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Laplace transform and semigroup approach to integrodifferential equations (**)

1 - Introduction

In this paper, we are interested in the initial value problem

$$\text{(IDE)} \qquad \begin{aligned} &\dot{u}(t) = Au(t) + \int_0^t k(t-s)\, Au(s)\, \mathrm{d}s\,, \\ &u(0) = u_0, \end{aligned}$$

where A is the generator of a C_0-semigroup on a Banach space X and $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is a scalar kernel. Such equations describe many processes such as, for example, heat conduction in materials with memory (see [Mil78] or [Pru93], Chapter 5.5). Another example from the theory of viscoelasticity will be discussed in Section **5**.

There are two main approaches to handle equation (IDE). The first one employs the Laplace transform. This method has been studied by Da Prato and Iannelli [DaPI80], [DaPI85], Da Prato and Lunardi [DaPL88], Grimmer and Prüss [GP85], and others. Many results obtained by this method can be found in the monograph by Prüss [Pru93], to which we will often refer. The second method rewrites (IDE) in the form of an abstract Cauchy problem $\dot{\mathcal{U}}(t) = \mathcal{A}\mathcal{U}(t)$ on the pro-

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duct space $X \times L^1(\mathbb{R}_+, X)$. The solution of (IDE) is then given by the first component of the semigroup generated by $\mathcal{A}$. This method was introduced by Miller in [Mil74] and later investigated by Desch and Grimmer [DG85], Desch and Schappacher [DS85], Grimmer [Gri82], Nagel and Sinestrari [NS93], and others. Concerning this method, we will often refer to [Bar01-1], [Bar01-2]. More references for both of the methods can be found in [Pru93].

The paper is organised as follows. The basic definitions and facts and some helpful technical lemmas are contained in Section **2**. In Section **3**, the operator matrix approach to (IDE) is presented, while in Section **4**, there are similar results obtained by the Laplace transform. In Section **5**, the results of Sections **3** and **4** are applied to a viscoelastic problem and compared. The abstract results of Sections **3** and **4** are compared in Section **6**.

2 - Preliminaries

A function $u \in C(\mathbb{R}_+, D(A)) \cap C^1(\mathbb{R}_+, X)$ satisfying (IDE) is called a *classical solution.* If $u \in C(\mathbb{R}_+, X)$,

$$t \mapsto \int_0^t u(s)\, \mathrm{d}s \in C(\mathbb{R}_+, D(A)),$$

and u satisfies

$$u(t) = A\int_0^t u(s)\, \mathrm{d}s + \int_0^t k(t-s)\, A\int_0^s u(\sigma)\, \mathrm{d}\sigma\, \mathrm{d}s + u_0,$$

then u is called a *mild solution* of (IDE).

We say that (IDE) is *well-posed* if there exists a unique classical solution to (IDE) for every $u_0 \in D(A)$ and this solution depends continuously on the initial value u_0. This means that for every sequence $(u_n)_{n=1}^{\infty} \subset D(A)$ of initial values satisfying $u_n \to u_0 \in D(A)$ (in the norm of X) the corresponding solutions $u(t; u_n)$ converge to $u(t; u_0)$ uniformly on compact intervals.

If this is the case, we can define a family of operators

$$S(t)u_0 := u(t; u_0)$$

and from the continuous dependence it follows that these operators (defined on $D(A)$) can be extended (uniquely) to bounded operators on X. Then the maps

$t \mapsto S(t)x$ are continuous and

$$S(t)\,x = A\int_0^t S(s)\,x\,\mathrm{d}s + \int_0^t k(t-s)\,A\int_0^s S(\sigma)\,x\,\mathrm{d}\sigma\,\mathrm{d}s + x \tag{2.1}$$

holds for each $x \in X$ (this follows from the operator matrix approach).

A family of operators S is called a *solution operator* for (IDE) if $S(0) = I$, the mappings $t \mapsto S(t)x$ are continuous for all $x \in X$, and for $u_0 \in D(A)$ the equality $AS(t)u_0 = S(t)Au_0$ holds and $S(t)u_0$ is a classical solution of (IDE).

It can be shown (see [Pru93], Proposition 1.1) that the well-posedness of (IDE) is equivalent to the existence of a solution operator. The solution operator is unique and $S(t)u_0$ is a mild solution for all $u_0 \in X$.

The *growth bound* of a family $(S(t))_{t \geqslant 0}$ of operators is

$$\omega_0(S) := \inf\{\omega \in \mathbb{R} : \exists M \geqslant 1;\, t > 0 \Rightarrow \|S(t)\| \leqslant Me^{\omega t}\}.$$

The family S is called *uniformly exponentially stable* if $\omega_0(S) < 0$ and *uniformly stable* if $\|S(t)\| \to 0$ as $t \to \infty$. Moreover, we define

$$\delta(S) := \inf\{\omega \in \mathbb{R} : \exists T \in C(\mathbb{R}_+, B(X)),\, M \geqslant 1;\, \|S(t) - T(t)\| \leqslant Me^{\omega t}\ \forall t \geqslant 0\}$$

and the *spectral bound* of an operator A by

$$s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}.$$

We will use weighted spaces such as

$$L^1_w(\mathbb{R}_+, X) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X) : \int_0^\infty e^{wt} f(t)\,\mathrm{d}t < +\infty \right\}$$

and

$$W^{1,1}_w(\mathbb{R}_+, X) := \{f \in L^1_w : f' \in L^1_w\} \text{ for } \omega > 0.$$

