'' \mathcal{U}, \tag{2.47}$$

with the notation (2.10). Similarly to (2.12) and (2.14), using the action of the group (2.44) and a method of descent, in [RV2] we prove that

(2.48) $$I(h,j,k,l,m,q,r,s)=a-2b\zeta(3) \text{ with } b\in\mathbb{Z} \text{ and } d_M d_N d_Q a\in\mathbb{Z},$$

where

$$b=-\left(\frac{1}{2\pi\sqrt{-1}}\right)^3\oint_C\oint_{C_x}\oint_{C_{x,y}}\frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)\,z)^{q+h-r}}\,\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)\,z},$$

with $C=\{x\in\mathbb{C}:|x|=\varrho_1\}$, $C_x=\{y\in\mathbb{C}:|y-1/x|=\varrho_2\}$ and $C_{x,y}=\{z\in\mathbb{C}:|z-(1-xy)^{-1}|=\varrho_3\}$ for any $\varrho_1,\varrho_2,\varrho_3>0$.

We remark that the linear conditions (2.39) and (2.40) are essential for the validity of (2.48), because one can show that if $j+q>l+s$, the integral (2.35) in general is a linear combination of 1, $\zeta(2)$ and $\zeta(3)$ with rational coefficients.

From now on, we assume the non-negative integers (2.45) to be such that (2.46) are also non-negative. Then we can enlarge the permutation group (2.44) by introducing two hypergeometric permutations $\boldsymbol{\varphi}$ and $\boldsymbol{\chi}$, as follows. In (1.6) we change y into $-yz/(1-z)$ and choose

$$\alpha=q+h-r+1,\qquad \beta=h+1,\qquad \gamma=h+l+2.$$

Then we obtain

$$\int_0^1\frac{x^h(1-x)^l}{(1-(1-xy)\,z)^{q+h-r+1}}\,\mathrm{d}x=(1-z)^{r-q}\,\frac{h!\,l!}{q'!\,r'!}\int_0^1\frac{x^{q'}(1-x)^{r'}}{(1-(1-xy)z)^{h+1}}\,\mathrm{d}x.$$

Multiplying by $y^k(1-y)^s z^j(1-z)^q$ and integrating in $0\leqslant y\leqslant 1$, $0\leqslant z\leqslant 1$, we get

$$I(h,j,k,l,m,q,r,s)=\frac{h!\,l!}{q'!\,r'!}\,I(q',j,k,r',m,r,q,s),$$

whence

(2.49) $$\frac{I(h,j,k,l,m,q,r,s)}{h!\,j!\,k!\,l!\,m!\,q!\,r!\,s!}=\frac{I(q',j,k,r',m,r,q,s)}{q'!\,j!\,k!\,r'!\,m!\,r!\,q!\,s!}.$$

Let φ be the integral transformation acting on the quotient

(2.50) $$\frac{I(h,j,k,l,m,q,r,s)}{h!\,j!\,k!\,l!\,m!\,q!\,r!\,s!}$$

as in (2.49), and let $\boldsymbol{\varphi}$ be the corresponding permutation, mapping h, j, k, l, m, q, r, s respectively to q', j, k, r', m, r, q, s and extended to any linear combination of h, j, k, l, m, q, r, s by linearity. Again this is possible because $\boldsymbol{\varphi}$ preserves (2.39) and (2.40):

$$\boldsymbol{\varphi}(h) + \boldsymbol{\varphi}(m) = q' + m = k + q = \boldsymbol{\varphi}(k) + \boldsymbol{\varphi}(r),$$

$$\boldsymbol{\varphi}(j) + \boldsymbol{\varphi}(q) = j + r = r' + s = \boldsymbol{\varphi}(l) + \boldsymbol{\varphi}(s).$$

