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Subelliptic F-harmonic maps (**)

1 - Introduction and statement of results

Let $(M, T_{1,0}(M))$ be a strictly pseudoconvex CR manifold, of CR dimension n, and θ a contact form on M such that the *Levi form*

$$G_\theta(X, Y) := (d\theta)(X, JY)$$

is positive-definite, $X, Y \in H(M) := Re\{T_{1,0}(M) \oplus T_{0,1}(M)\}$, where $J: H(M) \to H(M)$, $J(Z + \overline{Z}) = i(Z - \overline{Z})$, $Z \in T_{1,0}(M)$, $i = \sqrt{-1}$, and $T_{0,1}(M) := \overline{T_{1,0}(M)}$ (an overbar indicates complex conjugation). Let $F: [0, \infty) \to [0, \infty)$ a C^2 function, such that $F'(t) > 0$. For a smooth map $\phi: (M, \theta) \to (N, h)$ and a compact domain $D \subseteq M$ we consider the energy function

$$E_F(\phi; D) = \int_D F\left(\frac{1}{2}\, trace_{G_\theta}(\pi_H \phi^* h)\right) \theta \wedge (d\theta)^n. \tag{1}$$

Here (N, h) is a Riemannian manifold. Then ϕ is *F-pseudoharmonic* if, for any compact domain $D \subseteq M$, it is an extremal of the energy $E_F(\cdot; D)$ with respect to all variations of ϕ supported in D.

For $F(t) = t$, (1) is the energy function in [3] (and extremals were referred to as *pseudoharmonic maps*). If $\phi: (M, \theta) \to (N, h)$ is pseudoharmonic then for any point $x \in M$ there is a coordinate system $(U, \varphi = (x^1, \ldots, x^{2n+1}))$ on M at x

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such that $\phi \circ \varphi^{-1}: \Omega \to (N, h)$ is a *subelliptic harmonic map* on $\Omega := \varphi(U) \subseteq \boldsymbol{R}^{2n+1}$ in the sense of J. Jost & C.-J. Xu, [**12**], or Z.-R. Zhou, [**20**].

The cases $F(t) = (2t)^{p/2}$ $(p \geq 4)$ and $F(t) = \exp(t)$ (familiar in the theory of harmonic maps of Riemannian manifolds, cf. e.g. P. Baird & S. Gudmundson, [**2**], L.F. Cheung & P.F. Leung, [**4**], and M.C. Hong, [**10**], S.E. Koh, [**13**]) have not been studied from the point of view of CR and pseudohermitian geometry. However, if $F(t) = (2t)^{p/2}$ and $\phi : (M, \theta) \to (S^m, h_0)$ is F-pseudoharmonic [where S^m is the unit sphere in $\boldsymbol{R}^{m+1}$ and h_0 the standard Riemannian metric on S^m] then at any point $x \in M$ there is a local coordinate system (U, φ) such that $\phi \circ \varphi^{-1}$: $\Omega \to S^m$ is subelliptic p-harmonic in the sense of P. Hajlasz & P. Strzelecki, [**9**].

We obtain the following *first variation formula* [stated for simplicity in the case M is compact (and then one writes $E_F(\phi) := E_F(\phi; M)$)]

Theorem 1. *Let M be a compact strictly pseudoconvex CR manifold, of CR dimension n, and θ a contact form on M such that the Levi form G_θ is positive definite. Let (N, h) be a Riemannian manifold. Let $F : [0, \infty) \to [0, \infty)$ be a C^2 map such that $F'(s) > 0$ and set $\varrho(s) := F'(s/2)$. Let $\{\phi_t\}_{|t| < \varepsilon}$ be a 1-parameter variation of a smooth map $\phi = \phi_0 : M \to N$. Then*

$$\frac{d}{dt}\{E_F(\phi_t)\}_{t=0} = -\int_M \tilde{h}(V, \tau_F(\phi; \theta, h))\, \theta \wedge (d\theta)^n,$$

where

$$\tau_F(\phi; \theta, h) := \sum_{a=1}^{2n} [(\phi^{-1}\nabla^N)_{X_a}(\varrho(Q)\, \phi_* X_a) - \varrho(Q)\, \phi_* \nabla_{X_a} X_a]$$

and $Q := trace_{G_\theta}(\pi_H \phi^* h)$. Here $\{X_a\}$ is a local G_θ-orthonormal frame of $H(M)$. Also we set $\widetilde{M} := (-\varepsilon, \varepsilon) \times M$ and

$$\Phi : \widetilde{M} \to N, \quad \Phi(t, x) := \phi_t(x), \quad x \in M, \quad |t| < \varepsilon,$$

$$V_x := (d_{(0,x)}\Phi) \frac{\partial}{\partial t}\bigg|_{(0,x)} \in T_{\phi(x)}(N), \quad x \in M.$$

Then ϕ is F-pseudoharmonic if

(2) $$\tau_F(\phi; \theta, h) = 0.$$

Moreover, for each smooth map $\phi : M \to N$ the tension field $\tau_F(\phi; \theta, h)$

$\in \Gamma^\infty(\phi^{-1} TN)$ *is also given by*

$$\tau_F(\phi;\, \theta,\, h) = \left\{ div(\varrho(Q)\, \nabla^H \phi^i) + \sum_{a=1}^{2n} \varrho(Q) \left(\left| \begin{matrix} i \\ jk \end{matrix} \right| \circ \phi \right) X_a(\phi^j)\, X_a(\phi^k) \right\} Y_i. \tag{3}$$

on $U := \phi^{-1}(V)$, *where* (V, y^i) *is a local coordinate system on* N, $\phi^j := y^j \circ \phi$, *and* $Y_j(x) := (\partial/\partial y^j)(x)$, $x \in U$, $1 \leq j \leq m$.