In the following, $\widehat{a}$ denotes the Laplace transform of a. A function $a : \mathbb{R}_+ \to \mathbb{R}$ is said to be *of positive type* if $\operatorname{Re}\widehat{a}(\lambda) \geqslant 0$, whenever $\operatorname{Re}\lambda \geqslant 0$. A function $a : \mathbb{R}_+ \to \mathbb{R}$ is called *k-monotone* if

$$(-1)^n a^{(n)}(t) \geqslant 0 \text{ for } 0 \leqslant n \leqslant k-2,\ t > 0$$

and $(-1)^{k-2} a^{(k-2)}$ is non-increasing and convex. Hence 2-monotone functions are positive, non-increasing and convex. A function a is called *k-regular* if there exists $c > 0$ such that

$$\|\lambda^n \widehat{a}^{(n)}(\lambda)\| \leqslant c\|\widehat{a}(\lambda)\|$$

for all $\operatorname{Re}\lambda > 0$, $1 \leqslant n \leqslant k$. All $(k+1)$-monotone functions are k-regular and of positive type for $k \geqslant 1$, see [Pru93], Proposition 3.3.

The following lemmas will be useful.

Lemma 2.1. *Let $a \in L^1(\mathbb{R}_+)$ and $\varepsilon > 0$. Then there exists $K > 0$ such that $|\widehat{a}(\lambda)| < \varepsilon$, whenever $\operatorname{Re}\lambda \geqslant 0$, $|\lambda| > K$.*

Lemma 2.2. *Let $a \in L^1(\mathbb{R}_+)$, $M := \{\widehat{a}(\lambda) : \operatorname{Re}\lambda \geqslant 0\}$. Then $C := M \cup \{0\}$ is* compact.

Corollary 2.3. *Let $a \in L^1(\mathbb{R}_+)$ and $\widehat{a}(\lambda) \neq -1$ for all $\operatorname{Re}\lambda \geqslant 0$. Then there exists $m > 0$ such that $|1 + \widehat{a}(\lambda)| \geqslant m$ for all $\operatorname{Re}\lambda \geqslant 0$.*

Lemma 2.4. *Let $k \in L^1(\mathbb{R}_+)$ and $\varepsilon > 0$. Denote $k_{-\varepsilon} : t \mapsto k(t)e^{-\varepsilon t}$. If $1 + 1 * k$ is 1-regular and $\widehat{k} \neq -1$ for all $\operatorname{Re}\lambda \geqslant 0$, then $1 + 1 * k_{-\varepsilon}$ is* 1-*regular.*

Proof. We have $b(\lambda) := (1 \widehat{+1 * k}_{-\varepsilon})(\lambda) = (1 + \widehat{k}(\lambda + \varepsilon))/\lambda$, hence

$$|\lambda b'(\lambda)| = \left| \frac{\lambda \widehat{k}'(\lambda + \varepsilon) - (1 + \widehat{k}(\lambda + \varepsilon))}{\lambda} \right| \leqslant |\widehat{k}'(\lambda + \varepsilon)| + |b(\lambda)|.$$

It follows from the 1-regularity of $1 + 1 * k$ that

$$\left| \frac{(\lambda + \varepsilon)\, \widehat{k}'(\lambda + \varepsilon) - (1 + \widehat{k}(\lambda + \varepsilon))}{\lambda + \varepsilon} \right| \leqslant C \left| \frac{1 + \widehat{k}(\lambda + \varepsilon)}{\lambda + \varepsilon} \right|,$$

hence

$$|\widehat{k}'(\lambda + \varepsilon)| < (C + 1) \left| \frac{1 + \widehat{k}(\lambda + \varepsilon)}{\lambda + \varepsilon} \right| < (C + 1) \left| \frac{1 + \widehat{k}(\lambda + \varepsilon)}{\lambda} \right| = (C + 1)\,|b(\lambda)|.$$

So, we have $|\lambda b'(\lambda)| \leqslant (C + 2)\,|b(\lambda)|$. ■

Lemma 2.5. *Let $k \in W^{1,1}_w$ for some $w > 0$ and $\widehat{k}(\lambda) \neq -1$ for all $\operatorname{Re}\lambda \geqslant 0$. Then $1 + 1 * k$ is* 1-*regular.*

Proof. By taking $b = (1 + \widehat{k}(\lambda))/\lambda$ we obtain

$$|\lambda b'(\lambda)| \leqslant |\widehat{k}'(\lambda)| + |b(\lambda)|.$$

Write

$$|\widehat{k}'(\lambda)| = \left| -\int_0^\infty tk(t)\, e^{-\lambda t}\, \mathrm{d}t \right|$$

$$\leqslant \left| \left[tk(t) \frac{e^{-\lambda t}}{\lambda} \right]_0^\infty \right| + \left| \frac{1}{\lambda} \int_0^\infty (tk'(t) + k(t))\, e^{-\lambda t}\, \mathrm{d}t \right| \leqslant 0 + \frac{M}{|\lambda|},$$

where $M = \int_0^\infty (tk'(t) + k(t))\, \mathrm{d}t$. Since $m := \inf |1 + \widehat{k}(\lambda)| > 0$ for $\operatorname{Re}\lambda \geqslant 0$, we obtain

$$|\lambda b'(\lambda)| \leqslant \left(1 + \frac{M}{m}\right) |b(\lambda)|. \quad \blacksquare$$

For the stability of (IDE), the mapping

$$F : \lambda \mapsto \frac{\lambda}{1 + \widehat{k}(\lambda)}, \qquad \operatorname{Re}\lambda \geqslant 0,$$

is of great importance. So, we prove some properties of F in the following lemmas.