Therefore $\boldsymbol{\varphi}$ acts on (2.45)-(2.46) as follows:

$$\boldsymbol{\varphi} = (h\ \ q')(l\ \ r')(q\ \ r)(m'\ \ s').$$

We now change in (1.6) x into z and y into $1 - xy$, and choose

$$\alpha = q + h - r + 1, \qquad \beta = j + 1, \qquad \gamma = j + q + 2.$$

Thus

$$\int_0^1 \frac{z^j (1-z)^q}{(1-(1-xy)\,z)^{q+h-r+1}}\,\mathrm{d}z = \frac{j!\,q!}{q'!\,j'!} \int_0^1 \frac{z^{q'}(1-z)^{j'}}{(1-(1-xy)\,z)^{j+1}}\,\mathrm{d}z.$$

Multiplying by $x^h(1-x)^l y^k (1-y)^s$ and integrating in $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, we get

$$I(h, j, k, l, m, q, r, s) = \frac{j!\,q!}{q'!\,j'!} I(h, q', k, l, m, j', r, s),$$

whence

$$\frac{I(h, j, k, l, m, q, r, s)}{h!\,j!\,k!\,l!\,m!\,q!\,r!\,s!} = \frac{I(h, q', k, l, m, j', r, s)}{h!\,q'!\,k!\,l!\,m!\,j'!\,r!\,s!}. \tag{2.51}$$

Let χ be the integral transformation acting on the quotient (2.50) as in (2.51), and let $\boldsymbol{\chi}$ be the corresponding permutation, mapping h, j, k, l, m, q, r, s respectively to h, q', k, l, m, j', r, s and extended by linearity. Again, $\boldsymbol{\chi}$ preserves (2.39) and (2.40):

$$\boldsymbol{\chi}(h) + \boldsymbol{\chi}(m) = h + m = k + r = \boldsymbol{\chi}(k) + \boldsymbol{\chi}(r),$$

$$\boldsymbol{\chi}(j) + \boldsymbol{\chi}(q) = q' + j' = l + s = \boldsymbol{\chi}(l) + \boldsymbol{\chi}(s).$$

Therefore

$$\boldsymbol{\chi} = (j\ q')(q\ j')(h'\ r')(k'\ m'),$$

and the value of the quotient (2.50) is invariant under the action of the permutation group

$$\boldsymbol{\Phi} = \langle \boldsymbol{\varphi}, \boldsymbol{\chi}, \boldsymbol{\vartheta}, \boldsymbol{\sigma} \rangle \tag{2.52}$$

generated by $\boldsymbol{\varphi}$, $\boldsymbol{\chi}$, $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$. In [RV2], p. 283, we show that there exists an exact sequence of multiplicative groups:

$$1 \to K \hookrightarrow \boldsymbol{\Phi} \to \mathfrak{S}_5 \to 1, \tag{2.53}$$

where K is isomorphic to the additive group $(\mathbb{Z}/2\mathbb{Z})^4$, and $\mathfrak{S}_5$ is the symmetric group of permutations of five elements. This implies in particular $|K| = 2^4$, $|\mathfrak{S}_5| = 5!$, whence, by the exact sequence (2.53), the order of $\boldsymbol{\Phi}$ is

$$|\boldsymbol{\Phi}| = 2^4 \cdot 5! = 1920.$$

The rest of this theory proceeds on the same lines of the discussion in Section 2.1. With any permutation $\boldsymbol{\varrho} \in \boldsymbol{\Phi}$ we associate the quotient

$$\frac{h!\, j!\, k!\, l!\, m!\, q!\, r!\, s!}{\boldsymbol{\varrho}(h)!\, \boldsymbol{\varrho}(j)!\, \boldsymbol{\varrho}(k)!\, \boldsymbol{\varrho}(l)!\, \boldsymbol{\varrho}(m)!\, \boldsymbol{\varrho}(q)!\, \boldsymbol{\varrho}(r)!\, \boldsymbol{\varrho}(s)!}, \tag{2.54}$$

and if $\boldsymbol{\varrho}$, $\boldsymbol{\varrho}'$ lie in the same left coset of $\boldsymbol{\Theta} = \langle \boldsymbol{\vartheta}, \boldsymbol{\sigma} \rangle$ in $\boldsymbol{\Phi}$, the quotient (2.54) equals the analogous quotient for $\boldsymbol{\varrho}'$. We say that $\boldsymbol{\varrho}$ is a permutation of level v, or that the left coset $\boldsymbol{\varrho}\boldsymbol{\Theta}$ has level v, if, after simplifying (2.54), we have v factorials in the numerator and v in the denominator.

