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Periodic solutions of electrically heated conductors (**)

1 - Introduction

We study a system of two parabolic equations modelling the heat conduction in a conductor body, in the presence of the electrical heating due to the Joule effect. If u denotes the temperature and φ is the electric potential, an external variable magnetic field induces the Foucalt currents into the conductor, generating the so called Joule effect. Our problem consists to resolve a coupled system of nonlinear parabolic differential equations for u and φ.

We are interested to prove the existence of the weak periodic solutions for the system

$$\varphi_t = \operatorname{div}(\sigma(u)\nabla\varphi), \qquad \text{in } S := \Omega \times P \tag{1a}$$

$$\varphi(x, t) = h(x, t), \qquad \text{on } \Sigma := \partial\Omega \times P \tag{1b}$$

$$\varphi(x, t+\omega) = \varphi(x, t), \qquad \text{in } S,\ \omega > 0 \tag{1c}$$

$$u_t - \Delta u = \sigma(u)\,|\nabla\varphi|^2 + f(x, t), \qquad \text{in } S \tag{2a}$$

$$u(x, t) = 0, \qquad \text{on } \Sigma \tag{2b}$$

$$u(x, t+\omega) = u(x, t), \qquad \text{in } S,\ \omega > 0 \tag{2c}$$

where Ω is a bounded open set of R^N with regular boundary (e.g. C^∞ boundary)

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and $P := \dfrac{R}{\omega Z}$ is the period interval $[0, \omega]$, so that in viewing functions defined in S, we are imposing the time ω-periodicity.

We denote with $\sigma(u)$ the heat depending electric conductivity, while the quadratic term in (2a) represents the Joule heating. We recall that the time independent problem, corresponds to the thermistor problem that has been studied by several authors (see [**3**], [**5**], [**8**]). The study of an evolution system has been enterprised in [**1**], [**4**], [**7**], [**13**], [**17**], [**18**].

In [**13**], a derivation on physical grounds of the thermo-magnetic system (1a)-(2c) is given. It is worth to note that in the literature, to our knowledges, the study of the periodic solutions for a model as (1a)-(2c) is not been treated previously.

Papers related to the periodicity of the solutions for an electrical heated conductors regard the time-dependent thermistor models.

We are acquainted with the paper [**6**], [**7**]. In [**6**], is studied a system where the function u depends only on the spatial variable x, while in [**7**], the function φ_t does not appear. We study the system (1a)-(2c) under the following assumptions

(3) $\sigma \in C(R) : 0 < m \leqslant \sigma(\xi) \leqslant M, \ \forall \xi \in R$;

(4) $f \in L^2(P; W^{-1,2}(\Omega)), \ f(x, t) \geqslant 0$;

(5) h is a t-periodic bounded function and $\widehat{h}$ is an extension of h to S .

Our approach to the periodicity of the solutions, shall be of *static* type namely we look among solutions for which belong to some suitable space of t-periodic functions, rather than to look for a fixed point for the Poincaré period map. For certain approximating problems, we shall prove the existence of the periodic solutions and derive uniform estimates which allow, passing to the limit, to show the existence of the periodic solutions for our problem (1a)-(2c).

Lastly, in Section 4, utilizing the results of [**19**], we will consider the regularity of the periodic weak solutions for (1a)-(2c).

2 - Principal results

We begin our study solving (1a)-(1c). To this goal, we fix $w \in L^2(S)$ and look for a function v defined in S which satisfies

$$v_t = \operatorname{div}(\sigma(w)(\nabla v + \Phi)) - g, \qquad \text{in } S \tag{6}$$

$$v(x, t) = 0, \qquad \text{on } \Sigma \tag{7}$$

where $\varphi = v + \widehat{h}$, $\Phi = \nabla \widehat{h}$ and $g = \widehat{h}_t$.

For this reason, introduce the spaces $V := L^2(P; W^{1,2}(\Omega))$ and $V_0 := L^2(P; W_0^{1,2}(\Omega))$, endowed with the norm

$$\|v\|_V := \left(\int_S |v(x,t)|^2 \, dx \, dt + \int_S |\nabla v(x,t)|^2 \, dx \, dt \right)^{\frac{1}{2}}. \tag{8}$$

Then, V is the completation with respect to the norm (8) of $C^\infty(S)$, while V_0 is the closure in V of $C_0^\infty(S)$, the space of the periodic functions vanishing near Σ.