Here $\left| \begin{matrix} i \\ jk \end{matrix} \right|$ are the Christoffel symbols of (N, h). As a consequence of (3) the Euler-Lagrange equations (2) (the *F-pseudoharmonic map equation*) for $\phi : (M, \theta) \to (\boldsymbol{R}^m, h_0)$ may be written

$$div(\varrho(Q)\, \nabla^H \phi^j) = 0, \qquad 1 \leq j \leq m\,. \tag{4}$$

Compare to (0.1) in [**16**]. Here h_0 is the natural flat metric on $\boldsymbol{R}^m$. J. Jost & C-J. Xu study (cf. op. cit.) the existence of weak solutions to the pseudoharmonic map equation [i.e. (2) with $F(t) = t$] on $\Omega \subset \boldsymbol{R}^{2n+1}$. Precisely, they solve the Dirichlet problem on a domain $\omega \subset\subset \Omega$ whose boundary $\partial\omega$ is smooth and noncharacteristic for the system of vector fields $\{X_a\}$, when the given boundary values have values in regular balls of (N, h). Moreover, they prove continuity up to the boundary of bounded weak solutions $\phi : \overline{\omega} \to (N, h)$, a result which, together with a result of C.-J. Xu & C. Zuily, [**19**] (showing that continuous solutions to a class of quasilinar subelliptic systems covering the pseudoharmonic map system on Ω are actually smooth) proves the local existence of pseudoharmonic maps (whose boundary values have values in regular balls of (N, h)). We emphasize that the hypothesis adopted in [**12**] are that $\{X_a\}$ is a Hörmander system on Ω (and this is always satisfied, as a consequence of the fact that $(M, T_{1,0}(M))$ is nondegenerate) and that the boundary $\partial\omega$ is noncharacteristic for $\{X_a\}$ (this holds if and only if $T_x(\partial\omega) \neq H(M)_x$, for any $x \in \partial\omega$). The local result of J. Jost & C.-J. Xu is slightly more general than needed here (it holds for $\Omega \subset \boldsymbol{R}^N$ with N not necessarily odd, and a Hörmander system $\{X_a\}$ on Ω with $\{X_a\}$ not necessarily linearly independent at the points of Ω).

The apparently more general concept of a subelliptic harmonic map considered by Z.-R. Zhou, [**20**] (involving a positive-definite matrix of smooth functions $\gamma_{ij}(x)$ on Ω, which is the unit matrix in [**12**]) is but another local manifestation of our pseudoharmonic maps (corresponding to the case where the local frame $\{X_a\}$ on $U \subseteq M$ is not necessarily G_θ-orthonormal). Z-R. Zhou proves (cf. op. cit.) a local uniqueness result for pseudoharmonic maps on Ω [two pseudoharmonic maps $\phi_1, \phi_2 : \overline{\omega} \to N$, having the same boundary values $\phi_1|_{\partial\omega} = \phi_2|_{\partial\omega}$ (lying in regular balls of N), coincide ($\phi_1 = \phi_2$ in ω)].

The relationship among F-pseudoharmonicity and pseudoharmonicity is clarified in the following

Theorem 2. *Let* (M, θ) *be a CR manifold,* (N, h) *a Riemannian manifold, and* $F:[0, \infty) \to [0, \infty)$ *a* C^2 *function, be as in Theorem* 1. *Then*

$$\tau_F(\phi; \theta, h) = F'\left(\frac{Q}{2}\right)^{1+1/n} \tau\left(\phi; F'\left(\frac{Q}{2}\right)^{1/n} \theta, h\right).$$

Thus $\phi:(M, \theta) \to (N, h)$ *is a* F*-pseudoharmonic map if and only if* $\phi:(M, F'(Q/2)^{1/n}\theta) \to (N, h)$ *is a pseudoharmonic* (*with respect to the data* $(F'(Q/2)^{1/n}\theta, h)$) *map.*

The tension field $\tau(\phi; \theta, h)$ in Theorem 2 is obtained from $\tau_F(\phi; \theta, h)$ for $F(t) = t$.

Pseudohermitian maps, that is CR maps $\phi: M \to N$ of two strictly pseudoconvex CR manifolds M and N, preserving – up to a multiplicative constant – the given contact forms θ and θ_N, on M and N respectively, are examples of pseudoharmonic maps $\phi:(M, \theta) \to (N, g_{\theta_N})$, where g_{θ_N} is the *Webster metric* of (N, θ_N) (and these are also the only CR maps which are pseudoharmonic, cf. Theor. 1.1, p. 724, in [**3**]). New examples, as obtained in this paper, are the *pseudoharmonic morphisms*. Let M be a nondegenerate CR manifold and θ a contact form on M. The *sublaplacian* Δ_b: $C^\infty(M) \to C^\infty(M)$ is the second order differential operator

$$\Delta_b u := -div(\nabla^H u), \qquad u \in C^\infty(M).$$

Let $\phi: M \to N$ be a smooth map into a Riemannian manifold (N, h). We say ϕ is a *pseudoharmonic morphism* if for each local harmonic function $v: V \to \boldsymbol{R}$ ($V \subseteq N$ open, $\Delta_N v = 0$, where Δ_N is the Laplace-Beltrami operator of (N, h)) one has $\Delta_b(v \circ \phi) = 0$ in $U := \phi^{-1}(V)$. We shall prove the following

Theorem 3. *Let* M *be a nondegenerate CR manifold, of CR dimension* n, *and* θ *a contact form on* M. *Let* (N, h) *be a* m*-dimensional Riemannian manifold. If* $m > n$ *there is no pseudoharmonic morphism of* (M, θ) *into* (N, h), *except for the constant maps. If* $m \leqslant n$ *then any pseudoharmonic morphism* $\phi:(M, \theta) \to (N, h)$ *is a pseudoharmonic map and a* C^∞ *submersion and there* *is a unique* C^∞ *function* $\lambda : M \to [0, +\infty)$ *such that*

$$g_\theta^*(d_H \phi^i, d_H \phi^j)_x = 2\lambda(x)\, \delta^{ij}, \qquad 1 \leq i, j \leq m, \tag{5}$$

for any $x \in M$ *and any normal coordinate system* (V, y^i) *at* $\phi(x) \in N$.

The Riemannian counterpart of Theorem 3 is a result of T. Ishihara, [**11**] (thought of as foundational for the theory of harmonic morphisms, cf. e.g. J.C. Wood, [**18**]). The notion of a F-pseudoharmonic map admits the following geometric interpretation

Theorem 4. *Let* (M, θ) *be a compact strictly pseudoconvex CR manifold, with a contact form* θ, (N, h) *a Riemannian manifold, and* $F : [0, +\infty) \to [0, +\infty)$ *a* C^2 *function, as in Theorem* 1. *Let* $S^1 \to C(M) \xrightarrow{\pi} M$ *be the canonical circle bundle and* $\mathcal{F}_\theta$ *the Fefferman metric of* (M, θ). *Let* $\phi : M \to N$ *be a smooth map. Then* $\phi : (M, \theta) \to (N, h)$ *is a F-pseudoharmonic map if and only if its vertical lift* $\phi \circ \pi : (C(M), \mathcal{F}_\theta) \to (N, h)$ *is a F-harmonic map in the sense of M. Ara,* [1], *i.e. a critical point of the energy*

$$\boldsymbol{E}(\Phi) = \int_{C(M)} F\left(\frac{1}{2}\, trace_{\mathcal{F}_\theta} \Phi^* h\right) d\,\mathrm{vol}\,(\mathcal{F}_\theta),$$

on the class of all smooth functions $\Phi : C(M) \to N$. *Here* $d\,\mathrm{vol}\,(\mathcal{F}_\theta)$ *is the natural volume form on the Lorentzian manifold* $(C(M), \mathcal{F}_\theta)$.