Lemma 2.6. *Let* $k \in L^1(\mathbb{R}_+)$, $\varepsilon > 0$. *Then there exists* $K > 0$ *such that*

$$F(\lambda) \in \Sigma_{\pi/2 + \varepsilon} := \{\lambda \in \mathbb{C} : |\arg \lambda| \leqslant \pi/2 + \varepsilon\}$$

for all $\operatorname{Re}\lambda \geqslant 0$, $|\lambda| > K$.

Proof. It follows from Lemma 2.1 that $|\widehat{k}(\lambda)| < \delta$ for $\operatorname{Re}\lambda \geqslant 0$, $|\lambda| > K$, if K is large enough. Hence $|\arg(1 + \widehat{k}(\lambda))| < \varepsilon$ and $|\arg F(\lambda)| < \pi/2 + \varepsilon$. $\blacksquare$

Lemma 2.7. *Let* $k \in L^1(\mathbb{R}_+)$ *be of positive type and* $M := \int_0^\infty |k(t)|\, \mathrm{d}t$. *Then* $F(\lambda) \in \Sigma_{\pi/2 + \operatorname{arctg} M}$ *holds for all* $\operatorname{Re}\lambda \geqslant 0$.

Proof. The assertion follows from $|\arg(1 + \widehat{k}(\lambda))| \leqslant \operatorname{arctg} M$. $\blacksquare$

Lemma 2.8. *Let* $k \in L^1(\mathbb{R}_+)$ *and* $1 > M := \int_0^\infty |k(t)|\,\mathrm{d}t$. *Then* $F(\lambda) \in \Sigma_{\pi/2+\arcsin M}$ *holds for all* $\operatorname{Re}\lambda \geq 0$.

Proof. The assertion follows from $|\arg(1+\widehat{k}(\lambda))| \leq \arcsin M$. ∎

3 - Operator matrix method

In this section, we present some stability results obtained using operator matrices and semigroup theory. Take the following initial value problem

$$\begin{pmatrix} \dot{u} \\ \dot{F} \end{pmatrix} = \mathcal{A}\begin{pmatrix} u \\ F \end{pmatrix}, \qquad \begin{pmatrix} u(0) \\ F(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ f \end{pmatrix},$$

where

$$\mathcal{A} := \begin{pmatrix} A & \delta \\ k(\cdot)A & d/ds \end{pmatrix}, \qquad D(\mathcal{A}) := D(A) \times W^{1,1}(\mathbb{R}_+, X)$$

is the generator of a C_0-semigroup on $X \times L^1(\mathbb{R}_+, X)$, $\delta f := f(0)$ is the Dirac operator and d/ds is the first derivative with $D(d/ds) := W^{1,1}(\mathbb{R}_+, X)$.

Assume $k \in W^{1,1}(\mathbb{R}_+, X)$. Then it is easy to show (see [Bar01-2] or [NS93]) that

$$\mathcal{A}_0 := \begin{pmatrix} A & \delta \\ 0 & d/ds \end{pmatrix}, \qquad D(\mathcal{A}) := D(A) \times W^{1,1}(\mathbb{R}_+, X)$$

is the generator of the C_0-semigroup

$$\mathcal{T}_0(t) = \begin{pmatrix} T(t) & R(t) \\ 0 & T_0(t) \end{pmatrix}, \tag{3.1}$$

where T is the C_0-semigroup generated by A, $R(t)f := \int_0^t T(t-s)f(s)\,\mathrm{d}s$ and $(T_0(t)f)(s) := f(t+s)$ is the translation semigroup generated by d/ds. Moreover, if $k \in W^{1,1}(\mathbb{R}_+)$, then the operator

$$\mathcal{B} := \mathcal{A} - \mathcal{A}_0 = \begin{pmatrix} 0 & 0 \\ k(\cdot)A & 0 \end{pmatrix}$$

is bounded on $D(\mathcal{A})$ with the graph norm. Hence, with the help of perturbation theory (see [EN00], Section III.1), we obtain that $\mathcal{A}$ is the generator of a C_0-semigroup $\mathcal{T}$ given by the Dyson-Phillips series

$$(3.2) \qquad \mathcal{T}(t) = \mathcal{T}_0(t) + \sum_{n=1}^{\infty} \mathcal{T}_n(t), \qquad \mathcal{T}_{n+1}(t) := \int_0^t \mathcal{T}_n(t-s)\, \mathcal{B}\mathcal{T}_0(s)\, \mathrm{d}s\,.$$

It is shown in [Bar01-2] that if A generates an immediately norm continuous semigroup, i.e., the mapping $t \to T(t)$ is norm continuous for all $t > 0$, then all $\mathcal{T}_n$ are norm continuous for $n \geq 1$, $t > 0$. Hence, the infinite sum in (3.2) is norm continuous for $t > 0$. Since T is norm continuous as well, the only terms which are not norm continuous is the translation semigroup T_0 (see definition 3.4 of $\mathcal{T}_0$) and R. So, the first component of $\mathcal{T}$ is norm continuous (it is the solution operator for (IDE)), but the semigroup $\mathcal{T}$ itself is not norm continuous.