Let $W := W^{1,2}(P; W_0^{1,2}(\Omega))$, be the closure of $C_0^\infty(S)$ with respect to the norm

$$\left(\int_S |v(x,t)|^2 \, dx \, dt + \int_S |\nabla v(x,t)|^2 \, dx \, dt + \int_S |v_t(x,t)|^2 \, dx \, dt \right)^{\frac{1}{2}}.$$

Besides, we assume that

$$\Phi \in L^2(S), \qquad g \in L^2(S). \tag{9}$$

Since the existence of the weak solutions for the problem (6)-(7) shall follow from a result of Browder [**2**], we define for any $v \in V$ the linear functional

$$A(v)\,\xi := \int_S \sigma(w)(\nabla v + \Phi)\, \nabla \xi \, dx \, dt, \qquad \forall\, \xi \in V_0 \tag{10}$$

and the linear operator $L : D \to V_0^*$ that is a closed skew-adjoint extension of the linear operator L_0 defined on $C_0^\infty(S)$ to V_0^* by setting

$$L_0(v)\,\xi := \int_S v_t \, \xi \, dx \, dt, \qquad \forall\, \xi \in V_0 \tag{11}$$

where the domain D is taken as

$$D := \{ v \in L^2(P; W_0^{1,2}(\Omega)),\ v_t \in L^2(P; W^{-1,2}(\Omega)) \}.$$

The operator L is densely defined on a domain D, with $C_0^\infty(S) \subset D \subset V_0$.

Finally, let $\hat{g} \in V_0^*$ be the linear functional defined by

$$\hat{g}\,\xi := \int_S g \xi \, dx \, dt, \qquad \forall\, \xi \in V_0, \tag{12}$$

thus, the problem (6)-(7) becomes

$$(L+A)(v) = -\widehat{g} \tag{13}$$

and we can give the following

Definition. *By a weak periodic solution to* (6)-(7), *we mean a function* $v \in D \subset V_0$, *such that* (13) *holds i.e.*

$$\int_S (v_t \xi + \sigma(w)(\nabla v + \Phi)\nabla\xi + g\xi)\,dx\,dt = 0, \qquad \text{for all } \xi \in V_0. \tag{14}$$

The equation (13) shall be solved appealing to a result of [2], that we recall below

Theorem 0 ([2]). *Let L be a closed, densely defined skew-adjoint linear operator from a reflexive Banach space X to its dual X^* and let $A: X \to X^*$ be coercive, monotone and hemicontinuous. Then for any Φ, g verifying* (9) *there exists a weak solution v of* (13).

Now, we can prove the following result

Theorem 1. *Assume* (3)-(5) *and* (9). *Then there exists a weak periodic solution to* (13).

Proof To apply the above Browder's result, we need to verify that the operator A satisfies:

$$A(v)(.) \text{ is continuous on } V_0. \tag{15}$$

$$A: V \to V_0^* \text{ is continuous} \tag{16}$$

$$A \text{ is monotone} \tag{17}$$

$$A \text{ is coercive}. \tag{18}$$

The (15) is obtained by means of the Hölder inequality, which implies the estimate

$$\begin{aligned} &|A(v)\,\xi(x,t)| \\ &\leq M\left(\int_S |\nabla v(x,t) + \Phi(x,t)|^2\,dx\,dt\right)^{\frac{1}{2}}\left(\int_S |\nabla\xi(x,t)|^2\,dx\,dt\right)^{\frac{1}{2}} \\ &\leq M\left(\int_S |\nabla v(x,t) + \Phi(x,t)|^2\,dx\,dt\right)^{\frac{1}{2}}\|\xi\|_V \end{aligned} \tag{19}$$

by which we deduce that

$$\|A(v)\| \leqslant M\sqrt{2}(\|\nabla v\|_{L^2(S)} + \|\Phi\|_{L^2(S)}) \leqslant M\sqrt{2}(\|v\|_V + \|\Phi\|_{L^2(S)}) . \tag{20}$$

Now, a standard argument (see [**9**], Thm. 2.1), involves that the operator A is continuous in V, thus (16) it follows.

