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On two theorems of Bertini
for infinite-dimensional projective spaces (**)
the target finite-dimensional. Even worst: by [B2] on each 1\(^p\) space, 1\(\leq p < + \infty\), there are continuous real and complex polynomials whose set of critical values has a non-empty interior. There are very interesting results concerning differential Fredholm maps ([S], [QS], [PR]) but they cannot be applied in our set-up because we consider maps from an infinite-dimensional domain to a finite-dimensional target. In section two we prove the following result.

**Theorem 1.1.** Fix positive integers \(s, d_1, \ldots, d_s\) and a separable Hilbert space \(V\). Then there exists a smooth codimension \(s\) complete intersection \(X \subset P(V)\) of \(s\) hypersurfaces of degree \(d_1, \ldots, d_s\).

One could hope that adding an algebricity condition, still some weak form of Bertini theorem may hold. In general this is not true: there are projective spaces \(P(V)\) with \(V\) Fréchet nuclear space such that every homogeneous hypersurface of degree at least two of \(P(V)\) is singular (see Example 3.3). In section three we consider the case \(V = C^{(N)}\) and prove the following result.

**Theorem 1.2.** Fix an integer \(d \geq 2\) and a subset \(S\) of \(\mathbb{C}P^1\) with at most countable elements. Then there exist linearly independent homogeneous degree \(d\) polynomials \(F\) and \(G\) on \(C^{(N)}\) such that a hypersurface \(\{\lambda F + \mu G = 0\}\) of \(P = C^{(N)}\) with \((\lambda; \mu) \in \mathbb{C}P^1\) is singular if and only if \((\lambda; \mu) \in S\).

The set-up of Theorem 1.2 (varieties over a complex vector space with countable algebraic dimension) is essentially the set-up for infinite-dimensional algebraic geometry introduced in [S] and [T]. Theorem 1.2 just describes the singular members of the pencil of degree \(d\) hypersurfaces of \(P = C^{(N)}\) generated by the hypersurfaces \(\{F = 0\}\) and \(\{G = 0\}\).

**Remark 1.3.** We stress that in the statement of Theorem 1.2 we allow the case \(S = \emptyset\), i.e. for every integer \(d \geq 2\) we prove the existence of a pencil of degree \(d\) hypersurfaces of \(P = C^{(N)}\) without any singular member. This is in striking contrast with the case of pencils on \(\mathbb{C}P^n\): in that case if \(d \neq 1\) every pencil has a singular member because the set of degree \(d\) singular hypersurfaces is a hypersurface in the big projective space parametrizing all degree \(d\) hypersurfaces of \(\mathbb{C}P^n\).

2 - Proof of Theorem 1.1

**Lemma 2.1.** Fix positive integers \(n, s, d_1, \ldots, d_s\) such that \(n \geq s\). Fix homogeneous coordinates \(z_0, \ldots, z_n\) on \(P^n\). Let \(A(s, n, d_1, \ldots, d_s)\) be the subset of \(C^{(n + 1)}\) formed by all \(a_i \in C, 1 \leq i \leq s, 0 \leq j \leq n, \) such that \(\{F_1 = \ldots = F_s = 0\}\) is a
smooth codimension $s$ complete intersection of $P^n$, where $F_i = \sum_{0 \leq j \leq n} a_{ij} x_j^s$. Then $A(s, n, d_1, \ldots, d_s)$ is a non-empty Zariski open subset of $C^{\binom{n+1}{s}}$.

Proof. Since smoothness is an open condition in the Zariski topology and the same is true for the codimension $s$ (i.e. the maximal possible codimension) condition, it is sufficient to show that $A(s, n, d_1, \ldots, d_s) \neq \emptyset$. We will use induction on $s$. If $s = 1$, set $B(s-1) = P^n$. If $s \geq 2$ take $a_{ij} \in C$, $1 \leq i \leq s-1$, $0 \leq j \leq n$, such that $B(s-l) := \{ F_1 = \ldots = F_{s-1} = 0 \}$ is a smooth codimension $s-1$ complete intersection of $P^n$ (inductive assumption). Let $V(d_s)$ be the linear system on $B(s-1)$ spanned by the restriction to $B(s-1)$ of the degree $d_s$ monomials $x_j^{d_s}$, $0 \leq j \leq n$. Since $V(d_s)$ has no base points and $B(s-1)$ is smooth, the general member of the linear system $V(d_s)$ is a smooth hypersurface of $B(s-1)$ by Bertini theorem ([H], Cor. III. 10.9, or [K]), i.e. for general $a_{ij} \in C^{n+1}$, $0 \leq j \leq n$, $B(s-1) \cap F_s$ is a smooth codimension $s$ complete intersection of $P^n$. Thus $A(s, n, d_1, \ldots, d_s) \neq \emptyset$.