2 - Basic objects and formulae

Let M be a $(2n+1)$-dimensional C^∞ differentiable manifold. An *almost CR structure* on M, of *CR dimension n*, is a complex subbundle $T_{1,0}(M) \subset T(M) \otimes \boldsymbol{C}$, of complex rank n, of the complexified tangent bundle over M, such that

$$T_{1,0}(M) \cap T_{0,1}(M) = (0).$$

An almost CR structure is (formally) *integrable* if

$$Z, W \in \Gamma^\infty(T_{1,0}(M)) \Rightarrow [Z, W] \in \Gamma^\infty(T_{1,0}(M)).$$

A *CR structure* is a formally integrable almost CR structure and a pair $(M, T_{1,0}(M))$, consisting of a manifold M and a CR structure of CR dimension n, is a *CR manifold* (of *hypersurface type*).

Let $(M, T_{1,0}(M))$ be a CR manifold. The *Levi distribution* is $H(M) := Re\{T_{1,0}(M) \oplus T_{0,1}(M)\}$. It carries the complex structure $J : H(M) \to H(M)$

given by $J(Z+\overline{Z})=i(Z-\overline{Z})$, $Z\in T_{1,0}(M)$. Set $H(M)_x^\perp := \{\omega\in T_x^*(M)\colon \operatorname{Ker}(\omega) \supseteq H(M)_x\}$, $x\in M$. When M is orientable the *conormal bundle* $H(M)^\perp$ is trivial hence admits everywhere nonzero globally defined sections $\theta\in\Gamma^\infty(H(M)^\perp)$, each of which is called a *pseudohermitian structure* on M. Given a pseudohermitian structure θ, the *Levi form* is

$$G_\theta(X, Y)=(d\theta)(X, JY), \qquad X, Y\in H(M).$$

Any other pseudohermitian structure is of the form $\widehat{\theta}=\lambda\theta$, for some C^∞ function $\lambda: M\to \boldsymbol{R}\setminus\{0\}$ and $G_{\lambda\theta}=\lambda G_\theta$. An orientable CR manifold is *nondegenerate* (respectively *strictly pseudoconvex*) if G_θ is nondegenerate (respectively positive definite), for some θ. The property of nondegeneracy is a *CR invariant*, i.e. invariant under a transformation $\widehat{\theta}=\lambda\theta$, while strict pseudoconvexity is not (if G_θ is positive definite, $G_{-\theta}$ is negative definite). When $(M, T_{1,0}(M))$ is nondegenerate any pseudohermitian structure θ is a contact form, i.e. $\theta\wedge(d\theta)^n$ is a volume form on M. Nondegeneracy also implies the existence and uniqueness of a globally defined vector field T on M such that $\theta(T)=1$ and $T\rfloor d\theta=0$ (the *characteristic direction* of $d\theta$).

The Levi form of a CR manifold may be recast as the complex bilinear form

$$L_\theta(Z, \overline{W})=-i(d\theta)(Z, \overline{W}), \qquad Z, W\in T_{1,0}(M),$$

(and then L_θ and (the $\boldsymbol{C}$-linear extension to $H(M)\otimes\boldsymbol{C}$ of) G_θ coincide). On any nongenerate CR manifold on which a contact form θ has been fixed, there is a unique linear connection ∇ satisfying the following axioms 1) $H(M)$ is ∇-parallel, 2) $\nabla g_\theta=0$, $\nabla J=0$, 3) the torsion T_∇ of ∇ is *pure*, i.e. $T_\nabla(Z, W)=0$, $T_\nabla(Z,\overline{W}) =2iL_\theta(Z,\overline{W})T$, for any $Z, W\in T_{1,0}(M)$, and $\tau\circ J+J\circ\tau=0$, where $\tau X := T_\nabla(T, X)$, $X\in T(M)$ (the *pseudohermitian torsion* of ∇). This is the *Tanaka-Webster connection* of (M, θ) (cf. N. Tanaka, [**15**] and S. Webster, [**17**]). Here g_θ is the *Webster metric* i.e.

$$g_\theta=\pi_H G_\theta+\theta\otimes\theta,$$

where, in general for a bilinear form B on $T(M)$, we set

$$(\pi_H B)(X, Y):=B(\pi_H X, \pi_H Y), \qquad X, Y\in T(M),$$

and $\pi_H\colon T(M)\to H(M)$ is the natural projection associated with the direct sum decomposition $T(M)=H(M)\oplus\boldsymbol{R}T$. Also, in the axioms 2)-3) above, J is the endomorphism of $T(M)$ obtained by requesting that $JT=0$.

Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold, of CR dimension n, and θ a contact form. The divergence of a smooth vector field X is given by

$$\mathcal{L}_X(\theta \wedge (d\theta)^n) = div(X)\, \theta \wedge (d\theta)^n,$$

where $\mathcal{L}$ is the Lie derivative. We set

$$\nabla^H u := \pi_H \nabla u\,, \qquad u \in C^\infty(M),$$

where ∇ is the gradient with respect to the Webster metric, i.e. $g_\theta(X, \nabla u) = X(u)$. The *sublaplacian* is the (second order) differential operator

$$\Delta_b u := -div(\nabla^H u), \qquad u \in C^\infty(M).$$

It is formally self adjoint

$$\langle \Delta_b u, v \rangle_{L^2} = \langle u, \Delta_b v \rangle_{L^2},$$

(with one of the functions u, v of compact support), where the L^2 inner product is

$$\langle u, v \rangle_{L^2} = \int_M uv\, \theta \wedge (d\theta)^n.$$

Δ_b is actually subelliptic of order $\varepsilon = 1/2$ (cf. e.g. [**7**]).