Since we now have $|\boldsymbol{\Phi}| = 1920$ and $|\boldsymbol{\Theta}| = 16$, there are 120 left cosets of $\boldsymbol{\Theta}$ in $\boldsymbol{\Phi}$, yielding 120 distinct quotients of factorials (2.54), which can be classified as follows (see [RV2], pp. 286-287):

1 coset of level 0,

12 cosets of level 2,

32 cosets of level 3,

30 cosets of level 4,

32 cosets of level 5,

12 cosets of level 6,

1 coset of level 8.

Let

$$\nabla = (h, j, k, l, m, q, r, s, h', j', k', l', m', q', r', s')$$

be the sequence of the integers (2.45)-(2.46). Similarly to (2.20), we change the definition (2.47) as follows:

$$M = \max \nabla, \qquad N = \max{}' \nabla, \qquad Q = \max{}'' \nabla.$$

For fixed h, j, k, l, m, q, r, s and $n = 1, 2, \dots,$ (2.48) yields

$$I(hn, jn, kn, ln, mn, qn, rn, sn) = a_n - 2b_n \zeta(3)$$

with $b_n \in \mathbb{Z}$ and $d_{Mn} d_{Nn} d_{Qn} a_n \in \mathbb{Z}$. Therefore

$$d_{Mn} d_{Nn} d_{Qn} I(hn, jn, kn, ln, mn, qn, rn, sn) = A_n - 2B_n \zeta(3) \tag{2.55}$$

with $A_n, B_n \in \mathbb{Z}$.

As in Section 2.1, one can divide (2.55) by a large common divisor of A_n and B_n, thus obtaining

$$D_n I(hn, jn, kn, ln, mn, qn, rn, sn) = D_n a_n - 2D_n b_n \zeta(3) \in \mathbb{Z} + 2\mathbb{Z}\zeta(3), \tag{2.56}$$

where, as with (2.29),

$$\lim_{n\to\infty} \frac{1}{n} \log D_n = M + N + Q - (\text{arithmetical correction}). \tag{2.57}$$

Assuming the integers (2.45)-(2.46) to be all strictly positive, the function

$$f(x, y, z) := \frac{x^h (1-x)^l y^k (1-y)^s z^j (1-z)^q}{(1-(1-xy)\,z)^{q+h-r}}$$

has exactly two stationary points (x_0, y_0, z_0) and (x_1, y_1, z_1) for which $x(1-x)\,y(1-y)\,z(1-z) \neq 0$, and these points satisfy $0 < x_0, y_0, z_0 < 1$, $x_1, y_1, z_1 < 0$, $x_1 y_1 > 1$, $z_1 < (1 - x_1 y_1)^{-1}$. Then we have

$$\lim_{n\to\infty} \frac{1}{n} \log I(hn, jn, kn, ln, mn, qn, rn, sn) = \log f(x_0, y_0, z_0) \tag{2.58}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log |b_n| \leq \log |f(x_1, y_1, z_1)|. \tag{2.59}$$

