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Bitopological α-compact spaces (**)

1 - Introduction

In 1965, O.Njåstad [**9**] introduced the notion of α-sets. Since then, a large number of topologists studied various properties of point set topology with the help of α-sets. In 1985, utilizing α-sets, Maheswari et al. [8] defined the notion of α-compactness in spaces with single topology. In 1988, Noiri et al. [**11**] obtained further properties of this kind of spaces. The purpose of the present paper is to generalize the concept of α-compactness in bitopological setting and examine how far the properties of α-compact space remain valid in this new setting.

The notion «pairwise compactness» is current in the existing literature. «Pairwise open cover» defined by Fletcher et al. [**3**] is instrumental for the introduction of this concept. In like manner, defining «pairwise α-cover», we have introduced pairwise α-compact (briefly *pac*) spaces. In Section 2 of this paper some known definitions and results necessary for presentation of the subject in bitopological setting are reproduced. Section 3 gives the definition and examples of *pac* space, which is a stronger notion - substantiated by an example - than pairwise compact spaces. Fletcher et al. [**3**], the leading exponents of pairwise compact spaces, did not examine the cases whether bi-compact [**13**] spaces can generate a pairwise compact space or two non-compact spaces with single topologies can produce a pairwise compact space. In this section queries parallel to these have been answered for a *pac* space. Section 4 deals with some bitopological separation axioms which are interesting in their own right, but are necessary in this paper to unveil

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the richness of *pac* space and demonstrate its properties. Section 5 is concerned with the interrelationship among the new separation axioms on one hand, while the queries that crop up as a natural consequence of our study have been met, with appropriate examples, on the other hand. Last section is concerned with the properties of *pac* space developed in the light of the axioms introduced in Section 4.

Throughout the paper, the triple (X, τ_1, τ_2), where X is a set and τ_1, τ_2 are topologies on X, will always denote a bitopological space [**6**], while (X, τ) or simply X denotes a single topological space. In (X, τ), the family of all τ-closed sets are denoted by $\mathcal{F}(\tau)$. The τ_i-closure (resp. τ_i-interior) of a set A is denoted by τ_i-Cl(A) or $\mathrm{Cl}_{\tau_i}(A)$ (resp. τ_i-Int(A) or $\mathrm{Int}_{\tau_i}(A)$). The abbreviation b.t.s. for bitopological space is used in this paper. Notations not explained here, but used in this paper are obtained from Dugundji [**2**] and Pervin [**12**].

2 - Known definitions and results

We shall require the following known definitions and results.

Definition 2.1 [**9**]. *In* (X, τ), $A \subset X$ *is called an* α*-set iff* $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$.

Njåstad [**9**] used the symbol τ^α to denote the family of all α-set in X and showed that τ^α is a topology on X.

Definition 2.2 [**10**]. *The complement of an* α*-set is called* α*-closed. The family of all* α*-closed sets in* X *is denoted by* $\mathcal{F}(\tau^\alpha)$.

Definition 2.3 [**13**]. *In* (X, τ_1, τ_2), $A \subset X$ *is termed bi-compact iff* A *is both* τ_1*-compact and* τ_2*-compact.*

Definition 2.4 [**3**]. *A cover* $\mathcal{U}$ *of* (X, τ_1, τ_2) *is called pairwise open if* $\mathcal{U} \subset \tau_1 \cup \tau_2$, $\mathcal{U} \cap \tau_i \supset \{A \neq \phi\}$, $i = 1, 2$. *If every pairwise open cover of* (X, τ_1, τ_2) *has a finite subcover, then the space is called pairwise compact.*

Definition 2.5 [**6**]. (X, τ_1, τ_2) *is pairwise Hausdorff iff for each pair of distinct points* x *and* y *of* X *there are a* τ_1*-open set* U *and a* τ_2*-open set* V *such that* $x \in U$, $y \in V$ *and* $U \cap V = \phi$.

This definition was first given by Weston [**15**] who used the term consistent.

Definition 2.6 [6]. *In* (X, τ_1, τ_2), τ_i *is said to be regular with respect to* τ_j *iff for each point* x *in* X *and each* $F \in \mathcal{F}(\tau_i)$ *such that* $x \notin F$, *there exist* $U \in \tau_i$ *and* $V \in \tau_j$ *with* $x \in U$, $F \subset V$ *and* $U \cap V = \phi$, $i, j = 1, 2;\ i \neq j$.
(X, τ_1, τ_2) *is called pairwise regular iff* τ_1 *is regular with respect to* τ_2 *and vice-versa.*

Definition 2.7 [6]. (X, τ_1, τ_2) *is termed pairwise normal iff for each* $F_1 \in \mathcal{F}(\tau_1)$ *and* $F_2 \in \mathcal{F}(\tau_2)$ *with* $F_1 \cap F_2 = \phi$, *there exist* $U \in \tau_2$ *and* $V \in \tau_1$ *such that* $F_1 \subset U$, $F_2 \subset V$ *and* $U \cap V = \phi$.

Definition 2.8 [4]. (X, τ_1, τ_2) *is said to bi-Hausdorff if both* (X, τ_1) *and* (X, τ_2) *are Hausdorff.*

Definition 2.9 [8]. *X is called* α*-compact iff every cover of* X *by* α*-set has a finite subcover.*

Theorem 2.1.. (*Theorem 12* [3]). *If* (X, τ_1, τ_2) *is pairwise Hausdorff and pairwise compact, then it is pairwise regular.*

Theorem 2.2. (*Theorem 13* [3]). *If* (X, τ_1, τ_2) *is pairwise compact and either* τ_1 *is regular with respect to* τ_2 *or* τ_2 *is regular with respect to* τ_1, *then it is pairwise normal.*

Theorem 2.3. (*Theorem 10* [3]). *If* (X, τ_1, τ_2) *is pairwise Hausdorff and bi-compact, then* $\tau_1 = \tau_2$.

Theorem 2.4. (*Theorem 11* [3]). *If* (X, τ_1, τ_2) *is bi-Hausdorff and pairwise compact, then* $\tau_1 = \tau_2$.

3 - Pairwise α-compact spaces

Following Flectcher et al. [3], we introduce the definitions given below.

Definition 3.1. *A cover* $\mathcal{U}$ *of* (X, τ_1, τ_2) *is termed pairwise* α*-cover if* $\mathcal{U} \subset \tau_1^\alpha \cup \tau_2^\alpha$ *and* $\mathcal{U} \cap \tau_i^\alpha \supset \{A \neq \phi\}$, $i = 1, 2$.