Proof of Theorem 1.1. Fix an orthonormal basis $\{ x_n \}_{n \geq 1}$ of $V$. For any $z = \sum_{n \geq 1} z_n x_n \in V$ set $a_n(z) := z_n$. Thus $z = \sum_{n \geq 1} a_n(z) x_n$ for every $z \in V$ and $a_n \in V(1)$. Fix complex numbers $\mu_{i,j}$, $i \geq 1$, $1 \leq j \leq s$, which are algebraically independent over the field $Q$ of rational numbers and such that $0 < |\mu_{i,j}| \leq 1$ for all $i, j$. This is possible because $C$ has infinite (and even uncountable) trascendence degree over $Q$. For every $z = \sum_{n \geq 1} a_n(z) x_n \in V$ set $F_j(z) := \sum_{i \geq 1} \mu_{i,j} a_n(z)^j$. Hence $F_j$ is a continuous homogeneous degree $d_j$ polynomial on $V$. Set $A_j := \{ F_j = 0 \}$ and $X = A_1 \cap \ldots \cap A_s$. Obviously $X$ has codimension exactly $s$ in $P(V)$. It is sufficient to prove that $X$ is smooth. Fix $P \in X$ and take $z = \sum_{i \geq 1} a_i(z) x_i \in V \setminus \{ 0 \}$ representing $P$. Let $M$ be the matrix with $s$ rows and countable columns, say $M(P) = (b_{ij})$, $i \geq 1$, $1 \leq j \leq s$, with $b_{ij} = \partial^2 a_i(F_j)(P) = d_j \mu_{i,j} a_i(z)^j$. It is sufficient to prove that for every $P \in X$ the matrix $M(P)$ has rank $s$. First assume the existence of indices $i_1, \ldots, i_s$ such that $a(z)_{ik} \neq 0$ for every $k$ with $1 \leq k \leq s$. Call $M(P)(i_1, \ldots, i_s)$ the minor of $(M(P))$ formed by the columns $i_1, \ldots, i_s$. We have $\det(M(P)(i_1, \ldots, i_s)) = d_1 \ldots d_s a_i(z)^j \ldots a_i(z)^j \det(B)$ where $B$ is the $s \times s$ matrix $(\mu_{i_k,j})$, $1 \leq k \leq s$, $1 \leq j \leq s$. Since $a_i(z)^j \neq 0$ for every $k$ and $\det(B) \neq 0$ by the algebraic independence over $Q$ of the complex numbers $\mu_{i,j}$, we obtain $\det(M(P)) \neq 0$. Now assume that no such indices $i_1, \ldots, i_s$ do exists. Hence at most the first $s$ homogeneous coordinates of $P$ are non-zero. We apply Lemma 2.1 to the case $n = s$, in which we see $P^i$ as $P(W)$, where $W \subset V$ is the linear span of the vectors $x_1, \ldots, x_{s+1}$. By Lemma 2.1 the submatrix of $M(P)$ formed by the first $s+1 \times s$ columns has rank $s$ at $P$ and hence rank $(M(P)) = s$, proving the theorem.
3 - Proof of Theorem 1.2

Proof of Theorem 1.2. First we will prove the case \( S \) infinite (and countable). Then in a completely different way we will prove the case \( S = \emptyset \). Then we will adapt the proof of the case \( S = \emptyset \) to the case \( S \neq \emptyset \) and \( S \) finite.

Step 1. Here we assume \( S \) infinite and countable. Up to a projective transformation we may assume that \( (1; 0) \notin S \), \((-1; 1) \notin S \) and \((0; 1) \notin S \). Hence \( S = \{-a_i \in \mathbb{N} \} \) with \( a_i \in \mathbb{C} \), \( a_i \notin \{0, 1\} \). Choose homogeneous coordinates \( z_i \), \( i \geq 0 \), on \( P(C^{(n)}) \). Set \( F := \sum_{i \geq 0} z_i^d \) and \( G := \sum_{i \geq 0} a_i z_i^d \). Hence \( \{ F = 0 \} \) is a Fermat hypersurface and \( G \) is in diagonal form. Since every point of \( P(C^{(n)}) \) has only finitely many non-zero entries, it is very easy to check as in the finite-dimensional case that the hypersurface \( \{ \lambda F + \mu G \} \) has a singular point if and only if \( \lambda + \mu a_i = 0 \) for some \( i \), i.e. if and only if \( \{ (\lambda; \mu) \in S \} \).