3 - The first variation formula

Let $\{Z_1, \ldots, Z_n\}$ be a local frame of $T_{1,0}(M)$, defined on an open set $U \subseteq M$, such that $L_\theta(Z_\alpha, Z_{\overline{\beta}}) = \delta_{\alpha\beta}$ (with $Z_{\overline{\beta}} = \overline{Z}_\beta$). Then, for any bilinear form B on $T(M)$

$$trace_{G_\theta}(\pi_H B) = \sum_{\alpha=1}^{n} \{B(X_\alpha, X_\alpha) + B(JX_\alpha, JX_\alpha)\},$$

where

$$X_\alpha := \frac{1}{\sqrt{2}}(Z_\alpha + Z_{\overline{\alpha}}),$$

(hence $G_\theta(X_\alpha, X_\beta) = \delta_{\alpha\beta}$). If X is a tangent vector field on M, $\phi_* X$ denotes the section in $\phi^{-1}TN \to M$ given by $(\phi_* X)(x) := (d_x \phi)\, X_x \in T_{\phi(x)}(N) = (\phi^{-1}TN)_x$,

$x \in M$. Note that

$$trace_{G_\theta}(\pi_H \phi^* h) = \sum_{a=1}^{2n} \tilde{h}(\phi_* X_a, \phi_* X_a) \geqslant 0 ,$$

(hence the definition of $E_F(\phi; D)$ makes sense ($F(t)$ is defined only for $t \geqslant 0$)) where $\{X_a : 1 \leqslant a \leqslant 2n\} := \{X_\alpha, JX_\alpha : 1 \leqslant \alpha \leqslant n\}$.

Let $\phi : M \to N$ be a smooth map. Then ϕ is F-pseudoharmonic if

$$\frac{d}{dt}\{E_F(\phi_t)\}_{t=0} = 0 ,$$

for any compactly supported 1-parameter variation $\{\phi_t\}_{|t|<\varepsilon}$ of $\phi_0 = \phi$. To write the first variation formula we set $\widetilde{M} := (-\varepsilon, \varepsilon) \times M$ and

$$\Phi : \widetilde{M} \to N, \qquad \Phi(t, x) := \phi_t(x), \qquad x \in M, \ |t| < \varepsilon .$$

Also

$$V_x := (d_{(0,x)} \Phi) \frac{\partial}{\partial t}\Big|_{(0,x)} \in T_{\phi(x)}(N), \qquad x \in M .$$

Then $V \in \Gamma^\infty(\phi^{-1} TN)$. Moreover, let

$$\boldsymbol{V} := \Phi_* \frac{\partial}{\partial t} \in \Gamma^\infty(\Phi^{-1} TN),$$

(hence $\boldsymbol{V}_{(0,x)} = V_x$). Let $\tilde{\nabla} := \Phi^{-1} \nabla^N$ be the connection in $\Phi^{-1} TN \to \widetilde{M}$ induced by ∇^N (the Levi-Civita connection of (N, h)). We have

$$\frac{d}{dt}\{E_F(\phi_t)\} = \frac{d}{dt} \int_M F\left(\frac{1}{2}\, trace_{G_\theta}(\pi_H \phi_t^* h)\right) \Psi$$

$$= \int_M \frac{d}{dt} F\left(\frac{1}{2} \sum_{a=1}^{2n} (\phi_t^* h)(X_a, X_a)\right) \Psi ,$$

where $\Psi := \theta \wedge (d\theta)^n$. Let $\alpha_t : M \to \widetilde{M}$ be given by $\alpha_t(x) := (t, x)$, $x \in M$. If X is a tangent vector field on M we set $\tilde{X}_{(t,x)} := (d_x \alpha_t) X_x$. The symbol $\tilde{h}$ denotes the bundle metric $\Phi^{-1} h$ (induced by h) in $\Phi^{-1} TN \to \widetilde{M}$, as well. Then

$$(\phi_t^* h)(X, Y) = \tilde{h}(\Phi_* \tilde{X}, \Phi_* \tilde{Y}) \circ \alpha_t .$$

Set

$$Q_t := trace_{G_\theta}(\pi_H \phi_t^* h) \in C^\infty(M), \qquad |t| < \varepsilon .$$

Then

$$\frac{d}{dt}\{E_F(\phi_t)\} = \frac{1}{2}\int_M F'\left(\frac{Q_t}{2}\right)\sum_{a=1}^{2n}\frac{d}{dt}\{\tilde{h}(\Phi_*\tilde{X}_a, \Phi_*\tilde{X}_a)\circ\alpha_t\}\,\Psi$$

$$= \int_M F'\left(\frac{Q_t}{2}\right)\sum_{a=1}^{2n}\tilde{h}(\tilde{\nabla}_{\partial/\partial t}\Phi_*\tilde{X}_a, \Phi_*\tilde{X}_a)\,\Psi$$

$$= \int_M F'\left(\frac{Q_t}{2}\right)\sum_{a=1}^{2n}\tilde{h}\left(\tilde{\nabla}_{\tilde{X}_a}\Phi_*\frac{\partial}{\partial t}, \Phi_*\tilde{X}_a\right)\Psi ,$$

as $\tilde{\nabla}\tilde{h} = 0$, ∇^N is torsion-free, and $[\tilde{X}, \partial/\partial t] = 0$. Next

$$\frac{d}{dt}\{E_F(\phi_t)\} = \int_M F'\left(\frac{Q_t}{2}\right)\sum_a\{\tilde{X}_a(\tilde{h}(\boldsymbol{V}, \Phi_*\tilde{X}_a)) - \tilde{h}(\boldsymbol{V}, \tilde{\nabla}_{\tilde{X}_a}\Phi_*\tilde{X}_a)\}\,\Psi .$$

For $|t| < \varepsilon$ fixed, let $X_t \in \Gamma^\infty(H(M))$ be defined by

$$G_\theta(X_t, Y) = \tilde{h}(\boldsymbol{V}, \Phi_*\tilde{Y})\circ\alpha_t, \qquad Y \in \Gamma^\infty(H(M)).$$

