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Edge-bipancyclicity of the extended and the widened Fibonacci cubes (**)

1 - Introduction

The *hypercube* Q_n is the graph with 2^n vertices, each corresponding to a binary string of length n, where two vertices are adjacent if and only if the corresponding binary strings differ in exactly one bit.

These graphs have been used extensively as architectural models for parallel processors where each vertex represents a processor and each edge represents a direct link between two processors. In recent years various subgraphs of Q_n have been proposed as alternative models. Among these are the Fibonacci cubes, proposed by Hsu [4], the extended Fibonacci cube, proposed by Wu [8], the widened Fibonacci cube [1].

A *Fibonacci string* is a binary string with no two consecutive ones. Let B_n and C_n denote the sets of binary strings and Fibonacci strings of length n respectively; thus B_n is the set of vertices of Q_n, while C_n is the set of vertices of a subgraph Γ_n of Q_n, called *Fibonacci cube* of order n.

If α and β denote two strings, then $\alpha\beta$ is the string obtained by concatenating α and β. More generally if S is a set of strings, then $\alpha S\beta$ denotes the set of strings $\alpha\gamma\beta$, where $\gamma \in S$.

The set C_{n+2} can be partitioned into two disjoint subsets, depending whether

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the first element of a string is 0 or 1. If the first element is 1, then the second has to be 0. Then we obtain the relation

$$C_{n+2} = 0C_{n+1} + 10C_n \tag{1}$$

with initial conditions $C_0 = \{\emptyset\}$, $C_1 = \{0, 1\}$, where $\emptyset$ denotes the empty string.

It is well known that the cardinality of C_n is the Fibonacci number F_n, with the initial values $F_0 = 1$ and $F_1 = 2$. Thus the *Fibonacci cube* Γ_n is the bipartite graph whose set of vertices is C_n and whose edges are the pairs of vertices with unit Hamming distance.

The *extended Fibonacci cube* is constructed by the same recursive relation as the Fibonacci cube, but with different initial conditions.

For positive integers $i, n, i \leq n$, the ith extended Fibonacci cube of order n, denoted by Γ^i_n, is a vertex induced subgraph of Q_n, where $V(\Gamma^i_n) = V^i_n$ is defined recursively by the relation

$$V^i_{n+2} = 0V^i_{n+1} + 10V^i_n,$$

with initial conditions $V^i_i = B_i$, $V^i_{i+1} = B_{i+1}$. Thus $\Gamma^i_i = Q_i$ and $\Gamma^i_{i+1} = Q_{i+1}$.

The *widened Fibonacci cube* WFC_{n+4}, $n \geq 0$, is the graph whose set of vertices W_{n+4} satisfies the recursive relation

$$W_{n+4} = 00C_{n+2} + 10C_{n+2} + 0100C_n + 0101C_n \tag{2}$$

and whose edges are again the pairs of vertices with unit Hamming distance.

Clearly this graph is embedded in the hypercube Q_{n+4} and contains Q_n. Moreover it mantains all the desirable properties of the Fibonacci cube, having in addition the hamiltonicity, proved in [1], not satisfied by all the Fibonacci cubes.

In this paper we prove that the extended Fibonacci cubes Γ^i_n, $n \geq 5$, $i \geq 1$ and the widened Fibonacci cubes WFC_n, $n \geq 6$ satisfy the property that every edge belongs to cycles of every even length.

If G is a simple graph, with vertex set $V(G)$ and edge set $E(G)$, an edge e of G is said *pancyclic* in G when it belongs to cycles of all lengths in G.

G is said *edge-pancyclic* when every edge is pancyclic. In the case in which G is bipartite, the lengths have to be even and G is also said bipancyclic.

Let C be a cycle of G, (a, b) an edge of C and (a', b') an edge of G which does not belong to C; moreover let us assume a' adjacent to a and b' adjacent to b. We say that we *widen* C from (a, b) to (a', b') when we replace (a, b) with the 3-path $aa'b'b$.

For other definitions and notations, the reader is referred to [2].