To prove (17), we observe that

$$(A(v_1) - A(v_2))(v_1 - v_2) = \int_S \sigma(w)\,|\nabla(v_1 - v_2)\,|^2\,dx\,dt \geqq 0 .$$

Instead of, the coercivity of A descends from

$$A(v)\,v = \int_S \sigma(w)(\nabla v + \Phi)\,\nabla v\,dx\,dt = \int_S \sigma(w)\,|\nabla v|^2\,dx\,dt$$

$$+ \int_S \sigma(w)\,\Phi\nabla v\,dx\,dt .$$

By the Poincaré inequality, the L^2-norm of v is dominated by the L^2-norm of ∇v for $v = 0$ on Σ, because Ω is bounded. Thus, the $L^2(S)$ norm of ∇v is equivalent to the norm (8) for $v \in V_0$. Therefore,

$$\begin{aligned} A(v)\,v &\geqq m\|\nabla v\|^2_{L^2(S)} - M\|\Phi\|_{L^2(S)}\,\|\nabla v\|_{L^2(S)} \\ &\geqq mC\|v\|^2_V - M\|\Phi\|_{L^2(S)}\,\|v\|_V \end{aligned}$$

by which we get

$$A(v)\,\frac{v}{\|v\|_V} \geqq mC\|v\|_V - M\|\Phi\|_{L^2(S)} \to +\infty , \qquad \text{as } \|v\|_V \to +\infty .$$

Finally, it is possible to invoke the result of [**2**] to conclude that for any $\hat{g} \in V_0^*$, there exists a $v \in D$ such that solves (13). ■

The solution is unique because if one had two solutions v_1, v_2, then $\nabla v_1 = \nabla v_2$. For $v_1, v_2 \in V_0$ this means $v_1 = v_2$.

Thus, φ is a weak periodic solution of

$$\varphi_t = \operatorname{div}(\sigma(w)\,\nabla\varphi), \qquad \text{in } S \tag{21}$$

$$\varphi(x, t) = h(x, t), \qquad \text{on } \Sigma \tag{22}$$

$$\varphi(x, t + \omega) = \varphi(x, t), \quad \text{in } S \tag{23}$$

corrisponding to w.

3 - Existence of periodic solutions

The direct approach to the solvibility of (2a)-(2c) with the usual energy estimates, encounters a difficulty due to the presence of the quadratic term $|\nabla\varphi|^2$, as we only know that $|\nabla\varphi|^2 \in L^1(S)$. For this reason, it is necessary to consider a sequence of approximating problems. Set $p(x_1, x_2, \dots x_N) = \sum_{i=1}^{N} x_i^2$, for $(x_1, x_2, \dots x_N) \in R^N$, for each $n \in N$ we define

$$p_n(x) = n\,, \qquad \text{if } p(x) \geqq n \qquad (24)$$
$$= p(x)\,, \qquad \text{if } p(x) < n$$

and consider the approximating problems for $w \in L^2(S)$

$$u_t - \Delta u = \sigma(w)\, p_n(\nabla\varphi) + f(x, t), \qquad \text{in } S \qquad (25)$$

$$u(x, t) = 0, \qquad \text{on } \Sigma \qquad (26)$$

$$u(x, t+\omega) = u(x, t), \qquad \text{in } S \qquad (27)$$

where $p_n(\nabla\varphi) \in L^1\ (S)$.

We prove the result

Theorem 2. *Under the assumptions* (3)-(5) *and* (24), *for each n there are weak periodic solutions to* (25)-(27), *where φ solves* (21)-(23).

Proof. The weak periodic solution to (25)-(27) is a function $u \in D \subset V_0$ such that

$$\int_S (u_t\,\zeta + \nabla u\,\nabla\zeta - (\sigma(w)\, p_n(\nabla\varphi) + f(x, t))\,\zeta)\, dx\, dt = 0\,, \qquad \forall\, \zeta \in V_0.$$

holds.

Setting

$$L(u)\,\zeta = \int_S u_t\,\zeta\, dx\, dt\,;$$

$$A(u)\,\zeta = \int_S \nabla u\,\nabla\zeta\, dx\, dt$$

and

$$K_n \zeta = \int_S (\sigma(w)\, p_n(\nabla \varphi) + f(x,\,t))\,\zeta\,dx\,dt$$

for any $u \in D$, $\zeta \in V_0$, we can again apply the Browder result, to extablish that for any $K_n \in V_0^*$ there exists a solution $u \in D$ of

$$(L + A)(u) = K_n. \qquad \blacksquare \tag{28}$$

Let

$$U := \{\xi \in L^2(S) : \|\xi\|_{L^2(S)} \leq R\},$$

we define the nonlinear operator Φ on U by

$$\Phi(w) = u$$

where u is a weak periodic solution of the problem (25)-(27), so Φ is a well-defined mapping.