Definition 3.2. *A b.t.s* (X, τ_1, τ_2) *is called pairwise* α*-compact (briefly pac) if every pairwise* α*-cover of* (X, τ_1, τ_2) *has a finite subcover.*

A reformulation of this definition is:

$$(X, \tau_1, \tau_2) \text{ is pac iff } (X, \tau_1^\alpha, \tau_2^\alpha) \text{ is pairwise compact.}$$

Remark 3.1. Since $\tau \subset \tau^{\alpha}$ for every topology τ, it follows that every *pac* space is pairwise compact. But the converse is not, in general, true. This is seen from the following example.

Example 3.1. Let R be the real line and $A = (0, \infty) \subset R$. Consider the b.t.s. (R, τ_1, τ_2), where $\tau_1 = \{R\} \cup \{G : G \subset R$ and $G \cap A = \phi\}$, $\tau_2 = \{\phi, R, \{1\}, R \backslash \{1\}\}$. Let $\mathcal{U} = \{V_\beta : \beta \in \Delta\}$ be any pairwise open cover for (R, τ_1, τ_2). Since $\mathcal{U}$ is a cover for R, it is so for A. τ_1 is the set exclusion topology with the excluding set A. So, two cases arise.

Either $R \in \mathcal{U}$ or $R \backslash \{1\}$ and $\{1\} \in \mathcal{U}$. Since in each case we obtain a finite subcover of $\mathcal{U}$, (X, τ_1, τ_2) is pairwise compact. If we consider the family of sets $\mathcal{V}$, defined by $\mathcal{V} = \{V_n : n \in N$ ($=$ the set of positive integers)$\}$, where $V_n = (-\infty, n)$, then it is easy to check that $V_n \in \tau_i^{\alpha}$. Also, $\{1\} \in \tau_2$ and hence $\{1\} \in \tau_2^{\alpha}$. Let $\mathcal{U} = \mathcal{V} \cup \{\{1\}\}$. Then $\mathcal{U} \subset \tau_1^{\alpha} \cup \tau_2^{\alpha}$. Also, $\mathcal{U} \cap \tau_i^{\alpha} \supset \{A \neq \phi\}$, $i = 1, 2$. Hence $\mathcal{U}$ is a pairwise α-cover of (R, τ_1, τ_2). But no finite subfamily of $\mathcal{U}$ can cover R. Hence $\mathcal{U}$ has no finite subcover. Consequently, (R, τ_1, τ_2) is not pairwise α-compact.

The immediate problem that occurs to our mind is: Is α-compactness for the individual topologies equivalent to *pac* of the b.t.s. (X, τ_1, τ_2)? Or, can two non α-compact spaces with single topology generate a *pac* b.t.s.? Examples 3.2 and 3.3 offer answer to these two queries.

Example 3.2. Let R be the real line with $\tau_1 = \{R\} \cup \{G \subset R : 1 \notin G\}$ and $\tau_2 = \{R\} \cup \{G \subset R : 2 \notin G\}$. We assert that only α-set containing 1 in (R, τ_1) is R. For, if possible, let G be an α-set in (R, τ_1) such that $1 \in G$ but $G \neq R$. Then $\mathrm{Int}_{\tau_1}(G) \neq \phi$ and coincides with $G \backslash \{1\}$, so that $\mathrm{Cl}_{\tau_1}(\mathrm{Int}_{\tau_1}(G)) = G$ and $\mathrm{Int}_{\tau_1}(\mathrm{Cl}_{\tau_1}(\mathrm{Int}_{\tau_1}(G))) = G \backslash \{1\}$, that is, $G \notin \tau_1^{\alpha}$. Thus the only α-set containing 1 is the set R. Hence, any α-cover $\mathcal{C}$ of (R, τ_1) surely contains R. So, $\{R\}$ is a finite subcover of $\mathcal{C}$ so that (R, τ_1) is α-compact. Pursuing similar reasoning, we see that (R, τ_2) is α-compact. But the b.t.s. (R, τ_1, τ_2) is not *pac*. For, if we consider the family

$$\mathcal{U} = \{\{x\} : x \in R \backslash \{1\}\} \cup \{\{1\}\}$$

then $\mathcal{U} \subset \tau_1^{\alpha} \cup \tau_2^{\alpha}$. Clearly, $\mathcal{U} \cap \tau_i^{\alpha} \supset \{A \neq \phi\}$, $i = 1, 2$. Hence $\mathcal{U}$ is a pairwise α-cover for (R, τ_1, τ_2). But it has no finite subcover. So, (R, τ_1, τ_2) is not *pac*.

Example 3.3. Let R be the real line with $\tau_1 = \{\phi\} \cup \{G \subset R : G \supset (-\infty, 0]\}$ and $\tau_2 = \{\phi\} \cup \{G \subset R : G \supset [0, \infty)\}$. Let $\mathcal{U}_1 = \{U_n : n \in N$($=$the set of positive integers)}, $\mathcal{U}_2 = \{V_n\colon n \in N\}$, where $U_n = (-\infty, n)$ and $V_n = (-n, \infty)$, $\forall n \in N$. For each $n \in N$, $U_n \supset (-\infty, 0]$ and $V_n \supset [0, \infty)$. So, $U_n \in \tau_1$, $V_n \in \tau_2$ for each $n \in N$. Thus $\mathcal{U}_1$ is an open cover for (R, τ_1). Since each open set is an α-set, $\mathcal{U}_1$ is a α-cover of (R, τ_1). But no finite subfamily of $\mathcal{U}_1$ covers R, so that (R, τ_1) is not α-compact. In like manner (R, τ_2) is not α-compact. But the b.t.s. (R, τ_1, τ_2) is *pac*. Let $\mathcal{U}$ be a pairwise α-cover for (R, τ_1, τ_2). Then $\mathcal{U} \subset \tau_1^\alpha \cup \tau_2^\alpha$ and $\mathcal{U} \cap \tau_i^\alpha \supset \{A \neq \phi\}$, for $i = 1, 2$. Take $\phi \neq U_1 \in \mathcal{U} \cap \tau_1^\alpha$, $\phi \neq U_2 \in \mathcal{U} \cap \tau_2^\alpha$ and set $\mathcal{W} = \{U_1, U_2\}$. It is easy to check that the finite family $\mathcal{W}$ covers R. Now $U_1 \in \tau_1^\alpha \Rightarrow \mathrm{Int}_{\tau_1}(U_1) \neq \phi$ and this, by definition of τ_1, implies $\mathrm{Int}_{\tau_1}(U_1) \supset (-\infty, 0]$, because $\mathrm{Int}_{\tau_1}(U_1) \in \tau_1$. Thus $U_1 \supset \mathrm{Int}_{\tau_1}(U_1) \supset (-\infty, 0]$. Similar argument yields $U_2 \supset [0, \infty)$. So $U_1 \cup U_2 \supset (-\infty, 0] \cup [0, \infty) = R$ whence $U_1 \cup U_2 = R$. Consequently, $\mathcal{W}$ covers R and it is a finite subcover of $\mathcal{U}$. So the b.t.s. (R, τ_1, τ_2) is *pac*.