According to [NP99] or [BBS03], for any C_0-semigroup $\mathcal{T}$ the growth bound is obtained as

$$(3.3) \qquad \omega_0(\mathcal{T}) = \max\{\delta(\mathcal{T}),\, s(\mathcal{A})\}\,.$$

In our case, since the only non-continuous terms are the translation semigroup T_0 and R,

$$(3.4) \qquad \delta(\mathcal{T}) \leq \max\{\omega_0(T_0),\, \omega_0(R)\} = \max\{s(d/ds),\, s(A)\}\,.$$

The last equality holds since $\omega_0(R) = \omega_0(T) = s(A)$, where $s(A)$ is the spectral bound of A. Since $s(d/ds) = 0$ we cannot obtain uniform exponential stability in this way. However, it is possible to take a weighted space $L^1_w(\mathbb{R}_+, X)$ instead of $L^1(\mathbb{R}_+, X)$ in case $k \in W^{1,1}_w(\mathbb{R}_+)$. Then $\omega_0(T_0) = -w$ in L^1_w.

To compute $s(\mathcal{A})$, the following lemma will be helpful. For its proof see [Bar01-1].

Lemma 3.1. *For every* $\lambda \in \varrho(\mathcal{A}_0)$ *we have*

$$\lambda \in \sigma(\mathcal{A}) \quad \text{if and only if} \quad \widehat{k}(\lambda) = -1 \quad \text{or} \quad \frac{\lambda}{1+\widehat{k}(\lambda)} \in \sigma(A)\,.$$

Moreover, it holds that $\varrho(\mathcal{A}_0) = \varrho(A) \cap \varrho(d/ds)$.

The facts mentioned above yield the following stability theorem.

Theorem 3.2. *Let A generate an immediately norm continuous semigroup and* $w > 0$. *Let* $k \in W^{1,1}_w$, $s(A) < -w$ *and* $s(\mathcal{A}) < 0$. *Then* (IDE) *is uniformly exponentially stable.*

By investigating $\sigma(\mathscr{A})$ we obtain the following corollaries.

Corollary 3.3. *Let A generate an immediately norm continuous semigroup with $-\sigma(A) \subset \Sigma_\theta$, $0 \in \varrho(A)$ and $k \in W_w^{1,1}$ such that $k_w : t \mapsto k(t) e^{wt}$ is of positive type for some $w > 0$. If*

$$\theta + \operatorname{arctg} M < \pi/2 \quad \textit{for} \quad M := \int_0^\infty |k(t)| \, \mathrm{d}t \tag{3.6}$$

holds, then S is uniformly exponentially stable.

Proof. Denote $C := \{\lambda : \operatorname{Re}\lambda \geqslant 0$ or $(\operatorname{Re}\lambda \geqslant -\varepsilon$ and $\operatorname{Im}\lambda \geqslant K)\}$. We show that $\sigma(\mathscr{A}) \cap C = \emptyset$. Then $s(\mathscr{A}) < 0$ follows since $\sigma(\mathscr{A})$ is a closed set, and Theorem 3.2 completes the proof.

Since $\widehat{k_w}(\lambda) = \widehat{k}(\lambda - w)$ and k_w is of positive type, we obtain $\operatorname{Re}\widehat{k}(\lambda) \geqslant 0$ for all $\operatorname{Re}\lambda \geqslant -w$. For $\operatorname{Re}\lambda \geqslant 0$ we have $F(\lambda) \in \Sigma_{\pi/2 + \operatorname{arctg} M}$ (by Lemma 2.7). If $0 > \operatorname{Re}\lambda > -\varepsilon$ and $\operatorname{Im}\lambda > K$, then $F(\lambda) \in \Sigma_{\pi/2+\varepsilon}$ by Lemma 2.6. Since (3.6) holds, we have $F(\lambda) \notin \sigma(A)$ if $\lambda \in C$. Hence $\sigma(\mathscr{A}) \cap C = \emptyset$ follows by Lemma 3.1. ■

Corollary 3.4. *Let A generate an immediately norm continuous semigroup with $-\sigma(A) \subset \Sigma_\theta$, $0 \in \varrho(A)$, and $k \in W_w^{1,1}$. If*

$$\theta + \arcsin M < \pi/2 \quad \text{for} \quad 1 > M := \int_0^\infty |k(t)| \, \mathrm{d}t \,, \tag{3.7}$$

then S is uniformly exponentially stable.