Lemma 3. *The nonlinear operator Φ is continuous on U and maps U into itself.*

Proof. Let $w_k \in U$ be such that $w_k \to w$ in $L^2(S)$ and show that $u_k \to u$ in $L^2(S)$. Let φ_k, u_k be the weak periodic solutions to

$$\varphi_{kt} = \mathrm{div}\,(\sigma(w_k)\,\nabla \varphi_k), \qquad \text{in } S \tag{29}$$

$$\varphi_k(x,\,t) = h(x,\,t), \qquad \text{on } \Sigma \tag{30}$$

$$\varphi_k(x,\,t+\omega) = \varphi_k(x,\,t), \qquad \text{in } S \tag{31}$$

$$u_{kt} - \Delta u_k = \sigma(w_k)\,p_n(\nabla \varphi_k) + f(x,\,t), \qquad \text{in } S \tag{32}$$

$$u_k(x,\,t) = 0, \qquad \text{on } \Sigma \tag{33}$$

$$u_k(x,\,t+\omega) = u(x,\,t), \qquad \text{in } S\,. \tag{34}$$

As a weak formulation of (29)-(31) and (32)-(34) we take

$$\varphi_k - h,\ u_k \in L^2(P;\, W_0^{1,2}(\Omega))$$

and

$$\int_S (-\varphi_k \zeta_t + \sigma(w_k)\, \nabla\varphi_k \nabla\zeta)\, dx\, dt = 0, \ \ \forall\, \zeta \in W^{1,2}(P;\, W_0^{1,2}(\Omega)). \tag{35}$$

respectively

$$\int_S (-u_k \zeta_t + \nabla u_k \nabla\zeta - (\sigma(w_k)\, p_n(\nabla\varphi_k) + f(x,t))\,\zeta)\, dx\, dt = 0, \quad \forall\, \zeta \in W^{1,2}(P;\, W_0^{1,2}(\Omega)). \tag{36}$$

The weak maximum principle implies that

$$\min_S h(x,t) \leqslant \varphi_k(x,t) \leqslant \max_S h(x,t), \ \ \forall\, t \in P \ \text{and a.e. in } \Omega. \tag{37}$$

Chosen $\zeta = h - \varphi_k$ as test function in (35), one has

$$\int_S -\varphi_k (h-\varphi_k)_t\, dx\, dt + \int_S \sigma(w_k)\, \nabla\varphi_k \nabla(h-\varphi_k)\, dx\, dt = 0,$$

which implies

$$\frac{1}{2}\int_0^\omega \frac{d}{dt}\int_\Omega (\varphi_k - h)^2\, dx\, dt + \int_0^\omega\int_\Omega h_t (h-\varphi_k)\, dx\, dt$$

$$+ \int_0^\omega\int_\Omega \sigma(w_k)\, \nabla\varphi_k \nabla h\, dx\, dt = \int_0^\omega\int_\Omega \sigma(w_k)\, |\nabla\varphi_k|^2\, dx\, dt.$$

Using the Young inequality, one obtains

$$\frac{1}{2}\int_0^\omega\int_\Omega \sigma(w_k)\, |\nabla\varphi_k|^2\, dx\, dt \leqslant \frac{1}{2}\int_0^\omega\int_\Omega |h_t|^2\, dx\, dt$$

$$+ \frac{1}{2}\int_0^\omega\int_\Omega |(\varphi_k - h)|^2\, dx\, dt + \frac{1}{2}\int_0^\omega\int_\Omega \sigma(w_k)\, |\nabla h|^2\, dx\, dt \leqslant C\,.$$

(Here and throughout, C denotes a generic positive constant independent of k and n).

Therefore

$$\int_0^\omega \int_\Omega |\nabla \varphi_k|^2 \, dx\, dt \leq C\,. \tag{38}$$

Combining (37) with (38), we get the energy estimate

$$\sup_{0 \leq t \leq \omega} \int_\Omega |\varphi_k(x, t)|^2 \, dx + \int_0^\omega \int_\Omega |\nabla \varphi_k|^2 \, dx\, dt \leq C\,. \tag{39}$$

By (39) and (29) follow that φ_{kt} is bounded in $L^2(P; W^{-1,2}(\Omega))$. Since $\varphi_k - h \in D$ and it is bounded with respect to the norm of D, then $\varphi_k \rightharpoonup \varphi$ in $L^2(P; W^{1,2}(\Omega))$ (passing to subsequences if necessary) and by [**12**], $\varphi_k \to \varphi$ in $L^2(S)$ where φ is a weak periodic solution to (21)-(23). We prove that $\nabla\varphi_k \to \nabla\varphi$ in $L^2(S)$. In fact, choosing $\zeta = \varphi_k - \varphi$ in (35) as a test function, we have