4 - Some new bitopological separation axioms

We now introduce three bitopological separation axioms, namely pairwise α-Hausdorffness, pairwise α-regularity and pairwise α-normality.

Pairwise α-Hausdorffness

In 1980, Maheswari and Thakur [7] introduced the notion of α-Hausdorff space by replacing in the definition of Hausdorff space the words «open sets» by «α-sets». Noiri [10] has given a lucid and simple proof of the fact that (X, τ) is Hausdorff iff (X, τ^α) is so. Thus the concept of α-Hausdorffness coincides with the usual notion of Hausdorffness as pointed out by Janković and Reilly [5]. As a result of this fact, in the sequel, we shall freely interchange the terms «Hausdorff space» and «α-Hausdorff space».

Definition 4.1. *A b.t.s.* (X, τ_1, τ_2) *is called pairwise α-Hausdorff if the b.t.s.* $(X, \tau_1^\alpha, \tau_2^\alpha)$ *is pairwise Hausdorff, that is if, for* $x, y \in X$, $x \neq y$, *there exist a* τ_i^α*-set* U *and* τ_j^α*-set* V *such that* $x \in U$, $y \in V$ *and* $U \cap V = \phi$, $i, j = 1, 2$; $i \neq j$.

Remark 4.1. Since $\tau_i \subset \tau_i^\alpha$ $(i = 1, 2)$, for a b.t.s. pairwise Hausdorffness implies pairwise α-Hausdorffness. On the other hand, the reverse implication does not, in general, hold for a b.t.s.. This remarkable departure from a single topological space has been demonstrated by the example below.

Example 4.1. Let (R, τ_1, τ_2) be a b.t.s. where R = the real line, τ_1 = the usual topology and $\tau_2 = \{\phi, R\} \cup \{(a, \infty)\colon a \in R\}$, called the right hand topology by Pervin [12]. For the points 2 and 1 of R, let $2 \in U \in \tau_1$ and $1 \in V \in \tau_2$. Then $(2-r, 2+r) \subset U$ and $(t, \infty) \subset V$, for some $r > 0$ and some $t < 1$ respectively. Consequently, $(t, \infty) \cap (2-r, 2+r) \neq \phi$, as $t < 1 < 2$. This implies that $U \cap V \neq \phi$ and so (R, τ_1, τ_2) is not pairwise Hausdorff. Again, for any two distinct points $x < y$ in R, $U_1 = (x-r, x+r) \in \tau_1^\alpha$, $V_1 = (x+r, \infty) \in \tau_2^\alpha$, where $r = \dfrac{y-x}{2}$, are two disjoint τ_1^α-set and τ_2^α-set containing x and y respectively. Recall that the interior of a set A is the largest open set contained in A. Then, for the same r as taken above, $U_2 = (y-r, y+r) \in \tau_1^\alpha$ and $V_2 = \{x\} \cup (y+r, \infty) \in \tau_2^\alpha$, where U_2, V_2 are disjoint but contain y and x respectively. Thus (R, τ_1, τ_2) is pairwise α-Hausdorff.

Before the introduction of bitopological separation axiom of pairwise α-regularity, we define α-regularity in a single topological space as follows:

Definition 4.2. *A space (X, τ) is said to be α-regular iff the space (X, τ^α) is regular, that is, for every $F \in \mathcal{F}(\tau^\alpha)$ and every $x \notin F$, there exist $U, V \in \tau^\alpha$ such that $x \in U$, $F \subset V$ and $U \cap V = \phi$.*

Theorem 4.1. *Every α-regular space is regular.*

Proof. Let (X, τ) be an α-regular space, F any τ-closed set and $x \notin F$. Since every closed set is α-closed, $F \in \mathcal{F}(\tau^\alpha)$. The α-regularity of X gives the existence of $U, V \in \tau^\alpha$ such that $x \in U$, $F \subset V$ and $U \cap V = \phi$.

Now

$$\begin{aligned} U \cap V = \phi &\Rightarrow \operatorname{Int}(U) \cap \operatorname{Int}(V) = \phi \\ &\Rightarrow \operatorname{Cl}(\operatorname{Int}(U)) \cap \operatorname{Int}(V) = \phi \\ &\Rightarrow \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(U))) \cap \operatorname{Int}(V) = \phi \\ &\Rightarrow \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(U))) \cap \operatorname{Cl}(\operatorname{Int}(V)) = \phi \\ &\Rightarrow \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(U))) \cap \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(V))) = \phi \,. \end{aligned}$$

Since U and V are α-sets, $U \subset \operatorname{Int}(Cl(\operatorname{Int}(U)))$, $V \subset \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(V)))$. Hence we obtain the disjoint sets $\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(U)))$, $\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(V))) \in \tau$, which contain x and F respectively. Thus (X, τ) is a regular space.

Remark 4.2. The following example shows that the converse of the above theorem is not true.

Example 4.2. Let $(R, \mathcal{U})$ be a topological space where R = the real line, $\mathcal{U}$ = the usual topology. Let F be any $\mathcal{U}$-closed set and $x \notin F$. Obviously $x \in R \backslash F \in \mathcal{U}$.

Consequently, the open interval $I_r(x) = (x - r, x + r)$, for some $r > 0$, is contained in $R\backslash F$, which, in its turn, implies that $F \subset R\backslash I_r(x)$. If we take $U = I_{\frac{r}{2}}(x)$ and $V = R\backslash \mathrm{Cl}(I_{\frac{r}{2}}(x))$, then $U, V \in \mathcal{U}$, with $x \in U$. Also, we observe that

$$\mathrm{Cl}(I_{\frac{r}{2}}(x)) = \left[x - \frac{r}{2}, x + \frac{r}{2}\right] \subset (x - r, x + r)$$
$$\Rightarrow R\backslash \mathrm{Cl}(I_{\frac{r}{2}}(x)) \supset R\backslash I_r(x) \supset F \Rightarrow V \supset F .$$

Moreover, $V \cap \mathrm{Cl}(I_{\frac{r}{2}}(x)) = \phi \Rightarrow V \cap I_{\frac{r}{2}}(x) = \phi \Rightarrow U \cap V = \phi$.

Hence $(R, \mathcal{U})$ is a regular space.