2 - Extended Fibonacci cubes

In [7] the following decomposition of the extended Fibonacci cubes is proved:

$$\Gamma_n^i = \Gamma_{n-1}^{i-1} \times K_2 \tag{3}$$

where $\Gamma_n^0 = \Gamma_n$, $n \geq 1$ and $i \geq 1$. By the above decomposition, Γ_{n+1}^1 consists of two vertex-disjoint copies of the Fibonacci cubes Γ_n, $n \geq 1$, having sets of vertices V and V', with the addition of the F_n edges connecting each vertex of V with its corresponding vertex in V'. Denote by G and G' the copies of Γ_n having sets of vertices V and V'. An edge (v, w) of G and an edge (v', w') of G' are said corresponding when the vertices v', w' are adjacent to v, w respectively.

Lemma 1. *Let F_n be even; then Γ_{n+1}^1, $n > 5$, is edge-pancyclic.*

Proof. By a result proved in [9], Γ_n, $n \geq 5$, is edge-pancyclic. Let e be an edge of G, C a hamiltonian cycle of G containing e, $g \neq e$ an edge of G and g' its corresponding in G'. Then widening C from g to g', and replacing g' by suitable paths of all possible odd lengths, we obtain e is pancyclic. A perfectly similar situation holds when e is an edge of G'.

Now consider an edge $e = (v, v')$, where $v \in G$ and $v' \in G'$. If w is a vertex of G adjacent to v and w' its corresponding in G', then e belongs to the 4-cycle $vv'w'wv$; then by replacing (v, w) and also (v', w') by suitable odd paths of all lengths, we obtain that also (v, v') is pancyclic. Then the result follows. ■

Lemma 2. *For $n \geq 4$, Γ_{n+1}^1 is edge-bipancyclic.*

Proof. First consider the case of $n > 4$, in which Γ_n turns out to be pancyclic. In the case of F_n even, the result follows from Lemma 1.

Thus assume F_n is odd. By a result proved in [9], every edge of G belongs to cycles of length $l : 4 \leq l \leq F_n - 1$; now we prove it is pancyclic in Γ_{n+1}^1. Let $e = (a, b)$ be an edge of G, D an $(F_n - 1)$-cycle of G containing e, w the vertex not in D and t a vertex adjacent to w in D; denote by D', w' and t' the corresponding elements of G'. Moreover denote by u_i, $u_{i'}$, where $1 \leq i \leq (F_n - 2)$, the vertices of D and D' distinct from t, t' respectively. The vertices u_1, u_k, where $k = F_n - 2$, are adjacent to t in D. We note that $\{u_1, u_k\} \neq \{a, b\}$, otherwise $(a, b) \notin D$. We may also assume (a,b)$\neq (u_k, t)$, because otherwise we may to reverse the order of the vertices of D. Then the cycle $u_1 \dots u_k u_k' \dots u_1' t' w' w t u_1$ is a Hamiltonian cycle of Γ_{n+1}^1 which contains e. In particular if $t \in \{a, b\}$, say $t = a$, then $u_1 = b$. A perfectly similar procedure holds for the edges of G'.

Similarly to the even case we are able to prove that every edge (v, v'), where v is a vertex of G and v' its corresponding in G', belongs to cycles of every length l, where $4 \leq l \leq 2(F_n - 1)$. Now we prove that (v, v') belongs to a Hamiltonian cycle. Consider the cycle C

$$u_1 u_1' u_2' u_2 u_3 u_3' \ldots u_k u_k' t' w' w t u_1 .$$

It contains the edge (u_j', u_{j+1}') for j odd and in particular the edge (u_k', t'), since k is odd. Moreover C contains all the edges (v, v'), but (t, t').

However, as the degree of w is greater than 1, it follows that w is adjacent to at least another vertex q of D and we may repeat the above procedure by replacing t by q. Thus every edge (v, v') is pancyclic in Γ_{n+1}^1.

Finally consider the case of $n = 4$. A representation of Γ_5^1 is shown in Fig. 1.