$$\int_0^\omega \int_\Omega -\varphi_k(\varphi_k - \varphi)_t \, dx\, dt + \int_0^\omega \int_\Omega \sigma(w_k)\, \nabla\varphi_k \nabla(\varphi_k - \varphi)\, dx\, dt = 0\,,$$

hence

$$-\frac{1}{2}\int_0^\omega \frac{d}{dt}\int_\Omega (\varphi_k - \varphi)^2 \, dx\, dt + \int_0^\omega \int_\Omega \varphi_t(\varphi_k - \varphi)\, dx\, dt$$

$$+ \int_0^\omega \int_\Omega \sigma(w_k)\, |\nabla(\varphi_k - \varphi)|^2 \, dx\, dt = -\int_0^\omega \int_\Omega \sigma(w_k)\, \nabla\varphi\nabla(\varphi_k - \varphi)\, dx\, dt,$$

this implies that

$$\int_0^\omega \int_\Omega \sigma(w_k)\, |\nabla(\varphi_k - \varphi)|^2 \, dx\, dt = -\int_0^\omega \int_\Omega \varphi_t(\varphi_k - \varphi)\, dx\, dt$$

$$- \int_0^\omega \int_\Omega \sigma(w_k)\, \nabla\varphi\nabla(\varphi_k - \varphi)\, dx\, dt.$$

Therefore,

$$\int_0^\omega \int_\Omega |\nabla(\varphi_k - \varphi)|^2 \, dx\, dt \to 0$$

because of the weak convergence of $\sigma(w_k)\,\nabla(\varphi_k-\varphi)\rightharpoonup 0$ in $(L^2(S))^N$. Moreover,

$$p_n(\nabla\varphi_k)\to p_n(\nabla\varphi),\qquad \text{in } L^1(S).$$

Now, we pass to consider the problem (32)-(34). If choose $\zeta=u_k$ in (36), we infer that

$$\int_0^\omega\int_\Omega|\nabla u_k|^2\,dx\,dt\leqq\int_0^\omega\int_\Omega nMu_k\,dx\,dt+\frac{1}{2\varepsilon}\int_0^\omega\int_\Omega f(x,t)^2\,dx\,dt+\frac{\varepsilon}{2}\int_0^\omega\int_\Omega u_k^2\,dx\,dt,$$

because of the Young inequality and $\sigma(w_k)\,p_n(\nabla\varphi_k)\leqq nM$.

An iterated application of Young's and Poincaré's inequalities gives us

$$\int_0^\omega\int_\Omega|\nabla u_k|^2\,dx\,dt\leqq\frac{(nM)^2}{2\varepsilon}\,|S|+\varepsilon\int_0^\omega\int_\Omega u_k^2\,dx\,dt+\frac{1}{2\varepsilon}\int_0^\omega\int_\Omega f(x,t)^2\,dx\,dt.$$

and

$$\int_0^\omega\int_\Omega|\nabla u_k|^2\,dx\,dt\leqq\frac{(nM)^2}{2\varepsilon}\,|S|+\varepsilon C\int_0^\omega\int_\Omega|\nabla u_k|^2\,dx\,dt+\frac{1}{2\varepsilon}\int_0^\omega\int_\Omega f(x,t)^2\,dx\,dt,$$

which implies

$$\int_0^\omega\int_\Omega|\nabla u_k|^2\,dx\,dt\leqq C$$

by which it follows that

$$\int_0^\omega\int_\Omega u_k^2\,dx\,dt+\int_0^\omega\int_\Omega|\nabla u_k|^2\,dx\,dt\leqq C,\tag{40}$$

this estimate yields that $\Phi(U)\subset U$.

As done above, u_k is bounded in D and $u_k\geqslant 0$ by a comparison argument, hence there exists a subsequence denoted again u_k such that $u_k\rightharpoonup u$ in D and by a result of [**12**], $u_k\to u$ in $L^2(S)$ where u is the weak solution of (25)-(27) corresponding to w and φ.