To show that $(R, \mathcal{U})$ is not α-regular, take $F = \left\{ \frac{1}{n} : n \in N(= \text{the set of positive integers}) \right\}$. Then $\mathrm{Cl}(F) = F \cup \{0\} \Rightarrow \mathrm{Int}(\mathrm{Cl}(F)) = \mathrm{Int}(F \cup \{0\}) = \phi$ $\Rightarrow \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F))) = \phi \subset F \Rightarrow F \in \mathcal{F}(\mathcal{U}^\alpha)$. Also, $0 \notin F$. We take any two sets $U, V \in \mathcal{U}^\alpha$ such that $0 \in U$ and $F \subset V$. We assert that $U \cap V \neq \phi$. Suppose, if possible, $U \cap V = \phi$. Then, in like manner as in above theorem, $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(V))) = \phi$. Now $0 \in U \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))$. This implies that $0 \in U \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))) \in \mathcal{U}$. Since $0 \in D(F)$ (= the derived set of F), $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))) \cap F \neq \phi \Rightarrow \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(V))) \neq \phi$, a contradiction. Therefore $U \cap V \neq \phi$. Thus $(R, \mathcal{U})$ is not α-regular, though it is regular.

Pairwise α-regularity

Following the technique of J. C. Kelly [**6**], we introduce the definitions given below.

Definition 4.3. *Let (X, τ_1, τ_2) be a b.t.s.. Then τ_i is said to be α-regular with respect to τ_j if the b.t.s. $(X, \tau_1^\alpha, \tau_2^\alpha)$ is such that τ_i^α is regular with respect to τ_j^α, that is, for each point x in X and each $F \in \mathcal{F}(\tau_i^\alpha)$ with $x \notin F$, there exist $U \in \tau_i^\alpha$, $V \in \tau_j^\alpha$ such that $x \in U$, $F \subset V$ and $U \cap V = \phi$, $i, j = 1, 2$; $i \neq j$.*

Definition 4.4. *A b.t.s. (X, τ_1, τ_2) is said to be pairwise α-regular if $(X, \tau_1^\alpha, \tau_2^\alpha)$ is pairwise regular.*

In view of Theorem 4.1 and Example 4.2 we observe that in a space with single topology, α-regularity implies regularity, though the reverse implication may not hold. Thus, in a single topological space, some sort of interconnection exists between regularity and α-regularity. The natural query is: Is there any relation between pairwise regularity and α-regularity in bitopological spaces? The answer is in the negative. Examples 4.3 and 4.4 reveal that the notions of pairwise regularity and pairwise α-regularity, in bitopological setting, are independent.

Example 4.3. Let (R, τ_1, τ_2) be a b.t.s. where R = real line, τ_1 = the point exclusion topology with the excluding point 0 and $\tau_2 = \{\phi, R, \{0\}\}$. If $A = \{1, 2\}$, then $F = R \backslash A$ is τ_1-closed and $1 \notin F$. But R is the only τ_2-open set with $F \subset R$ and R intersects every τ_1-open set containing 1. So, by Definition 2.6, τ_1 is not regular with respect to τ_2. Therefore (R, τ_1, τ_2) is not pairwise regular according to Definition 2.6.

On the other hand, take any $F \in \mathcal{F}(\tau_1^\alpha)$, $F \neq \phi, R$. Then, by definition of τ_1, $0 \in F$ and, if $x \in R$ is such that $x \notin F$, then $x \neq 0$. It is not difficult to check that $U = \{x\} \in \tau_1^\alpha$ and $V = F \in \tau_2^\alpha$. From the disjointness of U and V we deduce that τ_1 is α-regular with respect to τ_2. The definition of τ_2 shows that, if $F \in \mathcal{F}(\tau_2^\alpha)$, $F \neq \phi, R$, then $0 \notin F$ and so the definition of τ_1 guarantees that $F \in \tau_1 \subset \tau_1^\alpha$. If $x \notin F$, then x and F can be separated by $U = \{0\} \cup \{x\} \in \tau_2^\alpha$ and $V = F \in \tau_1^\alpha$, as the disjointness of U and V is obvious from their construction. Hence τ_2 is α-regular with respect to τ_1. Therefore, (R, τ_1, τ_2) is pairwise α-regular.

Example 4.4. In the b.t.s. of the previous example take τ_1 = the topology τ_2 of that example and $\tau_2 = \{\phi, R, R \backslash \{0\}\}$. Clearly, (R, τ_1, τ_2) is pairwise regular. Take $A = \{0, 1\}$. Then $A \in \tau_1^\alpha$, so that $F = R \backslash A \in \mathcal{F}(\tau_1^\alpha)$ and $1 \notin F$. Now if $V \in \tau_1^\alpha$ such that $F \subset V$, then $V = R$ or $R \backslash \{0\}$. In both cases, V intersects any τ_1-set U containing 1. Consequently, τ_1 is not α-regular with respect to τ_2. So (R, τ_1, τ_2) is not pairwise α-regular.

Pairwise α-normality

Definition 4.5. *A b.t.s.* (X, τ_1, τ_2) *is termed pairwise α-normal if the b.t.s.* $(X, \tau_1^\alpha, \tau_2^\alpha)$ *is pairwise normal, that is if, for each* $F_1 \in \mathcal{F}(\tau_1^\alpha)$ *and* $F_2 \in \mathcal{F}(\tau_2^\alpha)$, *disjoint from* F_1, *there exist* $U \in \tau_2^\alpha$ *and* $V \in \tau_1^\alpha$, *such that* $F_1 \subset U$, $F_2 \subset V$ *and* $U \cap V = \phi$.

For a space with single topology Mršević and Reilly [**9**] posed the question: Does normality of (X, τ^α) imply $\tau^\alpha = \tau$? J. Dontchev [**1**] nicely answered this question in affirmative showing that, if (X, τ^α) is normal, then $\tau^\alpha = \tau$. Thus «α-normality» always implies «normality» in a single topological space. But it is highly surprising that the concepts pairwise normality and pairwise α-normality are independent of each other. This assertion has been substantiated by Examples 4.5 and 4.6 below.

Example 4.5. In the b.t.s. of Example 4.1, take τ_1 = the right hand topology τ_2 of that example and $\tau_2 = \sup\{\sigma_1, \sigma_2\}$, (called the countable complement extension topology by Steen and Seebach [14]), where σ_1 = the usual topology and σ_2 = the cocountable topology. We first note that any set G is τ_2-open iff $G = U\backslash A$ where $U \in \sigma_1$ and A is countable. Take $F_1 = (-\infty, 0]$ and $F_2 = \left\{\frac{1}{n} : n \in N \text{ (= the set of positive integers)}\right\}$. Then $F_1 \in \mathcal{F}(\tau_1)$, $F_2 \in \mathcal{F}(\sigma_2) \subset \mathcal{F}(\tau_2)$ and $F_1 \cap F_2 = \phi$. Suppose $U \in \tau_2$, $V \in \tau_1$ such that $F_1 \subset U$ and $F_2 \subset V$. We claim that $U \cap V \neq \phi$.