Also, if $f \in C^\infty(\tilde{M})$ and $|t| < \varepsilon$ we set $f_t := f\circ\alpha_t \in C^\infty(M)$, so that

$$X(f_t) = \tilde{X}(f)\circ\alpha_t,$$

for any $X \in T(M)$. Therefore

$$\frac{d}{dt}\{E_F(\phi_t)\} = \int_M \varrho(Q_t)\sum_a\{X_a(G_\theta(X_t, X_a)) - \tilde{h}(\boldsymbol{V}, \tilde{\nabla}_{\tilde{X}_a}\Phi_*\tilde{X}_a)\}\,\Psi$$

$$= \int_M \varrho(Q_t)\sum_a\{G_\theta(\nabla_{X_a}X_t, X_a) + G_\theta(X_t, \nabla_{X_a}X_a) - \tilde{h}(\boldsymbol{V}, \tilde{\nabla}_{\tilde{X}_a}\Phi_*\tilde{X}_a)\}\,\Psi ,$$

where $\varrho(s) := F'(s/2)$ and ∇ is the Tanaka-Webster connection of (M, θ) (so that $\nabla G_\theta = 0$), cf. e.g. [**15**], p. 29-30. As Ψ is parallel with respect to ∇

$$div(X) = trace\{Y \mapsto \nabla_Y X\}.$$

Then

$$\frac{d}{dt}\{E_F(\phi_t)\} = \int_M \varrho(Q_t)\{div(X_t) - \sum_a \tilde{h}(\boldsymbol{V}, \tilde{\nabla}_{\tilde{X}_a} \Phi_* \tilde{X}_a - \Phi_* \widetilde{\nabla_{X_a} X_a})\} \Psi$$

$$= \int_M \{div(\varrho(Q_t) X_t) - X_t(\varrho(Q_t)) - \varrho(Q_t) \sum_a \tilde{h}(\boldsymbol{V}, \tilde{\nabla}_{\tilde{X}_a} \Phi_* \tilde{X}_a - \Phi_* \widetilde{\nabla_{X_a} X_a})\} \Psi,$$

because of $div(fX) = f\, div(X) + X(f)$. As ϕ_t is compactly supported, so does X_t. Therefore (by Green's lemma)

$$\frac{d}{dt}\{E_F(\phi_t)\}$$

$$= -\int_M \{G_\theta(X_t, \nabla^H \varrho(Q_t)) + \tilde{h}(\boldsymbol{V}, \varrho(Q_t) \sum_a (\tilde{\nabla}_{\tilde{X}_a} \Phi_* \tilde{X}_a - \Phi_* \widetilde{\nabla_{X_a} X_a}))\} \Psi$$

$$= -\int_M \tilde{h}\Big(\boldsymbol{V}, \sum_a (\tilde{\nabla}_{\tilde{X}_a}(\varrho(Q_t) \Phi_* \tilde{X}_a) - \varrho(Q_t) \Phi_* \widetilde{\nabla_{X_a} X_a})\Big) \Psi.$$

The last equality holds because of

$$\sum_{a=1}^{2n} \tilde{\nabla}_{\tilde{X}_a}(\varrho(Q_t) \Phi_* \tilde{X}_a) = \sum_a (\tilde{X}_a(\varrho(Q_t)) \Phi_* \tilde{X}_a + \varrho(Q_t) \tilde{\nabla}_{\tilde{X}_a} \Phi_* \tilde{X}_a)$$

$$= \Phi_* \nabla^H \varrho(Q_t) + \varrho(Q_t) \sum_a \tilde{\nabla}_{\tilde{X}_a} \Phi_* \tilde{X}_a.$$

Note that

$$((\Phi^{-1} \nabla^N)_X \Phi_* Y)_{(0,x)} = ((\phi^{-1} \nabla^N)_X \phi_* Y)_x.$$

We are left with the proof of (3). We have

$$div(\varrho(Q) \nabla^H \phi^j) = trace\{Y \mapsto \nabla_Y(\varrho(Q) \nabla^H \phi^j)\}$$

$$= \sum_{a=1}^{2n} G_\theta(\nabla_{X_a}(\varrho(Q) \nabla^H \phi^j), X_a) + \theta(\nabla_T(\varrho(Q) \nabla^H \phi^j))$$

$$= \sum_{a=1}^{2n} [X_a(G_\theta(\varrho(Q) \nabla^H \phi^j, X_a)) - G_\theta(\varrho(Q) \nabla^H \phi^j, \nabla_{X_a} X_a)] + T(\theta(\varrho(Q) \nabla^H \phi^j)),$$

because of $\nabla g_\theta = 0$ and $\nabla T = 0$. Also (as $H(M)$ is ∇-parallel)

$$G_\theta(\nabla^H \phi^j, \nabla_{X_a} X_a)\, Y_j = g_\theta(\nabla \phi^j, \nabla_{X_a} X_a)\, Y_j = (\nabla_{X_a} X_a)(\phi^j)\, Y_j = \phi_* \nabla_{X_a} X_a,$$

hence

$$div\,(\varrho(Q)\, \nabla^H \phi^j)\, Y_j = \sum_{a=1}^{2n} \{X_a(\varrho(Q)\, X_a(\phi^j))\, Y_j - \varrho(Q)\, \phi_* \nabla_{X_a} X_a\}.$$

Consequently (by the very definition of $\tau_F(\phi;\theta,h)$)

$$\tau_F(\phi;\theta,h) = div\,(\varrho(Q)\, \nabla^H \phi^j)\, Y_j + \varrho(Q) \sum_a \{\tilde{\nabla}_{X_a} \phi_* X_a - (X_a^2 \phi^j)\, Y_j\}$$

and the following calculation leads to (3)

$$\tilde{\nabla}_{X_a} \phi_* X_a - (X_a^2 \phi^j)\, Y_j = \tilde{\nabla}_{X_a}(X_a(\phi^j)\, Y_j) - (X_a^2 \phi^j)\, Y_j$$

$$= X_a(\phi^j)\, X_a(\phi^k) \left(\nabla^N_{\partial/\partial y^k} \frac{\partial}{\partial y^j} \right) \circ \phi = \left(\left| \begin{matrix} i \\ jk \end{matrix} \right| \circ \phi \right) X_a(\phi^j)\, X_a(\phi^k)\, Y_i.$$

Theorem 1 is proved.

To prove Theorem 2 we need to derive the transformation law for $\tau(\phi;\theta,h)$ under a change of contact form $\hat{\theta} = e^{2u}\theta$, $u \in C^\infty(M)$. Set

$$\beta_\phi(X, Y) := (\phi^{-1}\nabla^N)_X \phi_* Y - \phi_* \nabla_X Y,$$

for any $X, Y \in T(M)$, so that

$$\tau(\phi;\theta,h) = trace_{G_\theta}(\pi_H \beta_\phi).$$

Consequently

$$\tau_F(\phi;\theta,h) = F'\left(\frac{Q}{2}\right) \tau(\phi;\theta,h) + \phi_* \nabla^H F'\left(\frac{Q}{2}\right). \tag{6}$$

If $\{Z_\alpha\}$ is a local orthonormal frame of $T_{1,0}(M)$ we set $\hat{Z}_\alpha := e^{-u} Z_\alpha$. Note that

$$\tau(\phi;\theta,h) = \sum_{\alpha=1}^{n} \{\beta_\phi(Z_\alpha, Z_{\bar{\alpha}}) + \beta_\phi(Z_{\bar{\alpha}}, Z_\alpha)\}.$$