Proof. The proof is similar to the previous one applying Lemma 2.8 instead of Lemma 2.7. ■

4 - Laplace transform method

Applying the Laplace transform (formally) to (IDE), one obtains

$$\lambda \widehat{u}(\lambda) - u(0) = A\widehat{u}(\lambda) + \widehat{k}(\lambda) \, A \widehat{u}(\lambda) \, .$$

Hence,

$$(\lambda - A - \widehat{k}(\lambda)\, A)^{-1} u_0 = \widehat{u}(\lambda) = \widehat{S}(\lambda) \, u_0$$

if the inverse exists since $u(t) = S(t)\, u_0$. Denote $H(\lambda) := (\lambda - A - \widehat{k}(\lambda) A)^{-1}$. The generation theorem by Da Prato and Iannelli implies the following (see [Pru93], Theorem 1.3).

Theorem 4.1. *Let* $\int_0^\infty e^{-\omega t}\,|k(t)|\,\mathrm{d}t < \infty$. *Then there exists a solution operator S for (IDE) satisfying*

$$\|S(t)\| \leq Me^{\omega t}$$

if and only if $\widehat{k}(\lambda) \neq -1$, $(\lambda - A - \widehat{k}(\lambda)\,A)^{-1}$ *exists, and*

$$\|H^{(k)}(\lambda)\| \leq \frac{Mk!}{(\lambda-\omega)^{k+1}} \tag{4.1}$$

holds for $k = 0, 1, 2, \dots$ *and all* $\lambda > \omega$.

If A is the generator of a C_0-semigroup and $k \in W^{1,1}(\mathbb{R}_+, X)$, then (IDE) is well-posed according to [Pru93], Corollary 1.4. However, we do not know anything about the growth of the solution operator S. Since the estimates (4.1) are very difficult to verify, we look for another condition guaranteeing stability. We introduce the concept of parabolic problems and uniform integrability (see [Pru93], Section **3** and **10**).

The equation (IDE) is called *parabolic* if

$$\|H(\lambda)\| \leq M/|\lambda| \ \text{ for all } \ \mathrm{Re}\,\lambda > 0\,. \tag{4.2}$$

For parabolic problems, the following two stability results can be found in [Pru93] (Theorems 3.1 and 10.2).

Theorem 4.2. *If* (IDE) *is parabolic and* $1 + 1 * k$ *is 1-regular, then there exists a solution operator S for (IDE) which is norm continuous, bounded, and satisfies*

$$|tS(t) - sS(s)| \leq M|t-s|\left(1 + \log\frac{t}{t-s}\right), \qquad 0 \leq s < t < \infty\,, \tag{4.3}$$

for some $M \geq 1$.

Theorem 4.3. *Let* (IDE) *be parabolic,* $0 \in \varrho(A)$, *and* $1 + 1 * k$ *be 2-regular. Denote* $\varphi(\lambda) := 1/(1+\widehat{k}(\lambda))$. *If*

$$\lim_{|\lambda|\to 0} \varphi(\lambda) \in \mathbb{C} \quad \textit{and} \quad \varphi'(i\cdot) \in L^1(-1, 1)\,, \tag{4.4}$$

then $S \in L^1(\mathbb{R}_+, B(X))$.

First, we show that (IDE) is uniformly stable if the assumptions of Theorem 4.3 are satisfied.

Proposition 4.4. *If* $S \in L^1(\mathbb{R}_+, B(X))$ *and* (4.3) *holds, then* $\lim_{t \to +\infty} \|S(t)\| = 0$.

Proof. Assume that there is a sequence $t_n \nearrow +\infty$ satisfying $t_{n+1} - t_n \geqslant 1$ such that $\|S(t_n)\| \geqslant \varepsilon > 0$. We show that $\|S(t)\| \geqslant \varepsilon/2$ on a set of infinite measure, in particular on $G := \bigcup_{n \geqslant k} (t_n - 1, t_n)$. Taking $t := t_n$ and $s := t_n - \delta$ in (4.3) where $\delta \in (0, 1)$ and $n \in \mathbb{N}$ are arbitrary, we obtain

$$\|(t_n - \delta) S(t_n - \delta)\| \geqslant t_n \varepsilon - M\delta \left(1 + \log \frac{t_n}{\delta} \right),$$

hence

$$\|S(t_n - \delta)\| \geqslant \varepsilon - M\delta \, \frac{1 - \log \delta + \log t_n}{t_n} .$$

Since $M\delta(1 - \log \delta) \leqslant C$ for $\delta \in (0, 1)$, we have

$$M\delta \, \frac{1 - \log \delta + \log t_n}{t_n} \leqslant \frac{C}{t_n} + M \frac{\log t_n}{t_n} < \frac{\varepsilon}{2}$$

for all t_n large enough. Hence $\|S(t)\| \geqslant \varepsilon/2$ on G and $S \notin L^1(\mathbb{R}_+, B(X))$. ■

We now find some sufficient conditions for (IDE) to be parabolic.

Proposition 4.5. *Let A be the generator of a bounded analytic semigroup and* $k \in L^1(\mathbb{R}_+)$. *If* $H(\lambda) := (\lambda - A - \widehat{k}(\lambda) A)^{-1}$ *exists for every* $\operatorname{Re} \lambda \geqslant 0$, *then* (IDE) *is parabolic.*

Proof. We first show (4.2) for $\lambda \in F_1 := \{\lambda : \operatorname{Re} \lambda \geqslant 0,\ |\lambda| \geqslant K\}$ and then for $\lambda \in F_2 := \{\lambda : \operatorname{Re} \lambda \geqslant 0,\ |\lambda| \leqslant K\}$.