Since $\frac{1}{n} \in F_2 \subset V$, $\forall\, n \in N$, and $V \in \tau_1$, it follows that $(0, \infty) \subset V$. The fact that $U \in \tau_2$ assures that U must be of the form $U = G\backslash A$, where $G \in \sigma_1$ and A is countable. Since $(-\infty, 0] = F_1 \subset U = G\backslash A$, it follows that $0 \in G\backslash A \subset G$. This gives the existence of some $r > 0$ such that $0 \in (-r, r) \subset G \Rightarrow 0 \in (-r, r)\backslash A \subset G\backslash A = U$. Since A is countable, $\phi \neq (0, r)\backslash A \subset (-r, r)\backslash A \subset U$. But $[(0, r)\backslash A] \cap (0, \infty) \neq \phi$, which indicates that $[(0, r)\backslash A] \cap V \neq \phi$ and this gives $U \cap V \neq \phi$. Hence (R, τ_1, τ_2) is not pairwise normal. Suppose $F_1 \in \mathcal{F}(\tau_1^\alpha)$, $F_1 \neq \phi$, R and $F_2 \in \mathcal{F}(\tau_2^\alpha)$, with $F_1 \cap F_2 = \phi$. Then $\mathrm{Int}_{\tau_1}(R\backslash F_1) = (a, \infty)$ for some $a \in R$. If $b \in R$ such that $b > a$ then

$$(-\infty, b) \supset (-\infty, a] = R\backslash \mathrm{Int}_{\tau_1}(R\backslash F_1) = \mathrm{Cl}_{\tau_1}(F_1).$$

Hence $F_1 \subset \mathrm{Cl}_{\tau_1}(F_1) \subset (-\infty, b)$. Also $F_1 \subset R\backslash F_2$ and so,

$$F_1 \subset (-\infty, b) \cap (R\backslash F_2) = U, \text{ say}.$$

Since $(-\infty, b) \in \sigma_1 \subset \tau_2 \subset \tau_2^\alpha$ and τ_2^α is a topology, we observe that $U \in \tau_2^\alpha$. If we write $V = F_2 \cup [b, \infty)$, then $(b, \infty) \subset \mathrm{Int}_{\tau_1}(V) \Rightarrow R \subset \mathrm{Cl}_{\tau_1}(\mathrm{Int}_{\tau_1}(V)) \Rightarrow R = \mathrm{Int}_{\tau_1}(\mathrm{Cl}_{\tau_1}(\mathrm{Int}_{\tau_1}(V)))$. So $V \in \tau_1^\alpha$. Also $F_2 \subset V$. Now $U \cap V = [(-\infty, b) \cap (R\backslash F_2)] \cap [F_2 \cup [b, \infty)] = [(-\infty, b) \cap (R\backslash F_2) \cap F_2] \cup [(-\infty, b) \cap (R\backslash F_2) \cap [b, \infty)] = \phi$. Hence (X, τ_1, τ_2) is pairwise α-normal.

Example 4.6. Let (R, τ_1, τ_2) be a b.t.s. where R = the real line, $\tau_1 = \{\phi, R, \{0\}\}$ and $\tau_2 = \{\phi, R, \{0, 1\}\}$. From the definitions of τ_1 and τ_2, it is clear that every non-empty τ_1-closed set intersects every non-empty τ_2-closed set and hence the b.t.s. (R, τ_1, τ_2) is trivially pairwise normal. Now take $F_1 = \{2, 3\}$, $F_2 = \{4, 5\}$. We observe:

(1) $\mathrm{Cl}_{\tau_1}(F_1) = R\backslash\{0\}$
$\Rightarrow \mathrm{Cl}_{\tau_1}(\mathrm{Int}_{\tau_1}(\mathrm{Cl}_{\tau_1}(F_1))) = \mathrm{Cl}_{\tau_1}(\mathrm{Int}_{\tau_1}(R\backslash\{0\})) = \phi \Rightarrow F_1 \in \mathcal{F}(\tau_1^\alpha)$.

(2) In like manner, $F_2 \in \mathcal{F}(\tau_2^\alpha)$ and (3) $F_1 \cap F_2 = \phi$.

Suppose $F_1 \subset U \in \tau_2^\alpha$ and $F_2 \subset V \in \tau_1^\alpha$. Since $U \in \tau_2^\alpha$, $\mathrm{Int}_{\tau_2}(U) \neq \phi$. This implies, by definition of τ_2, that $\{0, 1\} \subset \mathrm{Int}_{\tau_2}(U) \subset U$. Similar reasoning produces that $\{0\} \subset \mathrm{Int}_{\tau_1}(V) \subset V$. This gives that $U \cap V \neq \phi$. Hence (R, τ_1, τ_2) is not pairwise α-normal.

5 - Interrelationships

So far we have devoted ourselves to the introduction of three bitopological separation axioms, viz pairwise α-Hausdorffness, pairwise α-regularity and pairwise α-normality. We have also investigated the interconnection between these notions and the corresponding notions via open sets, which already exist in literature. The outcome of our investigation can be summarized as follows:

(a) Pairwise Hausdorffness $\Rightarrow$ Pairwise α-Hausdorffness but the reverse implication does not hold.

(b) Pairwise regularity and Pairwise α-regularity are independent of each other.

(c) Pairwise normality and Pairwise α-normality are independent of each other.

Now we like to pay our attention to study the interrelationships between these new bitopological separation axioms. Examples 5.1 and 5.2 have been constructed below to reveal some relations existing among these axioms.

Example 5.1. There is a pairwise α-Hausdorff b.t.s. which is not pairwise α-regular. Let (R, τ_1, τ_2) be a b.t.s. where $R =$ the real line, $\tau_1 =$ the usual topology and $\tau_2 =$ the lower limit topology. We first note that $\tau_1 \subset \tau_2$. Since τ_1 is Hausdorff and $\tau_1 \subset \tau_2$, τ_2 is Hausdorff. Hence the b.t.s. (R, τ_1, τ_2) is pairwise Hausdorff and so, by (a), it is pairwise α-Hausdorff.

That (R, τ_1, τ_2) is not pairwise α-regular will be clear from the following argument. Consider the set $F = \left\{ \frac{1}{n} : n \in N \text{ (= the set of positive integers)} \right\}$ of Example 4.2. Then $F \in \mathcal{F}(\tau_1^\alpha)$ as shown in that example. Also $0 \notin F$. Take any two sets U, V such that $0 \in U \in \tau_1^\alpha$ and $F \subset V \in \tau_2^\alpha$. We assert that $U \cap V \neq \phi$. This can be checked as follows.