Therefore

$$\tau(\phi;\widehat{\theta},h)=\sum_{\alpha=1}^{n}\{\widetilde{\nabla}_{\widehat{Z}_\alpha}\phi_*\widehat{Z}_{\overline{\alpha}}-\phi_*\widehat{\nabla}_{\widehat{Z}_\alpha}\widehat{Z}_{\overline{\alpha}}+\widetilde{\nabla}_{\widehat{Z}_{\overline{\alpha}}}\phi_*\widehat{Z}_\alpha-\phi_*\widehat{\nabla}_{\widehat{Z}_{\overline{\alpha}}}\widehat{Z}_\alpha\},$$

where $\widetilde{\nabla}=\phi^{-1}\nabla^N$ and $\widehat{\nabla}$ is the Tanaka-Webster connection of $(M,\widehat{\theta})$. Set $\nabla_{Z_A}Z_B = \Gamma^C_{AB}Z_C$, where $A, B, \ldots \in \{1,\ldots,n,\overline{1},\ldots,\overline{n},0\}$ and $Z_0:=T$. Then

$$\widehat{\nabla}_Z\overline{W}=\nabla_Z\overline{W}-2L_\theta(Z,\overline{W})\nabla^{0,1}u, \qquad Z,W\in T_{1,0}(M),$$

(where $\nabla^{0,1}u:=\pi_{0,1}\nabla u$ and $\pi_{0,1}: T(M)\otimes\boldsymbol{C}\to T_{0,1}(M)$ is the projection (associated to $T(M)\otimes\boldsymbol{C}=T_{1,0}(M)\oplus T_{0,1}(M)\oplus\boldsymbol{C}T)$) leads to

$$e^{2u}\tau(\phi;\widehat{\theta},h)=\tau(\phi;\theta,h)+2n\,\phi_*\nabla^H u.$$

Set $\lambda:=e^u$. Then

$$\tau(\phi;\lambda^2\theta,h)=\lambda^{-2(n+1)}\{\lambda^{2n}\tau(\phi;\theta,h)+\phi_*\nabla^H(\lambda^{2n})\}. \tag{7}$$

This is the transformation law we looked for. Setting $\lambda:=\varrho(Q)^{1/(2n)}$, the formulae (6)-(7) lead to the identity in Theorem 2. Q.e.d.

4 - Pseudoharmonic morphisms

To prove Theorem 3 we shall need the following

Lemma 1 (T. Ishihara, [**11**]). *Let (N,h) be a m-dimensional Riemannian manifold and $p\in N$. Let $C_i, C_{ij}\in\boldsymbol{R}$ be constants such that $C_{ij}=C_{ji}$ and $\sum_{i=1}^m C_{ii}=0$. Let (V,y^i) be a normal coordinate system on N at p such that $y^i(p)=0$, $1\leq i\leq m$. There is a harmonic function $v:V\to\boldsymbol{R}$ such that*

$$\frac{\partial v}{\partial y^i}(p)=C_i, \qquad v_{i,j}(p)=C_{ij}.$$

Here $v_{i,j}$ are second order covariant derivatives with respect to the Levi-Civita connection ∇^N of (N,h), i.e.

$$v_{i,j}=\frac{\partial^2 v}{\partial y^i\partial y^j}-\left|\begin{matrix}k\\ ij\end{matrix}\right|\frac{\partial v}{\partial y^k}.$$

Let $\phi:(M,\theta)\to(N,h)$ be a pseudoharmonic morphism. Let $x\in M$ be an arbitrary point and set $p:=\phi(x)\in N$. Let $i_0\in\{1,\ldots,m\}$ be a fixed index and set

$C_i := \delta_{ii_0}$ and $C_{ij} := 0$, $1 \leqslant i, j \leqslant m$. By Ishihara's lemma (cf. Lemma 1 above), given normal coordinates (V, y^i) at p, there is a harmonic function $v : V \to \boldsymbol{R}$. Then $\Delta_b(v \circ \phi) = 0$ in $U := \phi^{-1}(V)$, by the very definition of pseudoharmonic morphisms. Let $\{Z_\alpha\}$ be an orthonormal frame of $T_{1,0}(M)$ on U. Note that

$$(v \circ \phi)_{\alpha,\overline{\beta}} = (v_j \circ \phi)\, \phi^j_{\alpha,\overline{\beta}} + \phi^i_\alpha \phi^j_{\overline{\beta}} \left(v_{i,j} + \left| \begin{matrix} k \\ ij \end{matrix} \right| v_k \right) \circ \phi\,, \qquad v_i := \frac{\partial v}{\partial y^i}\,,$$

$$\Delta_b(v \circ \phi) = -\sum_\alpha \{(v \circ \phi)_{\overline{\alpha},\alpha} + (v \circ \phi)_{\alpha,\overline{\alpha}}\}\,,$$

yield

$$\Delta_b(v \circ \phi) = (\Delta_b \phi^j)(v_j \circ \phi) - 2 \sum_\alpha \phi^j_\alpha \phi^k_{\overline{\alpha}} \left(v_{j,k} + \left| \begin{matrix} i \\ jk \end{matrix} \right| v_i \right) \circ \phi\,. \tag{8}$$

Let us apply (8) at the preferred point x

$$0 = (\Delta_b \phi^{i_0})(x) - 2 \sum_\alpha \phi^j_\alpha(x)\, \phi^k_{\overline{\alpha}}(x) \left| \begin{matrix} i_0 \\ jk \end{matrix} \right| (p) = \tau(\phi;\, \theta,\, h)^{i_0}_x,$$

i.e. ϕ is a pseudoharmonic map.