Step 2. Here we assume \( S = \emptyset \). Let \( N(d) \) be the set of all multi-indices \( \alpha_i, \ i \geq 0 \), of non-negative integers with \( \sum \alpha_i = d \). Every homogeneous degree \( d \) hypersurface of \( P(C^{(n)}) \) has an equation of the form \( \sum_{\alpha \in N(d)} a_\alpha z^\alpha \) for some complex numbers \( a_\alpha \). Set \( F = \sum_{\alpha \in N(d)} a_\alpha z^\alpha \) and \( G = \sum_{\alpha \in N(d)} b_\alpha z^\alpha \), where we assume that all \( a_\alpha \) and \( b_\alpha \) are trascendentally free over the field \( Q \) of rational numbers. This may be done because \( N(d) \) is countable, while \( C \) has even uncountable transcendence degree over \( Q \). For all \( (\lambda; \mu) \in \mathbb{C}^1 \), set \( X(\lambda, \mu) := \{ \lambda F + \mu G \} \) and call \( L \) this pencil of hypersurfaces. We need to check that every \( X(\lambda, \mu) \) is smooth. For any integer \( n \geq 0 \), set \( CP^n := \{ z \in P(C^{(n)}) : z_i = 0 \text{ for } i > n \} \). \( X(\lambda, \mu ; n) := X(\lambda, \mu) \cap CP^n \) and \( L(n) \) the associated pencil of \( CP^n \). Since every point of \( P(C^{(n)}) \) has only finitely many non-zero coordinates, every singular point, \( P \), of \( X(\lambda, \mu) \) must be contained in some \( X(\lambda, \mu ; n) \) for some large \( n \). It is easy to check that \( P \) must be a singular point of \( X(\lambda, \mu ; n) \). However, the converse is not true. Take a hypersurface \( Y \) of \( CP^{n+1} \), a hyperplane \( H \) of \( CP^{n+1} \) and \( Q \in H \) such that \( Q \) is an ordinary double point of \( Y \cap H \). A priori two cases may occur: either \( Q \) is an ordinary double point of \( Y \) and \( H \) is as transversal as possible to \( Y \) at \( Q \) or \( Y \) is smooth at \( Q \) and \( H \) is tangent to \( Y \) at \( Q \). By the genericity of the coefficients \( a_\alpha \) and \( b_\alpha \) for every finite integer \( n \geq 2 \) the pencil \( L(n) \) has only finitely many singular members, each singular hypersurface of \( Y(n) \) has a unique singular point and this point is an ordinary double point. Now we compare the singular members of \( L(n) \) and of \( L(n+1) \). No singular member of \( L(n) \) is the restriction of a singular member of \( L(n+1) \), i.e. if \( X(\lambda, \mu ; n) \) is singular at \( Q \), then \( (\lambda, \mu ; n+1) \) is smooth at \( Q \) and \( CP^n \) is tangent to \( (\lambda, \mu ; n+1) \) at \( Q \). Hence letting \( n \) going to \( + \infty \) we obtain that no \( X(\lambda, \mu) \) may be singular.
Step 3. Here we assume $S$ finite and $S \neq \emptyset$. Set $s := \text{card}(S)$. We may assume, up to a projective transformation that $S$ is given by the complex numbers $-a_i$, $1 \leq j \leq s := \text{card}(S)$, with $a_j \notin \{0, 1\}$ for every $j$. For any positive integer $n$ set $S(n, n) := \{(a_i) \in S(d) : a_i = 0 \text{ for } i > n\}$. Set $F(s) := \sum_{0 \leq i \leq s} a_i z_i$ and $G(s) := \sum_{0 \leq i \leq s} a_i z_i$. Let $L(s)$ the pencil of hypersurfaces generated by $F(s)$ and $G(s)$. The singular members of $L(s)$ are exactly the hypersurfaces $\{a_j F(s) + G(s) = 0\}$ of $\mathbb{CP}^s$ with $Q_j(0; \ldots; 1; 0; \ldots; 0)$ as unique singular points. Let $V(s + 1)$ be the set of all extensions of $L(s)$ to a pencil of $\mathbb{CP}^{s+1}$ with singular members for the parameters $a_j$, $1 \leq j \leq s$, and respectively with $Q_j$ as singular point. $V(s + 1)$ is a finite-dimensional linear space. $V(s + 1) \neq \emptyset$ (e.g. just take $a_0 = b_0 = 0$ if $\alpha \in N(d, n + 1) \setminus N(d, n)$). Call $L(s + 1)$ any general member of $V(s + 1)$. The hypersurface of $L(s + 1)$ corresponding to the parameter $(a_j; 1)$ have $Q_j$ as only singular point. There will be also finitely many singular hypersurfaces in $L(s + 1)$ but all of them with an ordinary double point as unique singular point. Call $L(s + 2)$ a general extension of $L(s + 1)$ to a pencil of hypersurfaces of $\mathbb{CP}^{s+2}$ with a singular member for each parameter $(a_j; 1)$ at the point $Q_j$. The other singular members of the pencil $L(s + 1)$ will not be singular in $\mathbb{CP}^{s+1}$ except on the points $Q_j$, $1 \leq j \leq s$, i.e. passing from $L(s + 1)$ to $L(s + 2)$ we have swallowed the singularities of the pencil $L(s + 1)$ which were not assign in advance. And so on as in Step 2.