It holds that

$$H(\lambda) = \frac{1}{1 + \widehat{k}(\lambda)} \left(\frac{\lambda}{1 + \widehat{k}(\lambda)} - A \right)^{-1} . \tag{4.5}$$

Hence,

$$\|H(\lambda)\| \leqslant \left| \frac{1}{1+\widehat{k}(\lambda)} \right| \cdot \frac{M}{\left| \dfrac{\lambda}{1+\widehat{k}(\lambda)} \right|} = \frac{M}{|\lambda|}$$

for all $\lambda \in F_1$, since

$$\frac{\lambda}{1+\widehat{k}(\lambda)} \in \varrho(A)$$

by Lemma 2.6.

Since $H(\lambda)$ exists for all $\operatorname{Re}\lambda \geqslant 0$, we have $\inf\{1+\widehat{k}(\lambda) : \lambda \in F_2\} = m \geqslant 0$. Hence, the set

$$C := \left\{ \frac{\lambda}{1+\widehat{k}(\lambda)} : |\lambda| \leqslant K, \ \ \operatorname{Re}\lambda \geqslant 0 \right\}$$

is compact. It follows that the resolvent $R(\mu, A)$ is bounded on C and we obtain the estimate

$$\|H(\lambda)\| = \left| \frac{1}{1+\widehat{k}(\lambda)} \right| \cdot \left\| R\left(\frac{\lambda}{1+\widehat{k}(\lambda)}, A \right) \right\| \leqslant M_1 \cdot \left| \frac{1}{1+\widehat{k}(\lambda)} \right| \leqslant \frac{M_1}{m} \leqslant \frac{KM_1}{m|\lambda|}$$

for $\lambda \in F_2$.

Corollary 4.6. *Let A generate a bounded analytic semigroup in the sector $\Sigma_{\pi/2-\theta}$, $k \in L^1(\mathbb{R}_+)$ be of positive type, and* (3.6) *hold. Then* (IDE) *is parabolic.*

Proof. It follows from Lemma 2.7 that $F(\lambda) \in \varrho(A)$ whenever $\operatorname{Re}\lambda \geqslant 0$. Hence, $H(\lambda)$ exists and the assertion follows from Proposition 4.5. ■

Corollary 4.7. *Let A generate a bounded analytic semigroup in the sector $\Sigma_{\pi/2-\theta}$, $k \in L^1(\mathbb{R}_+)$, and* (3.7) *hold. Then* (IDE) *is parabolic.*

Proof. We use Lemma 2.8 instead of Lemma 2.7 in the proof of Corollary 4.6. ■

The following theorem follows from Theorem 4.2, Theorem 4.3, Proposition 4.4 and Proposition 4.5.

Theorem 4.8. *Let A be the generator of a bounded analytic semigroup, $k \in L^1(\mathbb{R}_+)$, and $H(\lambda) := (\lambda - A - \widehat{k}(\lambda)\, A)^{-1}$ exist for every* $\operatorname{Re}\lambda \geqslant 0$. *If* $1 + 1 * k$ *is* 1*-regular, then S is bounded. Moreover, if* (4.4) *holds,* $1 + 1 * k$ *is* 2*-regular, and* $0 \in \varrho(A)$, *then* (IDE) *is uniformly stable.*

The condition (4.4) is satisfied if $tk(t) \in L^1$. In this case, $\widehat{k}'(\lambda) = -\int_0^\infty te^{-\lambda t} k(t)\, \mathrm{d}t$, hence $\widehat{k}'$ is continuous for all $\operatorname{Re}\lambda \geqslant 0$. Continuity of φ' follows. If k is 2-monotone, then (4.4) is satisfied according to computations on page 266 in [Pru93].

Corollary 4.9. *Let A be the generator of a bounded analytic semigroup in the sector $\Sigma_{\pi/2-\theta}$ and let $k \in L^1(\mathbb{R}_+)$ satisfy* (3.7) *or be of positive type satisfying* (3.6). *If* $1 + 1 * k$ *is* 1*-regular, then S is bounded. Moreover, if* (4.4) *holds,* $1 + 1 * k$ *is* 2*-regular, and* $0 \in \varrho(A)$, *then* (IDE) *is uniformly stable.*

If $0 \in \varrho(A)$, then $s(A) < 0$, and we can even obtain exponential stability under certain assumptions on k.