Using (2.57), (2.58) and (2.59), we can apply the Proposition in Section 1.2 to the linear form (2.56). As we show in [RV2], Section 5, the choice

$$h=16\,,\;\; j=17\,,\;\; k=19\,,\;\; l=15\,,\;\; m=12\,,\;\; q=11\,,\;\; r=9\,,\;\; s=13\,,$$

satisfying (2.39) and (2.40) as required, yields the irrationality measure

$$\mu(\zeta(3)) < 5.513890\ldots.$$

3 - Sorokin's integral

The success of our permutation group method in the treatment of the triple integral (2.41) essentially depends upon two «miracles»: one is the (highly non-trivial) three-dimensional birational transformation ϑ of period 8 defined by (2.36), which shows that, assuming (2.39) and (2.40), the value of (2.41) is invariant under the action of the cyclic permutation $\boldsymbol{\vartheta}$ in (2.42). The second miracle is the *a priori* unexpected phenomenon that the linear conditions (2.39) and (2.40) are preserved by all the four permutations $\boldsymbol{\vartheta}$, $\boldsymbol{\sigma}$, $\boldsymbol{\varphi}$, $\boldsymbol{\chi}$, so that, under no additional restrictive assumptions besides (2.39) and (2.40), the permutation group (2.52) automatically acts on the sixteen integers (2.45)-(2.46).

As a first contribution towards a possible interpretation of the first miracle, in [V3], Sections 5 and 6, we give a constructive method to derive ϑ from the two-dimensional transformation τ defined by (2.4), and we also show how our method can be used to construct suitable four-dimensional involutions which can be applied to quadruple integrals of Euler-Vasilyev's type.

At present, we have no convincing explanation of the second miracle, nor any method to predict for which integrals or transformations such a phenomenon occurs.

In order to illustrate this point, we consider a suitable generalization of Sorokin's integral. In [S] p. 51, Sorokin introduced the triple integral

$$c_n(t)=\int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{t^{n+1}x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(t-xy)^{n+1}(t-xyz)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

In the spirit of our paper [RV2] we choose here $t=1$, and we generalize $c_n(1)$ by taking different exponents for the eight factors appearing inside the resulting integral. For integers $h, j, k, l, m, q, r, s \geqslant 0$, let

$$J(h, j, k, l, m, q, r, s) \tag{3.1}$$

$$= \int_0^1\int_0^1\int_0^1 \frac{x^s(1-x)^k y^h(1-y)^q z^r(1-z)^j}{(1-xy)^{q+k-m}(1-xyz)^{m+j-l}}\,\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(1-xy)(1-xyz)}\,.$$

Moreover, let $\eta:(x, y, z)\mapsto(X, Y, Z)$ be the birational transformation defined by

$$\eta:\begin{cases} X = yz \\[1ex] Y = \dfrac{1-x}{1-xyz} \\[2ex] Z = \dfrac{1-z}{1-yz}\,, \end{cases} \tag{3.2}$$

whence

$$\eta^{-1}:\begin{cases} x = \dfrac{1-Y}{1-XY} \\[2ex] y = \dfrac{X}{1-(1-X)\,Z} \\[2ex] z = 1-(1-X)\,Z\,, \end{cases} \tag{3.3}$$

and, by a straightforward computation of the jacobian determinant,

$$\frac{\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z}{1-(1-XY)\,Z} = -\,\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(1-xy)(1-xyz)}\,. \tag{3.4}$$

From (3.2) and (3.3) it is plain that η maps the open unit cube $(0, 1)^3$ onto itself. If we apply the transformation η to the integral $J(h, j, k, l, m, q, r, s)$, i.e., if we make in (3.1) the change of variables (3.3) and then replace X, Y, Z with x, y, z respectively, by virtue of (3.4) we obtain the integral

$$\int_0^1\int_0^1\int_0^1 \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-xy)^{l+s-j-q}(1-(1-x)\,z)^{h+m-k-r}(1-(1-xy)\,z)^{q+k-m}} \tag{3.5}$$

$$\times\,\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)\,z}\,.$$

Therefore, if we assume the linear conditions (2.39) and (2.40) which we write again here:

$$\begin{aligned} h + m &= k + r, \\ j + q &= l + s, \end{aligned} \tag{3.6}$$

the parasite factors $1 - xy$ and $1 - (1 - x)\,z$ disappear from (3.5), and the exponent $q + k - m$ can be written as $q + h - r$. Hence (3.5) becomes (2.41). Thus, assuming the linear conditions (3.6), we get

$$J(h, j, k, l, m, q, r, s) = I(h, j, k, l, m, q, r, s),$$

so that all the results proved in [RV2] for $I(h, j, k, l, m, q, r, s)$ under the assumptions (3.6) hold for the integral $J(h, j, k, l, m, q, r, s)$ defined by (3.1).