The α-ness of U with respect to τ_1 indicates that 0 is an interior point of the τ_1-open set $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))$. Hence $r_1 (> 0)$ can be found such that

$$0 \in (-r_1, r_1) \subset \mathrm{Int}_{\tau_1}(\mathrm{Cl}_{\tau_1}(\mathrm{Int}_{\tau_1}(U))) \subset \mathrm{Cl}_{\tau_1}(\mathrm{Int}_{\tau_1}(U)).$$

Also we can find a positive integer n_0 such that the element $\frac{1}{n_0} \in F$ lies in

$(-r_1, r_1)$. But $\frac{1}{n_0}$ is an interior point of the τ_2-open set $\text{Int}_{\tau_2}(\text{Cl}_{\tau_2}(\text{Int}_{\tau_2}(V)))$ and τ_2 is the lower limit topology. So there exists $r_2(>0)$ such that

$$\left[\frac{1}{n_0}, \frac{1}{n_0}+r_2\right] \subset \text{Int}_{\tau_2}(\text{Cl}_{\tau_2}(\text{Int}_{\tau_2}(V))) \subset \text{Cl}_{\tau_2}(\text{Int}_{\tau_2}(V)).$$

If $r=\min\left\{r_1-\frac{1}{n_0}, r_2\right\}$, then $\left[\frac{1}{n_0}, \frac{1}{n_0}+r\right) \subset \left[\frac{1}{n_0}, \frac{1}{n_0}+r_2\right) \subset \text{Cl}_{\tau_2}(\text{Int}_{\tau_2}(V))$ $\Rightarrow \left[\frac{1}{n_0}, \frac{1}{n_0}+r\right) \cap \text{Int}_{\tau_2}(V) \neq \phi$. Suppose $x \in \left[\frac{1}{n_0}, \frac{1}{n_0}+r\right) \cap \text{Int}_{\tau_2}(V)$. Since $\text{Int}_{\tau_2}(V)$ is a τ_2-open set containing x, there exists $r_3>0$ such that $[x, x+r_3)$ $\subset \text{Int}_{\tau_2}(V)$. This implies that the set G defined by $G=\left[\frac{1}{n_0}, \frac{1}{n_0}+r\right)$ is such that $G \cap [x, x+r_3) \neq \phi = [x, k)$, where $k=\min\left\{\frac{1}{n_0}+r, x+r_3\right\}$. Obviously, the definitions of n_0, r and k give $(x, k) \subset [x, x+r_3) \subset \text{Int}_{\tau_2}(V)$ and $\left[\frac{1}{n_0}, \frac{1}{n_0}+r\right)$ $\subset (-r_1, r_1) \subset \text{Cl}_{\tau_1}(\text{Int}_{\tau_1}(U))$, whence we obtain $(x, k) \subset G \subset \left[\frac{1}{n_0}, \frac{1}{n_0}+r\right)$ $\subset \text{Cl}_{\tau_1}(\text{Int}_{\tau_1}(U)) \Rightarrow (x, k) \cap \text{Int}_{\tau_1}(U) \neq \phi \Rightarrow \text{Int}_{\tau_2}(V) \cap \text{Int}_{\tau_1}(U) \neq \phi \Rightarrow U \cap V \neq \phi$. Hence τ_1 is not α-regular with repect to τ_2. So (R, τ_1, τ_2) is not pairwise α-regular.

Example 5.2. There is a pairwise α-regular b.t.s. which is not pairwise α-normal. In this case, the ground set of the space we consider is taken from Example 94 in [**14**]. Let $X=\bigcup_{n=0}^{\infty} L_n$ be the union of the horizontal lines in the Euclidean plane where

(i) $L_0=\{(x, 0): 0<x<1\}$

and

(ii) $L_n=\left\{\left(x, \frac{1}{n}\right): 0 \leqslant x<1\right\}$, where $n \in N$, the set of positive integers.

A family τ of subsets of X is defined as follows:

(a) $\phi, X \in \tau$

(b) $G \subset X \backslash \left[\left\{\left(0, \frac{1}{n}\right): n \in N\right\} \cup L_0\right] \Rightarrow G \in \tau$.

(c) For $n \geqslant 1$, $G(\subset X)$ contains points of the form $\left(0, \frac{1}{n}\right) \Rightarrow G \in \tau$ iff $X \backslash G$ contains only a finite number of points of L_n.

(d) For $0 < x < 1$, $(x, 0) \in G \Rightarrow G \in \tau$ iff for each $(x, 0) \in G$ there exists $n_x \in N$ with

$$U_{n_x}(x, 0) = \{(x, 0)\} \cup \left\{\left(x, \frac{1}{n}\right) : n \geqslant n_x\right\} \subset G .$$

It is not hard to check that τ is a topology with the following properties:

$[P_1]$ Every singleton on the horizontal line L_n $(n \geqslant 1)$ with the point $\left(0, \frac{1}{n}\right)$ deleted is open and hence any subset of $L_n \setminus \left\{\left(0, \frac{1}{n}\right)\right\}$ is so. (This follows from (b) of the definition of τ). Also each L_n $(n \geqslant 1)$ is open, by (c) of the definition of τ.

$[P_2]$ No point of the line L_n $(n \geqslant 1)$, not to speak of other points of X, can be an accumulation point of a singleton of L_n and hence singletons on L_n $(n \geqslant 1)$ are closed.

$[P_3]$ The sets $U_{n_x}(x, 0)$ defined in (d) are open sets. On the other hand, no point of $X \setminus U_{n_x}(x, 0)$ is an accumulation point of $U_{n_x}(x, 0)$ and therefore $U_{n_x}(x, 0)$ is closed as well.

$[P_4]$ For the topological space (X, τ), $\tau^\alpha = \tau$, which is proved below.

Suppose $U \in \tau^\alpha$. Then the following cases are to be considered.

Case I. Let $U \subset X \setminus \left[\left\{\left(0, \frac{1}{n}\right) : n \in N\right\} \cup L_0\right]$. Then by (b) of the definition of τ, $U \in \tau$.