Theorem 4.10. *Let A be the generator of a bounded analytic semigroup,* $0 \in \varrho(A)$ *and* $w > 0$. *Denote* $k_w \colon t \mapsto e^{wt} k(t)$. *Let* $k \in L^1_w(\mathbb{R}_+)$ *and* $H(\lambda) := (\lambda - A - \widehat{k}(\lambda) A)^{-1}$ *exist for every* $\operatorname{Re}\lambda \geqslant -w$. *If* $1 + 1 * k_w$ *is* 1*-regular, then* (IDE) *is uniformly exponentially stable.*

Proof. Take $0 < \varepsilon < \max\{-s(A), w\}$ and $A_1 := a + \varepsilon$. Since A generates the analytic semigroup T satisfying $\|T(t)\| \leqslant M_c e^{ct}$ for every $c > s(A)$, A_1 generates the semigroup $T_1(t) := e^{\varepsilon t} T(t)$ which is analytic and bounded. Denote $k_1 \colon t \mapsto e^{\varepsilon t} k(t)$. According to Lemma 2.4, $1 + 1 * k_1$ is 1-regular since $k_1(t) = k_w(t)\, e^{\varepsilon - w}$. Then

$$
\begin{aligned}
&\dot{u}(t) = A_1 u(t) + \int_0^t k_1(t-s)\, Au(s)\, \mathrm{d}s\,, \\
&u(0) = u_0
\end{aligned}
\tag{IDE$_1$}
$$

has a bounded solution operator S_1, according to [Bar04] (The equation (IDE$_1$) is no longer an equation of the type studied in this paper, but it can be written in the form

$$
\dot{u}(t) = A_1 u(t) + \int_0^t (k_1(t-s)\, A_1 - \varepsilon k_1(t-s))\, u(s)\, \mathrm{d}s,
$$

where the norm of $b := \varepsilon k_1$ can be made arbitrarily small). Then $S(t) := e^{-\varepsilon t} S_1(t)$ is the solution operator for (IDE), hence S is uniformly exponentially stable. ■

Corollary 4.11. *Let A be the generator of a bounded analytic semigroup in the sector $\Sigma_{\pi/2-\theta}$, $0 \in \varrho(A)$, and $w > 0$. Denote $k_w : t \mapsto e^{wt} k(t)$. Let $k_w \in L^1(\mathbb{R}_+)$ be of positive type and* (3.6) *hold. If $1 + 1 * k_w$ is 1-regular, then* (IDE) *is uniformly exponentially stable.*

5 - Applications

In this section, we consider the following viscoelastic problem. Assume the space between two infinite parallel plates to be filled with a homogeneous fluid. Let one of the plates move along its tangent with constant velocity. In fact, this problem is one-dimensional.

Let $u(t, x)$ be the displacement function, $0 \leqslant x \leqslant 1$, $t \geqslant 0$ and $v(t, x) := u_t(t, x)$ be the velocity. The linearised strain is defined by

$$e(t, x) = u_x(t, x). \tag{5.1}$$

Denoting the stress by s, the balance of momentum in the body gives

$$\varrho u_{tt}(t, x) = s_x(t, x), \tag{5.2}$$

where ϱ is the density of mass. Assume $\varrho = 1$. The stress-strain relation has the form

$$s(t, x) := \int_0^\infty da(\tau, x)\, \dot{e}(t - \tau, x)\, \mathrm{d}\tau, \tag{5.3}$$

where a is the so called stress relaxation function and depends on the material of the body. Since the fluid is considered to be homogeneous, a is independent of x. Then (5.1), (5.2) and (5.3) give

$$u_{tt}(t, x) = \int_0^\infty da(\tau)\, u_{txx}(t - \tau, x)\, \mathrm{d}\tau. \tag{5.4}$$

Assuming the history of $u(t, x)$ for $t \leqslant 0$ to be known, we can rewrite (5.4) as

$$u_{tt}(t, x) = \int_0^t da(\tau)\, u_{txx}(t-\tau, x)\, \mathrm{d}\tau + g(t, x), \tag{5.5}$$

where $g(t, x) := \int_{-\infty}^{0} da(t-\tau) u_{txx}(\tau, x)\, \mathrm{d}\tau$. Since the stress relaxation function a is typically positive, non-decreasing and concave (see Chapter 5.2 in [Pru93]) with $a(0) > 0$ and $a_\infty := \lim_{t\to\infty} a(t) < \infty$, we can write $a(t) = a(0) + \int_0^t a_1(\tau)\, \mathrm{d}\tau$ and $da(t) = a(0)\, \delta + a_1(t)$, where $a_1 \in L^1(\mathbb{R}_+)$ and δ is the Dirac measure.

Assuming the history of u to be zero, (5.5) for the velocity function reads

$$v_t(t, x) = a(0)\, v_{xx}(t, x) + \int_0^t a_1(\tau)\, v_{xx}(t-\tau, x)\, \mathrm{d}\tau\, .$$

Moreover, we have the boundary conditions

$$v(t, 0) = v_0, \qquad v(t, 1) = 0$$

and the initial value

$$v(0, x) = 0\, .$$

Taking $w(t, x) := v(t, x) - (1-x)\, v_0$ we obtain Dirichlet boundary conditions and

$$\begin{aligned} \dot{w}(t) &= Aw(t) + \int_0^t k(s)\, Aw(t-s)\, \mathrm{d}s\, , \\ w(0) &= w_0, \end{aligned} \tag{5.6}$$

where $Af := a(0) f''$, $D(A) := W_0^{1,p}(0, 1) \cap W^{2,p}(0, 1)$, $k(t) := a_1(t)/a(0)$ and $w_0(x) := (x-1)\, v_0$.