Let us consider now $C_{ij} \in \boldsymbol{R}$ such that $C_{ij} = C_{ji}$ and $\sum_{i=1}^m C_{ii} = 0$. By Ishihara's lemma there is $v : V \to \boldsymbol{R}$ such that $\Delta_N v = 0$, $v_i(p) = 0$ and $v_{i,j}(p) = C_{ij}$. As ϕ is a pseudoharmonic morphism (by (8))

$$0 = \Delta_b(v \circ \phi)(x) = -\sum_\alpha \phi^j_\alpha(x)\, \phi^k_{\overline{\alpha}}(x)\, C_{jk}.$$

Set

$$X^{jk} := \sum_\alpha \phi^j_\alpha \phi^k_{\overline{\alpha}},$$

so that

$$C_{jk} X^{jk}(x) = 0\,.$$

Thus

$$\sum_{i \neq j} C_{ij} X^{ij}(x) + \sum_i \left(X^{ii}(x) - X^{11}(x)\right) C_{ii} = 0\,. \tag{9}$$

Let us choose for a moment $C_{ij} = 0$ for any $i \neq j$ and

$$C_{ii} = \begin{cases} 1, & i = i_0 \\ -1, & i = 1 \\ 0, & \text{otherwise} \end{cases},$$

for $i_0 \in \{2, \ldots, m\}$ fixed. Then (9) yields

$$X^{i_0 i_0}(x) - X^{11}(x) = 0\,,$$

i.e.

$$X^{11}(x) = \ldots = X^{mm}(x)$$

and (9) becomes

$$\sum_{i \neq j} C_{ij} X^{ij}(x) = 0\,.$$

Again we may fix $i_0, j_0 \in \{1, \ldots, m\}$ with $i_0 \neq j_0$ and choose

$$C_{ij} = \begin{cases} 1, & i = i_0,\ j = j_0 \\ 0, & \text{otherwise} \end{cases}$$

so that to get

$$X^{i_0 j_0}(x) = 0\,.$$

We proved that $X^{ij}(x) = 0$ for any $i \neq j$. If we set

$$\lambda := X^{11} = \sum_{\alpha=1}^{n} \phi^1_\alpha \phi^1_{\bar\alpha} \in C^\infty(U)\,,$$

then

$$\sum_{\alpha=1}^{n} \phi^i_\alpha(x)\, \phi^j_{\bar\alpha}(x) = \lambda(x)\, \delta^{ij}. \tag{10}$$

The contraction of i, j now leads to

$$m\lambda = \sum_{\alpha, i} |\phi^i_\alpha|^2 \geqslant 0\,,$$

hence $\lambda : M \to [0, +\infty)$ is a C^∞ function. To complete the proof of Theorem 3 assume that there is $x \in M$ such that $\lambda(x) \neq 0$ and set

$$w^i := (\phi^i_1(x), \dots, \phi^i_n(x)) \in \boldsymbol{C}^n.$$

Clearly $w^i \neq 0$ and (by (10)) $i \neq j \Rightarrow w^i \cdot \overline{w^j} = 0$, that is the rows of $[\phi^i_\alpha(x)] = (w^1, \dots, w^m)^t$ are mutually orthogonal nonzero vectors in $\boldsymbol{C}^n$. Hence $\operatorname{rank}[\phi^i_\alpha(x)] = m$ and then $m \leq n$. Therefore $m > n$ implies $\lambda = 0$ hence $\phi^i_\alpha = 0$. Thus $\phi^i_{\bar\alpha} = 0$ (by complex conjugation) or $\bar\partial_b \phi^i = 0$, i.e. ϕ^i is a $\boldsymbol{R}$-valued CR function on a nondegenerate CR manifold, hence $\phi^i = \text{const.}$, $1 \leq i \leq m$, i.e. ϕ is a constant map. Q.e.d.

5 - The Fefferman metric and F-pseudoharmonicity

Let $(M, T_{1,0}(M))$ be a strictly pseudoconvex CR manifold, of CR dimension n, and θ a contact form on M with G_θ positive definite. Let T be the characteristic direction of $d\theta$. A complex valued p-form ω on M is a $(p, 0)$*-form* if $T_{0,1}(M) \rfloor \omega = 0$. Let $\Lambda^{p,0}(M) \to M$ be the bundle of all $(p, 0)$-forms. The multiplicative group of positive real numbers $GL^+(1, \boldsymbol{R}) = (0, +\infty)$ acts on $K_0(M) := \Lambda^{n+1,0}(M) \setminus \{\textit{zero section}\}$ and the quotient space $C(M) := K_0(M)/GL^+(1, \boldsymbol{R})$ is (the total space of) a principle circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$. Let us extend G_θ to the whole of $T(M)$ by requesting that $G_\theta(X, T) = 0$, for all $X \in T(M)$. Consider the 1-form $\sigma \in \Omega^1(C(M))$

$$\sigma = \frac{1}{n+2}\left\{d\gamma + \pi^*\left(i\omega^\alpha_\alpha - \frac{i}{2} g^{\alpha\bar\beta} dg_{\alpha\bar\beta} - \frac{R}{4(n+1)}\theta\right)\right\}, \tag{11}$$

and set

$$\mathcal{F}_\theta = \pi^* G_\theta + 2(\pi^*\theta) \odot \sigma, \tag{12}$$

(the *Fefferman metric* of (M, θ)). Here γ is a local fibre coordinate on $C(M)$, ω^α_β are the connection 1-forms of the Tanaka-Webster connection ($\nabla T_\beta = \omega^\alpha_\beta \otimes T_\alpha$), and $R = g^{\alpha\bar\beta} R_{\alpha\bar\beta}$ is the *pseudohermitian scalar curvature* (cf. [5], p. 103). $\mathcal{F}_\theta$ is a Lorentz metric on $C(M)$ and $\mathcal{F}_{e^u\theta} = e^{u\circ\pi}\mathcal{F}_\theta$, for any $u \in C^\infty(M)$ (in particular $[\mathcal{F}_\theta] := \{e^{u\circ\pi}\mathcal{F}_\theta : u \in C^\infty(M)\}$ is a CR invariant).

Let $F(t) \geq 0$ be a C^2 map defined for $t \geq 0$, such that $F'(t) > 0$. For simplicity, assume for the rest of this section that M is compact (hence $C(M)$ is compact, as well). A smooth map $\Phi : (C(M), \mathcal{F}_\theta) \to (N, h)$ is F-harmonic if it is a critical point of

$$\boldsymbol{E}_F(\Phi) = \int_{C(M)} F\left(\frac{1}{2}\, trace_{\mathcal{F}_\theta}(\Phi^* h)\right) d\,\mathrm{vol}(\mathcal{F}_\theta)\,,$$

i.e. if $\tau_F(\Phi; \mathcal{F}_\theta, h) = 0$, where (cf. [**1**])

$$\tau_F(\Phi; \mathcal{F}_\theta, h) := F'\left(\frac{1}{2}\, trace_{\mathcal{F}_\theta}(\Phi^* h)\right) \tau(\Phi; \mathcal{F}_\theta, h) + \Phi_*\left\{\nabla F'\left(\frac{1}{2}\, trace_{\mathcal{F}_\theta}(\Phi^* h)\right)\right\}. \tag{13}$$