Denote by v_i and w_i, $1 \leq i \leq 8$, the vertices of G and G' respectively, both isomorphic to Γ_4. Notice that every edge e of G distinct from (v_2, v_5), (v_4, v_5) is pancyclic in G; then, by following the same procedure used in Lemma 1, it is pancyclic in Γ_5^1. Consider now the 4-cycle $v_1 v_2 v_5 v_4 v_1$. Widen it from (v_1, v_2) to (w_1, w_2) and then to (w_4, w_5), (w_7, w_8), (v_7, v_8); moreover from (w_2, w_5) to (w_3, w_6), then to (v_3, v_6). This implies that (v_2, v_5) and (v_4, v_5), which belong to the initial cycle, are pancyclic.

Consider an edge (v_i, w_i), $1 \leq i \leq 8$. Let v_j, $j \neq i$, $1 \leq j \leq 8$, be a vertex of G adjacent to v_i such that (v_i, v_j) is distinct from (v_2, v_5), (v_4, v_5) and then pancyclic in G. Starting from the 4-cycle $v_i v_j w_j w_i v_i$ and replacing (v_i, v_j) and (w_i, w_j) by suitable paths in G and G', we obtain that also the edges (v_i, w_i) are pancyclic. This completes the proof of the lemma. ■

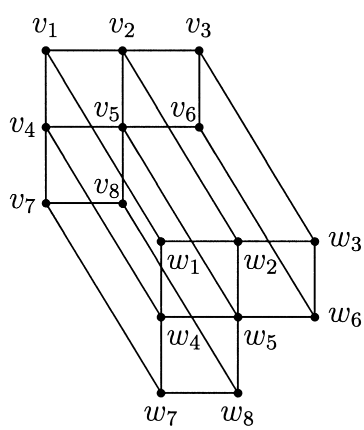

Figure 1

Theorem 1. *Let i, n be positive integers with $n \geqslant i$ and $n \geqslant 5$. Then the extended Fibonacci cube Γ_n^i is edge-pancyclic.*

Proof. Let Γ_n^i be an extended Fibonacci cube, where $0 < i \leqslant n$. We prove the result by induction on the value of i. For $i = 1$ the result follows from Lemma 2. Assume $i \geqslant 2$ and consider the decomposition (3). Since, by induction, Γ_{n-1}^{i-1} is edge-pancyclic, using the same procedure of Lemma 1, the result follows. ∎

3 - Widened Fibonacci cubes

In this section we study the edge-bipancyclity of the widened Fibonacci cubes, WFC_{n+4}, $n \geqslant 1$, the graphs, embedded in the hypercube Q_{n+4}, whose set of vertices W_{n+4} satisfies the relation (2) and whose edges are the pairs of vertices having unit Hamming distance. By iterating a suitable number of times the decomposition (1), equation (2) gives

$$W_{n+4} = 0010C_n + 0000C_n + 00010C_{n-1} + 1010C_n + 1000C_n$$

$$+ 10010C_{n-1} + 0100C_n + 0101C_n .$$

Denote by A, B, C, D, E, F the graphs, isomorphic to Γ_n, having as sets of vertices XC_n where X coincides with 1010, 0010, 1000, 0000, 0100, 0101 respectively. Moreover let G and H be the graphs, isomorphic to Γ_{n-1}, having as sets of vertices YC_{n-1}, where Y is 00010 and 10010 respectively. Thus WFC_{n+4} can be decomposed as the graph of Fig. 2, isomorphic to Γ_4.