Hence, has been proven that

$$\varphi_k \to \varphi, \qquad \text{in } L^2(S) \tag{41}$$

$$\nabla\varphi_k \to \nabla\varphi, \qquad \text{in } L^2(S) \tag{42}$$

$$u_k \to u, \qquad \text{in } L^2(S) \tag{43}$$

$$\nabla u_k \rightharpoonup \nabla u, \qquad \text{in } L^2(S) \tag{44}$$

$$w_k \to w, \qquad \text{in } L^2(S) \tag{45}$$

$$\sigma(w_k) \to \sigma(w), \qquad \text{in } L^2(S) \tag{46}$$

$$p_n(\nabla\varphi_k) \to p_n(\nabla\varphi), \qquad \text{in } L^1(S). \tag{47}$$

This show the continuity of the operator Φ in $L^2(S)$. In fact, $u_k = \Phi(w_k)$ strongly converges to $u = \Phi(w)$ in $L^2(S)$. ■

The compact imbedding of D into $L^2(S)$ implies that Φ is compact, so by the Schauder fixed point theorem, Φ has a fixed point $u_k = \Phi(u_k)$. Hence, we are able to prove our main

Theorem 4. *Assume* (3)-(5) *and* (24). *Then there exists a weak periodic solution to* (1a)-(2c).

Proof. For each n there is a weak periodic solution φ_n, u_n of the system

$$\varphi_{nt} = \operatorname{div}(\sigma(u_n)\,\nabla\varphi_n), \qquad \text{in } S \tag{48}$$

$$\varphi_n(x, t) = h(x, t), \qquad \text{on } \Sigma \tag{49}$$

$$\varphi_n(x, t+\omega) = \varphi_n(x, t), \qquad \text{in } S \tag{50}$$

$$u_{nt} - \Delta u_n = \sigma(u_n)\, p_n(\nabla\varphi_n) + f(x, t), \qquad \text{in } S \tag{51}$$

$$u_n(x, t) = 0, \qquad \text{on } \Sigma \tag{52}$$

$$u_n(x, t+\omega) = u_n(x, t), \qquad \text{in } S \tag{53}$$

$$\int_S (-\varphi_n \zeta_t + \sigma(u_n)\,\nabla\varphi_n \nabla\zeta)\, dx\, dt = 0, \quad \forall\, \zeta \in W^{1,2}(P;\, W_0^{1,2}(\Omega)). \tag{54}$$

$$\int_S (-u_n \zeta_t + \nabla u_n \nabla\zeta - (\sigma(u_n)\, p_n(\nabla\varphi_n) + f(x, t))\,\zeta)\, dx\, dt = 0, \tag{55}$$
$$\forall\, \zeta \in W^{1,2}(P;\, W_0^{1,2}(\Omega)).$$

Summing (48) up (51) and using $\left(\dfrac{\varphi_n^2}{2}+u_n-\dfrac{h^2}{2}\right)^+$ as a test function, one has

$$\frac{1}{2}\int_0^\omega \frac{d}{dt}\int_\Omega\left(\frac{\varphi_n^2}{2}+u_n-\frac{h^2}{2}\right)^{+2}dx\,dt+\int_0^\omega\int_\Omega hh_t\left(\frac{\varphi_n^2}{2}+u_n-\frac{h^2}{2}\right)^+dx\,dt$$

$$\leqslant\int_0^\omega\int_\Omega(\Delta u_n+\varphi_n\operatorname{div}(\sigma(u_n)\nabla\varphi_n) \tag{56}$$

$$+\sigma(u_n)|\nabla\varphi_n|^2+f(x,t))\left(\frac{\varphi_n^2}{2}+u_n-\frac{h^2}{2}\right)^+dx\,dt,$$

since $\sigma(u_n)\,p_n(\nabla\varphi_n)\leqslant\sigma(u_n)|\nabla\varphi_n|^2$. We observe that $\varphi_n\operatorname{div}(\sigma(u_n)\nabla\varphi_n)+\sigma(u_n)|\nabla\varphi_n|^2=\operatorname{div}(\varphi_n\sigma(u_n)\nabla\varphi_n)$ hence by (56) we obtain

$$\int_0^\omega\int_\Omega hh_t\left(\frac{\varphi_n^2}{2}+u_n-\frac{h^2}{2}\right)^+dx\,dt\leqslant$$

$$-\int_0^\omega\int_{\{u_n\geqslant M_1\}}\nabla u_n\nabla\left(\frac{\varphi_n^2}{2}+u_n-\frac{h^2}{2}\right)dx\,dt$$

$$-\int_0^\omega\int_{\{u_n\geqslant M_1\}}\sigma(u_n)\,\varphi_n\nabla\varphi_n\nabla\left(\frac{\varphi_n^2}{2}+u_n-\frac{h^2}{2}\right)dx\,dt$$

$$+\int_0^\omega\int_\Omega f(x,t)\left(\frac{\varphi_n^2}{2}+u_n-\frac{h^2}{2}\right)^+dx\,dt,$$

where we set $M_1=ess\sup\limits_S\left(\dfrac{h^2}{2}-\dfrac{\varphi_n^2}{2}\right)$.