Naturally we can transfer the action of the transformation ϑ given by (2.36) to the integral (3.1), by defining $\vartheta_* = \eta^{-1}\vartheta\eta$, the transformation obtained by applying first η, then ϑ, and then η^{-1}. By (2.36), (3.2) and (3.3), one easily sees that the transformation $\vartheta_* : (x, y, z) \mapsto (X, Y, Z)$ is given by the equations

$$\vartheta_* = \eta^{-1}\vartheta\eta : \begin{cases} X = y \\ Y = \dfrac{1 - z}{1 - yz} \\ Z = \dfrac{(1 - yz)\,x}{1 - xyz}\,. \end{cases} \tag{3.7}$$

Since ϑ has period 8, ϑ_* has also period 8. Moreover, by (2.37) and (3.4), under the action of ϑ_* we have

$$\frac{\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z}{(1 - XY)(1 - XYZ)} = -\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(1 - xy)(1 - xyz)}\,.$$

Thus, if we apply the transformation ϑ_* directly to the integral (3.1), i.e., if we make in (3.1) the change of variables

$$\vartheta_*^{-1} = \eta^{-1}\vartheta^{-1}\eta : \begin{cases} x = \dfrac{(1 - XY)\,Z}{1 - (1 - (1 - Y)\,Z)\,X} \\ y = X \\ z = \dfrac{1 - Y}{1 - XY} \end{cases}$$

and then replace X, Y, Z with x, y, z respectively, by the second of (3.6) we get the integral $J(j, k, l, m, q, r, s, h)$. Therefore with the action of the transformation ϑ_* on $J(h, j, k, l, m, q, r, s)$ we associate the permutation

$$\boldsymbol{\vartheta}_* = \boldsymbol{\vartheta} = (h\ j\ k\ l\ m\ q\ r\ s), \tag{3.8}$$

and if we assume both the linear conditions (3.6), we see that they are preserved by the permutation (3.8).

So far everything runs nicely, and one may even consider the pair (J, ϑ_*) given by (3.1) and (3.7) to be more convenient than (I, ϑ) given by (2.41) and (2.36). In fact, in contrast with (2.41), all the eight parameters h, j, k, l, m, q, r, s explicitly appear on the right side of (3.1), and moreover the factors $1 - xy$ and $1 - xyz$ in the denominator of (3.1) look simpler and more natural than the factor $1-(1-xy)z$ in (2.41).

However, if we apply to (3.1) the transformation σ given by (2.43) without employing η and η^{-1}, i.e., if we interchange the variables x, y directly in (3.1), we obtain the integral $J(s, j, q, l, m, k, r, h)$, so that the permutation associated with the action of σ on $J(h, j, k, l, m, q, r, s)$ is $(h\ s)(k\ q)$, which does *not* preserve the linear conditions (3.6).

Similarly, if we apply to (3.1) the hypergeometric integral transformation with respect to z, i.e., if in (1.6) we change x into z and y into xy, choose $\alpha = m + j - l + 1$, $\beta = r + 1$, $\gamma = j + r + 2$, and then multiply by

$$\frac{x^s (1-x)^k y^h (1-y)^q}{(1-xy)^{q+k-m+1}}$$

and integrate in $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, we get (with the notation (2.46)):

$$J(h, j, k, l, m, q, r, s) = \frac{j!\, r!}{l'!\, m'!} J(h, l', k, l, m, q, m', s).$$

Thus the associated permutation is $(j\ l')(r\ m')$, which again does not preserve the linear conditions (3.6).