Case II. Let U contain points of the form $\left(0, \frac{1}{n}\right)$. The α-ness of $U \Rightarrow \mathrm{Int}_\tau(U) \neq \phi$. Define the set A by

$$A = \left[L_n \setminus \left\{\left(0, \frac{1}{n}\right)\right\}\right] \cap [X \setminus \mathrm{Int}_\tau(U)]$$

which is a subset of $L_n \setminus \left\{\left(0, \frac{1}{n}\right)\right\}$. Hence, by $[P_1]$, A is open. We assert that A is finite. The very construction of A guarantees that A contains no point of $\mathrm{Int}_\tau(U)$. On the other hand, the openness of A indicates that no point of A is an accumulation point of $\mathrm{Int}_\tau(U)$. These facts give

$$\begin{aligned} & A \cap \mathrm{Int}_\tau(U) = \phi, \ A \cap D[\mathrm{Int}_\tau(U)] = \phi \\ \Rightarrow \ & A \cap \mathrm{Cl}_\tau(\mathrm{Int}_\tau(U)) = \phi \\ \Rightarrow \ & A \cap \mathrm{Int}_\tau(\mathrm{Cl}_\tau(\mathrm{Int}_\tau(U))) = \phi \\ \Rightarrow \ & A \subset X \backslash \mathrm{Int}_t(\mathrm{Cl}_\tau(\mathrm{Int}_\tau(U))). \end{aligned}$$

Again, $\left(0, \dfrac{1}{n}\right) \in U \subset \mathrm{Int}_\tau(\mathrm{Cl}_\tau(\mathrm{Int}_\tau(U)))$. This, together with (c) of the definition of τ, implies that $X \backslash \mathrm{Int}_\tau(\mathrm{Cl}_\tau(\mathrm{Int}_\tau(U)))$ contains only a finite number of points of L_n and contains A. Hence A is finite and this yields from the construction of A that $\mathrm{Int}_\tau(U)$ and, *a fortiori*, U contains all but finitely many points of L_n. So, by (c) of the definition of τ, $U \in \tau$.

Case III. Let U contain points of L_0, i.e. points of the form $(x, 0)$, where $0 < x < 1$. Clearly $(\mathrm{Int}_\tau(U)) \neq \phi$ and $\mathrm{Int}_\tau(\mathrm{Cl}_\tau(\mathrm{Int}_\tau(U)))$ contains $(x, 0) \in L_0$. Hence (d) of the definition of τ gives the existence of a positive integer n_x such that

(1) $\quad U_{n_x}(x, 0) \subset \mathrm{Int}_\tau(\mathrm{Cl}_\tau(\mathrm{Int}_\tau(U))) \subset \mathrm{Cl}_\tau(\mathrm{Int}_\tau(U)) = [\mathrm{Int}_\tau(U) \cup D[\mathrm{Int}_\tau(U)]]$.

By $[P_1]$, $\left\{\left(x, \dfrac{1}{n}\right)\right\} \subset L_n \backslash \left\{\left(0, \dfrac{1}{n}\right)\right\}$ is open, whence no point of $\left\{\left(x, \dfrac{1}{n}\right) : n \geqslant n_x\right\}$ is an accumulation point of $\mathrm{Int}_\tau(U)$ which then implies that

$$\left\{\left(x, \frac{1}{n}\right) : n \geqslant n_x\right\} \cap D[\mathrm{Int}_\tau(U)] = \phi \ .$$

Consequently, from (1) and the definition of $U_{n_x}(x, 0)$, it follows that $U_{n_x}(x, 0) \subset U$. So, by (d) of the definition of τ, $U \in \tau$. Thus in any case $U \in \tau$ and it indicates that $\tau^\alpha \subset \tau$. This gives $\tau^\alpha = \tau$.

Now we consider the b.t.s. (X, τ_1, τ_2), where $\tau_1 = \tau_2 = \tau =$ the topology constructed above.

Result 1) of Ex. 94 in [**14**], saying that (X, τ) is completely regular, implies that

$$(X, \tau_1, \tau_2) = (X, \tau_1^\alpha, \tau_2^\alpha)$$

is pairwise α-regular. Result 2) of Ex. 94 in [**14**], saying that (X, τ) is not normal, implies that

$$(X, \tau_1, \tau_2) = (X, \tau_1^{\alpha}, \tau_2^{\alpha})$$

is not pairwise normal.

Remark 5.1. The purpose of the above two examples can now be summed up as follows:

I. Pairwise α-Hausdorffness $\nRightarrow$ Pairwise α-regularity
II. Pairwise α-regularity $\nRightarrow$ Pairwise α-normality.

It is a natural question whether the reverse implications made in Remark 5.1 will hold or not. Since *pac* spaces have no role to play in that investigation, we did not pay our attention in that direction for the time being.

6 - Properties of *pac* spaces

Remark 5.1 raises the pertinent question: Can there be any space where pairwise α-Hausdorffness implies pairwise α-regularity or pairwise α-regularity implies pairwise α-normality? Next two theorems offer a positive answer to this query.

Theorem 6.1. *If a b.t.s.* (X, τ_1, τ_2) *is both pac and pairwise* α*-Hausdorff, then it is pairwise* α*-regular.*

Proof. It follows by Theorem 2.1 applied to the space $(X, \tau_1^{\alpha}, \tau_2^{\alpha})$.

Theorem 6.2. *If a b.t.s.* (X, τ_1, τ_2) *is pac and either* τ_1 *is* α*-regular with respect to* τ_2 *or* τ_2 *is* α*-regular with respect to* τ_1*, then* (X, τ_1, τ_2) *is pairwise* α*-normal.*

Proof. It follows by Theorem 2.2 applied to the space $(X, \tau_1^{\alpha}, \tau_2^{\alpha})$.

Corollary 6.1. *If a b.t.s.* (X, τ_1, τ_2) *is both pac and pairwise* α*-regular, then it is pairwise* α*-normal.*

Proof. Obvious.

Corollary 6.2. *If a b.t.s.* (X, τ_1, τ_2) *is both pac and pairwise* α*-Hausdorff, then it is pairwise* α*-normal.*

Proof. Follows from Theorem 6.1 and Corollary 6.1.

Replacing «τ_i-compact» $(i = 1, 2)$ by «τ_i-α-compact» $(i = 1, 2)$ in the Definition 2.3 one obtains a b.t.s. (X, τ_1, τ_2) which may be termed as bi-α-compact space. This notion has been utilized in the theorem to follow.

From Example 3.2. it has been observed that α-compactness for individual topologies does not necessarily imply the *pac* of a b.t.s.. But pairwise α-Hausdorffness provides a condition for a bi-α-compact space (X, τ_1, τ_2) to be *pac*, as seen from the following theorem.

Theorem 6.3. *If a b.t.s.* (X, τ_1, τ_2) *is both bi-α-compact and pairwise α-Hausdorff, then it is pac.*

Proof. Let (X, τ_1, τ_2) be a bi-α-compact space which is also pairwise α-Hausdorff. Consequently, $(X, \tau_1^\alpha, \tau_2^\alpha)$ is bi-compact and pairwise Hausdorff. So, by Theorem 2.3, $\tau_1^\alpha = \tau_2^\alpha$. Now, the compactness of (X, τ_1^α) and the equality $\tau_1^\alpha = \tau_2^\alpha$ together lead to the pairwise compactness of $(X, \tau_1^\alpha, \tau_2^\alpha)$, that is the *pac* of (X, τ_1, τ_2).

Corollary 6.3. *If a b.t.s.* (X, τ_1, τ_2) *is both bi-α-compact and pairwise α-Hausdorff then*

(i) (X, τ_1, τ_2) is *pac*
(ii) (X, τ_1, τ_2) is pairwise α-regular
(iii) (X, τ_1, τ_2) pairwise α-normal.