Then,

$$\int_0^\omega\int_{\{u_n\geqslant M_1\}}|\nabla u_n|^2dx\,dt\leqslant-\int_0^\omega\int_{\{u_n\geqslant M_1\}}\nabla u_n\varphi_n\nabla\varphi_n\,dx\,dt$$

$$+\int_0^\omega\int_{\{u_n\geqslant M_1\}}\nabla u_n h\nabla h\,dx\,dt-\int_0^\omega\int_{\{u_n\geqslant M_1\}}\sigma(u_n)\,\varphi_n^2|\nabla\varphi_n|^2dx\,dt$$

$$-\int_0^\omega \int_{\{u_n \geq M_1\}} \sigma(u_n)\, \varphi_n \nabla\varphi_n \nabla u_n \, dx\, dt + \int_0^\omega \int_{\{u_n \geq M_1\}} \sigma(u_n)\, \varphi_n \nabla\varphi_n h \nabla h \, dx\, dt$$

$$+\int_0^\omega \int_{\{u_n \geq M_1\}} f(x,\, t)\left(\frac{\varphi_n^2}{2} + u_n - \frac{h^2}{2}\right) dx\, dt - \int_0^\omega \int_{\{u_n \geq M_1\}} h h_t \left(\frac{\varphi_n^2}{2} + u_n - \frac{h^2}{2}\right) dx\, dt.$$

Now,

$$\int_0^\omega \int_{\{u_n \geq M_1\}} |\nabla u_n|^2 \, dx\, dt \leqslant -\int_0^\omega \int_{\{u_n \geq M_1\}} (\varphi_n \nabla\varphi_n - h\nabla h + \sigma(u_n)\, \varphi_n \nabla\varphi_n)\, \nabla u_n \, dx\, dt$$

$$-\int_0^\omega \int_{\{u_n \geq M_1\}} \sigma(u_n)\, \varphi_n^2\, |\nabla\varphi_n|^2 \, dx\, dt + \int_0^\omega \int_{\{u_n \geq M_1\}} \sigma(u_n)\, \varphi_n \nabla\varphi_n h\nabla h \, dx\, dt$$

$$+\int_0^\omega \int_{\{u_n \geq M_1\}} f(x,\, t)\left(\frac{\varphi_n^2}{2} + u_n - \frac{h^2}{2}\right) dx\, dt - \int_0^\omega \int_{\{u_n \geq M_1\}} h h_t \left(\frac{\varphi_n^2}{2} + u_n - \frac{h^2}{2}\right) dx\, dt.$$

By the Hölder and the Poincaré inequalities, we have

$$\int_0^\omega \int_{\{u_n \geq M_1\}} |\nabla u_n|^2 \, dx\, dt \leqslant \frac{1}{2\varepsilon} \int_0^\omega \int_{\{u_n \geq M_1\}} (\varphi_n \nabla\varphi_n - h\nabla h + \sigma(u_n)\, \varphi_n \nabla\varphi_n)^2 \, dx\, dt$$

$$+\frac{\varepsilon}{2} \int_0^\omega \int_{\{u_n \geq M_1\}} |\nabla u_n|^2 \, dx\, dt - \int_0^\omega \int_{\{u_n \geq M_1\}} \sigma(u_n)\, \varphi_n^2\, |\nabla\varphi_n|^2 \, dx\, dt$$

$$+\int_0^\omega \int_{\{u_n \geq M_1\}} \sigma(u_n)\, \varphi_n \nabla\varphi_n h\nabla h \, dx\, dt + \int_0^\omega \int_{\{u_n \geq M_1\}} f(x,\, t)\left(\frac{\varphi_n^2}{2} - \frac{h^2}{2}\right) dx\, dt$$

$$+\frac{1}{2\varepsilon} \int_0^\omega \int_{\{u_n \geq M_1\}} f(x,\, t)^2 \, dx\, dt + \frac{\varepsilon C}{2} \int_0^\omega \int_{\{u_n \geq M_1\}} |\nabla u_n|^2 \, dx\, dt$$

$$+\frac{1}{2\varepsilon} \int_0^\omega \int_{\{u_n \geq M_1\}} (h h_t)^2 \, dx\, dt + \frac{\varepsilon C}{2} \int_0^\omega \int_{\{u_n \geq M_1\}} |\nabla u_n|^2 \, dx\, dt$$

$$-\int_0^\omega \int_{\{u_n \geq M_1\}} h h_t \left(\frac{\varphi_n^2}{2} - \frac{h^2}{2}\right) dx\, dt.$$