We infer that there is no natural construction of a permutation group similar to (2.52) acting directly on

$$\frac{J(h, j, k, l, m, q, r, s)}{h!\, j!\, k!\, l!\, m!\, q!\, r!\, s!},$$

i.e., obtained without passing through $I(h, j, k, l, m, q, r, s)$ by means of the transformations η and η^{-1}.

There are some variations on this theme. We can introduce an equivalence relation $\sim$ by defining

$$\eta' \sim \eta$$

if there exist $\lambda \in \langle \vartheta, \sigma \rangle$ and $\nu \in \langle \sigma \rangle$ such that

$$\eta' = \lambda \eta \nu, \tag{3.9}$$

the transformation obtained by applying first ν (i.e., either the identity or (2.43)), then η given by (3.2), and then λ (i.e., any product of transformations (2.36) and (2.43)).

It is easily seen that $\eta \sigma \eta^{-1} \notin \langle \vartheta, \sigma \rangle$. Thus if $\lambda \eta \nu = \lambda_1 \eta \nu_1$ with $\lambda, \lambda_1 \in \langle \vartheta, \sigma \rangle$ and $\nu, \nu_1 \in \langle \sigma \rangle$, then $\nu = \nu_1$ whence $\lambda = \lambda_1$. Since $|\langle \vartheta, \sigma \rangle| = 16$ and $|\langle \sigma \rangle| = 2$, there are 32 transformations (3.9) equivalent to η, each of which, under the assumptions (3.6), plainly changes a Sorokin-type integral (3.1) (with a suitable reordering of h, j, k, l, m, q, r, s depending on λ and ν) into the integral (2.41).

As the referee kindly pointed out, Fischler ([F], Section 5.5.2) gave a transformation in any dimension n which for $n = 3$ (using a notation consistent with ours) can be written as

$$\begin{cases} X = 1 - z \\ Y = \dfrac{(1-x)\, y}{1 - xy} \\ Z = x, \end{cases} \tag{3.10}$$

whence

$$\begin{cases} x = Z \\ y = \dfrac{Y}{1 - (1-Y)\, Z} \\ z = 1 - X\,. \end{cases} \tag{3.11}$$

It is easy to check that Fischler's transformation (3.10) is given by (3.9) for $\lambda = \vartheta\sigma$

and $\nu = \sigma$, and hence is equivalent to η. Specifically, if we define

$$K(h, j, k, l, m, q, r, s) := J(k, h, m, r, q, s, l, j) \tag{3.12}$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{x^j(1-x)^m y^k(1-y)^s z^l(1-z)^h}{(1-xy)^{s+m-q}(1-xyz)^{q+h-r}} \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(1-xy)(1-xyz)}$$

with J given by (3.1), applying the transformation (3.10) to $K(h, j, k, l, m, q, r, s)$, i.e. making in (3.12) the change of variables (3.11) and then replacing X, Y, Z with x, y, z respectively, yields the integral

$$\int_0^1 \int_0^1 \int_0^1 \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-y)\,z)^{k+r-h-m}(1-(1-xy)\,z)^{q+h-r}} \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)\,z} \,.$$

By the first of (3.6), this integral is (2.41). Hence under the assumptions (3.6) we have

$$K(h, j, k, l, m, q, r, s) = I(h, j, k, l, m, q, r, s).$$

If we apply the transformation σ to (3.12), i.e. if we interchange x, y, we get the integral $K(h, k, j, l, s, q, r, m)$. Thus the associated permutation is $(j\ k)(m\ s)$, which again does not preserve the linear conditions (3.6).

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Abstract

We define double and triple Euler-type integrals generalizing the integrals considered by Beukers in 1979. We show how the permutation group method, recently introduced by Rhin and the author, applies to such integrals to yield the best known irrationality measures of $\zeta(2)$ *and* $\zeta(3)$. *In the last section we introduce a family of 32 three-dimensional birational transformations changing integrals of Sorokin's type into integrals of Beukers' type. Such a family includes a transformation recently given by Fischler.*

* * *