Proof. Follows from Theorems 6.3, 6.1 and Corollary 6.1.

We draw an end to our present treatment of *pac* space after the following results (Theorem 6.4 and Corollary 6.4) which establish an interconnection among bi-Hausdorffness, pairwise α-Hausdorffness and the *pac*.

Fukutake [4] introduced the notion of bi-Hausdorffness of a b.t.s. (X, τ_1, τ_2) (see Definition 2.8). In view of the observation made at page 5, the notion bi-Hausdorffness can be rephrased as bi-α-Hausdorffness.

The bi-α-Hausdorffness does not, in general, imply pairwise α-Hausdorffness as seen in the following example.

Example 6.1. In the b.t.s. of Example 4.1, take $\tau_2 = \{G \subset R : R \setminus G$ is finite or $0 \in R \setminus G\}$, which is a modification of Example 24 in [**14**]. Clearly (R, τ_1) is Hausdorff and hence α-Hausdorff. To show that (R, τ_2) is also α-Hausdorff we observe:

If $x, y\ (\neq x) \in R$ and $x \neq 0,\ y \neq 0$, then take $U, V \in \tau_2$, where $U = \{x\}$ and $V = \{y\}$, so that $U \cap V = \phi$. If $x = 0,\ y \neq 0$, then we take $V = \{y\}$ and $U = R \setminus V$ so that $U, V \in \tau_2$, $x \in U$, $y \in V$ and $U \cap V = \phi$. Thus, in any case two distinct points are strongly separated in (R, τ_2). Hence (R, τ_2) is Hausdorff and so it is α-Hausdorff.

On the other hand, that (R, τ_1, τ_2) is not pairwise α-Hausdorff can be checked as follows:

Take two points $x, y \in R$ with $x = 0$. Let $U \in \tau_2^\alpha$, $V \in \tau_1^\alpha$ such that $x \in U$ and $y \in V$. Now $\mathrm{Int}_{\tau_2}(U) \neq \phi$ and $\mathrm{Int}_{\tau_1}(V) \neq \phi$. The following cases are to be considered.

Case 1. Let $0 \notin \mathrm{Int}_{\tau_2}(U)$. Then, by definition of τ_2, $\mathrm{Int}_{\tau_2}(U) \cup \{0\}$ is a τ_2-closed set containing $\mathrm{Int}_{\tau_2}(U)$. Since $\mathrm{Cl}_{\tau_2}(\mathrm{Int}_{\tau_2}(U))$ is the smallest closed set containing $\mathrm{Int}_{\tau_2}(U)$, we obtain

$$\begin{aligned} &\mathrm{Cl}_{\tau_2}(\mathrm{Int}_{\tau_2}(U)) \subset \mathrm{Int}_{\tau_2}(U) \cup \{0\} \\ &\Rightarrow \mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(\mathrm{Int}_{\tau_2}(U))) \subset \mathrm{Int}_{\tau_2}(\mathrm{Int}_{\tau_2}(U) \cup \{0\}). \end{aligned} \tag{2}$$

The following subcases now deserve consideration.

Subcase (i). Suppose $\mathrm{Int}_{\tau_2}(U) \cup \{0\} \in \tau_2$. Then, by definition of τ_2, $R \backslash [\mathrm{Int}_{\tau_2}(U) \cup \{0\}]$ is finite, whence it follows that $R \backslash [\mathrm{Int}_{\tau_2}(U)]$ is finite. Therefore $\mathrm{Int}_{\tau_2}(U)$ and, *a fortiori*, U contain all but a finite number of elements of R.

Subcase (ii). Suppose $\mathrm{Int}_{\tau_2}(U) \cup \{0\} \notin \tau_2$. Then $\mathrm{Int}_{\tau_2}(\mathrm{Int}_{\tau_2}(U) \cup \{0\}) = \mathrm{Int}_{\tau_2}(U)$. Hence (2) gives

$\mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(\mathrm{Int}_{\tau_2}(U))) \subset \mathrm{Int}_{\tau_2}(U) \subset U$,

$\Rightarrow$ $U = \mathrm{Int}_{\tau_2}(\mathrm{Cl}_{\tau_2}(\mathrm{Int}_{\tau_2}(U)))$, by α-ness of U,

$\Rightarrow$ $U \in \tau_2$, $0 \in U$,

$\Rightarrow$ U contains, by definition of τ_2, all but a finite number of elements of R.

Case II. Suppose $0 \in \mathrm{Int}_{\tau_2}(U)$. Then the definition of τ_2 gives that $\mathrm{Int}_{\tau_2}(U)$ and hence U contains all but a finite number of elements of R. Thus, in any case U contains all but a finite number of elements of R.

Again, $\mathrm{Int}_{\tau_1}(V) \neq \phi \Rightarrow V$ contains an open interval. Hence V is infinite. From this we infer that $U \cap V \neq \phi$. Hence (R, τ_1, τ_2) is not pairwise α-Hausdorff.

Theorem 6.4. *If a b.t.s. (X, τ_1, τ_2) is both pac and bi-α-Hausdorff, then it is pairwise α-Hausdorff.*

Proof. Let (X, τ_1, τ_2) be a *pac* space which is also bi-α-Hausdorff. Consequently, $(X, \tau_1^\alpha, \tau_2^\alpha)$ is pairwise compact and bi-Hausdorff. So, by Theorem 2.4, $\tau_1^\alpha = \tau_2^\alpha$. Now, the Hausdorffness of (X, τ_1^α) and the equality $\tau_1^\alpha = \tau_2^\alpha$ together lead to the pairwise Hausdorffness of $(X, \tau_1^\alpha, \tau_2^\alpha)$ which, in its turn, implies that (X, τ_1, τ_2) is pairwise α-Hausdorff.

Corollary 6.4. *If a b.t.s.* (X, τ_1, τ_2) *is both pac bi-α-Hausdorff then*

(i) (X, τ_1, τ_2) *is pairwise α-Hausdorff*
(ii) (X, τ_1, τ_2) *is pairwise α-regular*
(iii) (X, τ_1, τ_2) *is pairwise α-normal.*

Proof. Follows from Theorem 6.4, 6.1 and Corollary 6.1.

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Abstract

The concept of α-compactness [8] *in single topological spaces has been generalized to bitopological spaces introduced by Kelly* [**6**]. *Some new bitopological separation axioms have been introduced to interpret the properties of α-compact spaces in the bitopological setting. Apart from this parallel to, some pertinent questions of the existing literature in respect to compactness have been solved. Profuse examples have been provided in support of every statement made in the form of Definition, Lemma or Remark.*

* * *