Here $\tau(\Phi; \mathcal{F}_\theta, h)$ is the ordinary tension field of $\Phi : (C(M), \mathcal{F}_\theta) \to (N, h)$ (cf. e.g. [**6**], p. 107) and $\nabla : C^\infty(C(M)) \to \mathfrak{X}(C(M))$ the gradient operator with respect to the Fefferman metric $\mathcal{F}_\theta$. At this point we may prove Theorem 4. Let (U, x^A) be a local coordinate system on M (the convention as to the range of indices is $A, B, \ldots \in \{0, 1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$) and set $u^A := x^A \circ \pi : \pi^{-1}(U) \to \boldsymbol{R}$. If $c = [\omega] \in \pi^{-1}(U)$ then $\omega \in K_0(M)$ may be written $\omega = \lambda(\theta \wedge \theta^1 \wedge \ldots \wedge \theta^n)_x$, for some $\lambda \in \boldsymbol{C}^* = \boldsymbol{C} \setminus \{0\}$ and $x = \pi(c) \in U$. Then $\gamma : \pi^{-1}(U) \to \boldsymbol{R}$ is given by $\gamma(c) = \arg(\lambda/|\lambda|)$ (with $\arg : S^1 \to [0, 2\pi)$) and $(\pi^{-1}(U), u^A, \gamma)$ is a local coordinate system on $C(M)$. Set $g_{ab} = \mathcal{F}_\theta(\partial/\partial u^a, \partial/\partial u^b)$ and $[g^{ab}] = [g_{ab}]^{-1}$, where $a, b, \ldots \{0, 1, \ldots, n, \overline{1}, \ldots, \overline{n}, 2n+2\}$ and $u^{2n+2} = \gamma$. Let $\phi : M \to N$ be a smooth map and set $\Phi := \phi \circ \pi$. Then

$$trace_{\mathcal{F}_\theta}(\Phi^* h) = g^{ab}\, \frac{\partial \Phi^j}{\partial u^a}\, \frac{\partial \Phi^k}{\partial u^b}\,(h_{jk} \circ \Phi) = g^{AB}\left(\frac{\partial \phi^j}{\partial x^A} \circ \pi\right)\left(\frac{\partial \phi^k}{\partial x^B} \circ \pi\right)(h_{jk} \circ \Phi)$$

(because $\partial \pi^A / \partial \gamma = 0$) and then, by the identity (9) in H. Urakawa & alt., [**3**], p. 730

$$g^{AB}\left(\frac{\partial \phi^j}{\partial x^A} \circ \pi\right)\left(\frac{\partial \phi^k}{\partial x^B} \circ \pi\right) = h^{\alpha\overline{\beta}}\{T_\alpha(\phi^j)\, T_{\overline{\beta}}(\phi^k) + T_{\overline{\beta}}(\phi^j)\, T_\alpha(\phi^k)\} \circ \pi$$

we have

$$trace_{\mathcal{F}_\theta}(\Phi^* h) = 2h^{\alpha\overline{\beta}}\, T_\alpha(\phi^j)\, T_{\overline{\beta}}(\phi^k) \circ \pi = \{trace_{G_\theta}(\pi_H \phi^* h)\} \circ \pi = Q \circ \pi\,.$$

Therefore (by (13))

$$\tau_F(\Phi; \mathcal{F}_\theta, h) = F'\left(\frac{1}{2}\, Q \circ \pi\right) \tau(\Phi; \mathcal{F}_\theta, h) + \Phi_* \nabla\left\{F'\left(\frac{1}{2}\, Q \circ \pi\right)\right\}$$

i.e. $\Phi = \phi \circ \pi$ is F-harmonic if and only if

$$(\varrho(Q) \circ \pi)\, \tau(\Phi; \mathcal{F}_\theta, h) + \Phi_* \nabla(\varrho(Q) \circ \pi) = 0 \tag{14}$$

where $\varrho(s) := F'(s/2)$. It is well known that, under a conformal change of metric, the tension tensor field transforms as

$$\tau(\Phi; \lambda^2 \mathcal{F}_\theta, h) = \lambda^{-(2n+2)} \{\lambda^{2n} \tau(\Phi; \mathcal{F}_\theta, h) + \Phi_* \nabla(\lambda^{2n})\}, \tag{15}$$

for any $\lambda \in C^\infty(C(M))$. By Theorem 2 ϕ is F-pseudoharmonic if and only if $\tau(\phi; \varrho(Q)^{1/n}\theta, h) = 0$, that is [by Theorem 2.1 in H. Urakawa & alt., [3], p. 729] if and only if $\phi \circ \pi$ is harmonic with respect to the Fefferman metric $\mathcal{F}_{\varrho(Q)^{1/n}\theta}$, i.e.

$$\tau(\phi \circ \pi; \mathcal{F}_{\varrho(Q)^{1/n}\theta}, h) = 0 .$$

Finally [by (15) with $\lambda := (\varrho(Q) \circ \pi)^{1/(2n)}$] ϕ is F-pseudoharmonic if and only if (14) holds. Q.e.d.

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Summary

Owing to the ideas of M. Ara (cf. [**1**]) *and K. Uhlenbeck (cf.* [**16**]) *we consider F-pseudoharmonic maps, i.e. critical points of the energy* $E_F(\phi) = \int_M F\left(\frac{1}{2}\, trace_{G_\theta}(\pi_H \phi^* h)\right) \theta \wedge (d\theta)^n$, *on the class of smooth maps* $\phi : M \to N$ *from a (compact) strictly pseudoconvex CR manifold* (M, θ) *to a Riemannian manifold* (N, h), *where* θ *is a contact form and* $F: [0, \infty) \to [0, \infty)$ *is a* C^2 *function such that* $F'(t) > 0$. *F-pseudoharmonic maps generalize both J. Jost & C-J. Xu's subelliptic harmonic maps (the case* $F(t) = t$, *cf.* [**12**]) *and P. Hajlasz & P. Strzelecki's subelliptic p-harmonic maps (the case* $F(t) = (2t)^{p/2}$, *cf.* [**9**]). *We obtain the first variation formula for* $E_F(\phi)$. *We investigate the relationship between F-pseudoharmonicity and pseudoharmonicity, by exploiting the analogy between CR and conformal geometry (cf.* [**1**] *for the Riemannian counterpart). We consider pseudoharmonic morphisms from a strictly pseudoconvex CR manifold and show that any pseudoharmonic morphism is a pseudoharmonic map (the CR analogue of T. Ishihara's theorem, cf.* [**11**]). *We give a geometric interpretation of F-pseudoharmonicity in terms of the Fefferman metrics of* (M, θ).

* * *