Remark 3.1. In the case $S$ infinite and countable we obtained a pencil in which all singular members have only one singular point and with a rather bad singularity (at least if $d \geq 3$). Just allowing repetitions among the complex numbers $a_i$, $i \geq 0$, we obtain in the same way examples in which $S$ is the set of all singular hypersurface, but each singular hypersurface is a cone over a smooth hypersurface and the vertex of the cone may have arbitrary dimension (finite or countable). If $S$ is finite and $S \neq \emptyset$ the construction of Step 3 of the proof of Theorem 1.2 gives hypersurfaces with a unique singular point and an ordinary one, because for every $m \geq 2s + 1$ the hypersurface of the pencil $L(m)$ corresponding to the parameter $(a_j; 1)$ have an ordinary double point at $Q_j$. However, we may even at each step of the induction to impose a bad singularity and find examples satisfying, the thesis of Theorem 1.2 but with prescribed multiplicity at the singular points.

Remark 3.2. Let $L$ be any pencil of degree $d$ hypersurfaces of $\mathbb{P}(\mathbb{C}^N)$ and call $L(n)$ its restriction to $\mathbb{CP}^s := \{z \in \mathbb{P}(\mathbb{C}^N) : z_i = 0 \text{ for } i > n\}$. Assume that for every $n \geq 2L(n)$ has no base points. Hence $L(n)$ has only finitely many singular members. Since every point of $\mathbb{P}(\mathbb{C}^N)$ has only finitely many non-zero coordina-
tes and the restriction to $CP^n$ of a hypersurface singular at $P \in CP^n$ is singular at $P$, we see that $L$ has at most countably many singular members.

Example 3.3. Let $I$ be any infinite set. Since every germ of holomorphic function on $C^I$ depends only from finitely many variables, every homogeneous polynomial on $C^I$ depends only from finitely many variables. Hence every zero-locus of a homogeneous polynomial of $C^I$ is a cone with infinite-dimensional vertex over a hypersurface of a finite-dimensional projective space. Hence every hypersurface of degree at least two of $C^I$ is singular. The space $C^N$ is a Fréchet nuclear space and hence it should be considered as a rather good locally convex space.

References

Summary

Here we prove the following two results. Fix positive integers $s, d_1, \ldots, d_s$ and a separable Hilbert space $V$; then there exists a smooth codimension $s$ complete intersection $X \subset P(V)$ of $s$ hypersurfaces of degree $d_1, \ldots, d_s$. Fix an integer $d \geq 2$ and a subset $S$ of $CP^1$ with at most countable elements; then there exist linearly independent homogeneous degree $d$ polynomials $F$ and $G$ on $C^{(N)}$ such that a hypersurface $\{\lambda F + \mu G = 0\}$ of $P(C^{(N)})$ with $(\lambda; \mu) \in CP^1$ is singular if and only if $(\lambda; \mu) \in S$; we allow the case $S = \emptyset$, which is in striking contrast with the corresponding problem in $CP^n$.

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