In all the section we denote by $\{x_i \,|\, 1 \leqslant i \leqslant F_n\}$, where $x \in \{a, b, \ldots, f\}$ the sets of vertices of A, B, C, D, E, F and by $\{x_j \,|\, 1 \leqslant j \leqslant F_{n-1}\}$, where $x \in \{g, h\}$, the sets of vertices of G, H respectively. Notice that both the subgraphs induced by $C \cup H$ and $D \cup G$ are each isomorphic to Γ_{n+1}, while the subgraphs induced by $V_1 = A \cup (C \cup H)$ and $V_2 = B \cup (D \cup G)$, denoted by H_1 and H_2 respectively,

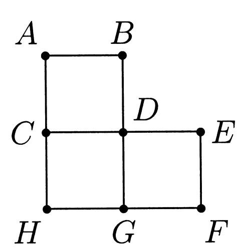

Figure 2

are isomorphic to Γ_{n+2}. Finally the subgraph, denoted by G_1, induced by $V_1 \cup V_2$ is isomorphic to Γ^1_{n+3}, while the subgraph, denoted by G_2, induced by $E \cup F$ is isomorphic to Γ^1_{n+1}.

Note that WFC_5 is not edge-pancyclic; indeed it is easy to see that the edges (c_1, d_1) and (d_1, g_1) do not belong to hamiltonian cycles.

Lemma 3. *Let $n \geqslant 4$ and U a hamiltonian cycle of G_1. Then U contains at least two edges which belong to the subgraph induced by $D \cup G$.*

Proof. Since every vertex of G is adjacent to only one vertex of H, it is adjacent in U to at most one vertex of H. Consequently in U every vertex of G is adjacent to a vertex of $G \cup D$. When F_{n-1} is odd at least one vertex of G is adjacent in U to a vertex of D; thus at least $\left\lceil \frac{F_{n-1}}{2} \right\rceil > 2$ edges of U belong to the subgraph induced by $G \cup D$. ■

Lemma 4. *Let $n \geqslant 4$ and T a hamiltonian cycle of G_2. Then T contains at least two edges having at least one vertex in E.*

Proof. Notice that every vertex of E is adjacent to only one vertex of G_1 and to only one vertex of F. Then in T every vertex of E is adjacent to at most a vertex of F. It follows that T contains at least $\left\lceil \frac{F_n}{2} \right\rceil > 2$ edges which connect either vertices of E or vertices of E and F. ■

Theorem 2. *For $n > 1$, WFC_{n+4} is edge-pancyclic.*

Proof. First assume that $n \geqslant 4$. Then by Lemma 2 both the subgraphs G_1 and G_2, respectively isomorphic to Γ^1_{n+3} and Γ^1_{n+1}, are edge-pancyclic. Let e be an edge of G_1, C_1 an hamiltonian cycle of G_1, which contains e, $g \neq e$ an edge of C_1 which belongs to the subgraph induced by $G \cup D$ by Lemma 3. Denoted by g' the corresponding of g in G_2, by widening C_1 from g to g', and replacing g' by suitable paths of odd lengths, it follows that e is edge-pancyclic in all the graph.

Let f be an edge of G_2 and C_2 an hamiltonian cycle containing f. It follows from Lemma 4 that C_2 contains at least one edge $t \neq f$ having one vertex in E; consequently t corresponds to an edge of G_1, say t'. Then by widening C_2 from t to t' and replacing t' by suitable paths, we obtain that f is pancyclic in all the graph.

Finally consider the edge (v, v') connecting a vertex v' of G_2 with its corresponding v in G_1; denoted by w' a vertex of G_2, adjacent to v', having corresponding in G_1, say w, adjacent to v, we see that (v, v') belongs to the 4-cycle

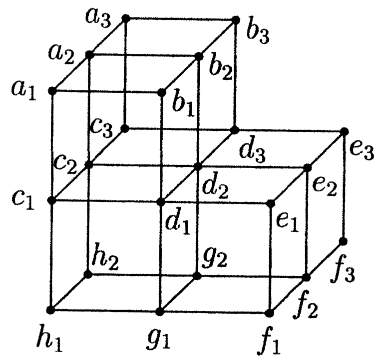

Figure 3

$vv'w'wv$. Replacing (v, w) and (v', w') by suitable paths, we obtain that also (v, v') is pancyclic.