Since Δ generates a bounded analytic semigroup in $\Sigma_{\pi/2}$ and $0 \in \varrho(\Delta)$, we can apply the results from the previous two sections. Since $\sigma(\Delta) \subset (-\infty, 0)$, the condition (3.6) is satisfied for every k.

In Corollary 3.3 we need $k \in W_w^{1,1}$ and k_w to be of positive type in order to obtain uniform exponential stability. In Corollary 4.11, we do not need k to be differentiable, $k \in L_w^1$ is sufficient. On the other hand, $1 + 1 * k_w$ has to be 1-regular. According to Lemma 2.5, 1-regularity follows from $k \in W_w^{1,1}$.

We know that k is positive and non-increasing. This follows from physical arguments, see the definition of k and a_1 above. If, moreover, k_w is positive, non-increasing and convex, then it is 2-monotone, hence of positive type and $1 + 1 * k_w$ is 1-regular. In this case, we can apply both Corollaries 3.3 and 4.11 to obtain uniform exponential stability.

If $t \mapsto e^{t\varepsilon} k(t)$ is not 2-monotone for any $\varepsilon > 0$, but k is 3-monotone, we can still obtain uniform stability by Corollary 4.9. In fact, if k is 3-monotone, then it is of positive type and 2-regular. Then it is easy to prove that $1 + 1 * k$ is 2-regular as well. If k is only 2-monotone, then we obtain boundedness of the solutions by the same corollary.

Here, the stability means $v(t) \to -w_0$ in $L^p(0, 1)$, if we consider the Laplacian on L^p. Since it is possible to work on $C[0, 1]$ instead of L^p, we obtain $v(t, x) \to -w_0(x) = (1 - x)\, v_0$ uniformly.

6 - Conclusions

We now compare the results obtained by operator matrices to the results obtained by Laplace transform. First, A has to be the generator of a bounded analytic semigroup if we use the Laplace transform, while in the other case generators of immediately norm continuous semigroups are allowed. We show an example such that Theorem 3.2 applies and Theorem 4.10 not.

There exist generators of immediately norm continuous semigroups which are not analytic, e.g. multiplication operators, see [EN00], II.4.32. However, such multiplication operators does not have their spectra in a sector. So, Corollary 3.3 does not apply. There are some operators with their spectra in a sector which generate non-analytic semigroups (e.g. nilpotent translation semigroup), but these semigroups are not immediately norm continuous. However, there is no example of a semigroup T with the following properties known to the author.

(i) T is immediately norm continuous and not analytic

(ii) the generator of T has its spectum contained in a sector.

We show that the condition $-\sigma(A) \subset \Sigma_\theta$ in Corollary 3.3 is not necessary. Since $k \in W^{1,1}$, we have

$$|\widehat{k}(\lambda)| \leq \frac{K}{|\lambda|} \tag{6.1}$$

for all $\mathrm{Re}\,\lambda \geqslant 0$ and some $K > 0$. It follows that

$$\left| \lambda - \frac{\lambda}{1 + \widehat{k}(\lambda)} \right| \leqslant K .$$

Hence $s(\mathcal{A}) < 0$ if $s(A) < -K$. Let A be a multiplication operator generating a semigroup with property (i) and let $s(A) < -K$ with K satisfying (6.1). Let $k \in W^{1,1}_w$ for some $w > 0$. Then (IDE) is uniformly exponentially stable by Theorem 3.2.

Concerning the spectral conditions in Theorems 3.2 and 4.10 we can see that $s(A)$ is assumed to be negative in both cases. The condition on $H(\lambda)$ in Theorem 4.10 is equivalent to $\sigma(\mathcal{A}) \cap \{\mathrm{Re}\,\lambda \geqslant -w\} = \emptyset$, so it is equivalent to $s(\mathcal{A}) < 0$ in Theorem 3.2, since w can be chosen arbitrarily.

The main difference consists in the assumptions on k. In Theorem 3.2 we need $k \in W^{1,1}_w$, while in Theorem 4.10 we want $k \in L^1_w$ and $1 + 1 * k$ to be 1-regular. Since $s(\mathcal{A}) < 0$, we have $k(\lambda) \neq -1$ for $\mathrm{Re}\,\lambda \geqslant 0$, hence 1-regularity of $1 + 1 * k$ follows from $k \in W^{1,1}_w$ according to Lemma 2.5. The converse is not true. Take $k_w(t) := k(t)\, e^{wt}$ negative, increasing and such that $\|k_w\|_1 < 1$. Then $1 + 1 * k_w$ is 2-monotone, hence 1-regular. On the other hand, k_w is not necessarily continuous, so $k_w \notin W^{1,1}_w$. Therefore, Theorem 4.10 is stronger in this point.

The above examples shows that the cases where one theorem applies and the other does not are rather artificial. The semigroups appearing in applications are usually analytic and the kernels smooth. So, the main advantage of Laplace transform approach remains the case $k \notin L^1_w$ for any $w > 0$. In this case we do not obtain exponential stability, but we can still obtain uniform stability or boundedness by Theorem 4.8. We have no such result by the operator matrix approach.

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Abstract

The purpose of this paper is to compare the results on abstract Volterra integrodifferential equations obtained by two different methods and to apply these results to a viscoelastic problem. In particular, we focus on the asymptotic behaviour of the solutions.

* * *