Consider the case of $n = 2$; a representation of WFC_6 is shown in Fig. 3. Now G_1 is isomorphic to Γ_5^1, edge-pancyclic by Lemma 2, while G_2 is isomorphic to Γ_3^1, edge-pancyclic with the exception of one edge, namely (e_2, f_2). Let e be an edge of G_1 and C_1 an hamiltonian cycle of G_1 which contains e. We note that this cycle has to contain at least one edge, say q, of the path $P = g_2 g_1 d_1$, because otherwise the vertex g_1 does not belong to the cycle. If $e \notin P$, then the edge q of C_1 is distinct from e and its corresponding q' in G_2 is distinct from (e_2, f_2). This implies that e is pancyclic in all the graph. If $e \in P$, then we may determine the following cycle of G_1 which contains P and then an edge q satisfying the above condition:

$$d_1 g_1 g_2 d_2 d_3 b_3 b_2 b_1 a_1 a_2 a_3 c_3 c_2 h_2 h_1 c_1 d_1 .$$

Let $e \neq (e_2, f_2)$ be an edge of G_2. It belongs to a 4-cycle of G_2 and then to the 6-cycle $C_2 = e_1 e_2 e_3 f_3 f_2 f_1$ which contains at least one edge $l \in \{(e_1, e_2), (e_2, e_3)\}$, distinct from e, having corresponding l' in G_1. If we widen C_2 from l to l' in G_1 and replace l' by suitable paths, we obtain that e is pancyclic in all the graph.

It therefore remains to consider the edge (e_2, f_2). It belongs to the 4-cycle $e_2 f_2 f_3 e_3 e_2$, which we may widen from (e_2, e_3) to (d_2, d_3), then to (b_2, b_3), (a_2, a_3), (c_2, c_3), from (a_2, b_2) to (a_1, b_1), (c_1, d_1), (h_1, g_1), (h_2, g_2) and finally from (g_1, d_1) to (f_1, e_1), thus proving (e_2, f_2) is pancyclic in WFC_6.

Now, consider the case of $n = 3$. The subgraph G_1 is isomorphic to Γ_6^1, pancyclic by Lemma 2, while G_2 is isomorphic to Γ_4^1, which is edge-pancyclic with the exception of the edge (e_2, f_2). Let e be an edge of G_1 and C_1 an hamiltonian cycle of G_1 which contains e. We note that this cycle has to contain at least one edge, say q, of the path $P = g_2 g_1 d_1$, because otherwise the vertex g_1 does not belong to the cycle. If $e \notin P$, then C_1 contains at least the edge $q \neq e$ having corresponding in G_2. This implies that e is pancyclic in all the graph WFC_7. If $e \in P$, then we may determine the following cycle of G_1 which contains P and then an edge q as before:

$$g_2 g_1 d_1 d_2 d_3 b_3 b_2 b_5 b_4 b_1 a_1 a_4 a_5 a_2 a_3 c_3 c_2 c_5 d_5 d_4 c_4 c_1 h_1 h_2 h_3 g_3 g_2.$$

In relation to an edge of G_2, distinct from (e_2, f_2), or an edge (v, v'), where $v' \in G_2$ and v is its corresponding in G_1, we may repeat the above procedure.

Finally consider the edge (e_2, f_2). It belongs to the 4-cycle $e_2 f_2 f_5 e_5 e_2$; widening this cycle from (e_2, e_5) to (e_1, e_4), then to (f_1, f_4), from (e_1, e_2), to (d_1, d_2), (d_4, d_5), (c_4, c_5), (c_1, c_2), (a_1, a_2), (b_1, b_2), (b_4, b_5), (a_4, a_5), from (a_2, b_2) to (a_3, b_3), (c_3, d_3), (h_3, g_3), (h_2, g_2), (h_1, g_1), from (d_3, g_3) to (e_3, f_3), we obtain that (e_2, f_2) is pancyclic, thus completing the proof. ■

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Summary

Extended Fibonacci cubes and widened Fibonacci cubes are generalizations of the Fibonacci cube, the subgraph of the usual hypercube induced by the set of binary strings with no two consecutive ones. Using particular decompositions of these bipartite graphs we prove that, except some initial cases, they satisfy the property that every edge belongs to cycles of any even length.

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