Finally, one has

$$\left(1-\frac{\varepsilon}{2}-\varepsilon C\right)\int_0^\omega\int_{\{u_n\geqslant M_1\}}|\nabla u_n|^2\,dx\,dt$$

$$\leqslant \frac{1}{2\varepsilon}\int_0^\omega\int_{\{u_n\geqslant M_1\}}(\varphi_n\nabla\varphi_n-h\nabla h+\sigma(u_n)\,\varphi_n\nabla\varphi_n)^2\,dx\,dt$$

$$-\int_0^\omega\int_{\{u_n\geqslant M_1\}}\sigma(u_n)\,\varphi_n^2\,|\nabla\varphi_n|^2\,dx\,dt+\int_0^\omega\int_{\{u_n\geqslant M_1\}}\sigma(u_n)\,\varphi_n\nabla\varphi_n\,h\nabla h\,dx\,dt$$

$$+\int_0^\omega\int_{\{u_n\geqslant M_1\}}f(x,\,t)\left(\frac{\varphi_n^2}{2}-\frac{h^2}{2}\right)dx\,dt+\frac{1}{2\varepsilon}\int_0^\omega\int_{\{u_n\geqslant M_1\}}f(x,\,t)^2\,dx\,dt$$

$$+\frac{1}{2\varepsilon}\int_0^\omega\int_{\{u_n\geqslant M_1\}}(hh_t)^2\,dx\,dt-\int_0^\omega\int_{\{u_n\geqslant M_1\}}hh_t\left(\frac{\varphi_n^2}{2}-\frac{h^2}{2}\right)dx\,dt.$$

That is

$$\int_0^\omega\int_{\{u_n\geqslant M_1\}}|\nabla u_n|^2\,dx\,dt\leqslant C. \tag{57}$$

When $0\leqslant u_n\leqslant M_1$, multiplying (51) by u_n one has

$$\int_0^\omega\int_\Omega|\nabla u_n|^2\,dx\,dt\leqslant\int_0^\omega\int_\Omega\sigma(u_n)\,p_n(\nabla\varphi_n)\,u_n\,dx\,dt+\int_0^\omega\int_\Omega f(x,\,t)\,u_n\,dx\,dt$$

$$\leqslant M_1M\int_0^\omega\int_\Omega|\nabla\varphi_n|^2\,dx\,dt+\frac{1}{2}\int_0^\omega\int_\Omega f(x,\,t)^2\,dx\,dt+\frac{1}{2}\int_0^\omega\int_\Omega|u_n|^2\,dx\,dt \tag{58}$$

$$\leqslant M_1MC+\frac{1}{2}\int_0^\omega\int_\Omega f(x,\,t)^2\,dx\,dt+\frac{1}{2}M_1^2\,|S|\,.$$

From (56) and (58) follow that

$$\int_0^{\omega}\int_{\Omega} |\nabla u_n|^2 \, dx\, dt \leq C,$$

hence

$$\int_0^{\omega}\int_{\Omega} u_n^2(x, t)\, dx\, dt + \int_0^{\omega}\int_{\Omega} |\nabla u_n|^2 \, dx\, dt \leq C. \tag{59}$$

Now we turn our attention to show the precompactness of u_n in $L^2(S)$. Following [**18**], we define $u_{n,\tau}(x, t) = u_n(x, t+\tau)$, for $0 < \tau < \omega$ with $u_n \in L^2(S)$, then integrate (51) over $(t, t+\tau)$ to get

$$u_{n,\tau}(x, t) - u_n(x, t) = \int_t^{t+\tau} (\Delta u_n + \sigma(u_n)\, p_n(\nabla \varphi_n) + f(x, s))\, ds. \tag{60}$$

Taking the scalar product of the formula (60) with $(u_{n,\tau} - u_n)^-$ and integrating over $(0, \omega - \tau)$, one establishes that (see [**18**])

$$\int_0^{\omega-\tau}\int_{\Omega} ((u_{n,\tau} - u_n)^-)^2 \, dx\, dt \leqq C' \tau^{\frac{1}{2}}$$

where C shall be a constant independent of h and n. An analogous calculation to what was been done in [**18**], yields

$$\int_0^{\omega-\tau}\int_{\Omega} ((u_{n,\tau} - u_n)^+)^2 \, dx\, dt \leqq C' \tau^{\frac{1}{2}}.$$

Using a result of [**14**], we conclude that $u_n \to u$, in $L^2(S)$. Proceeding as in the proof of Lemma 3, it is possible to show that $\varphi_n \to \varphi$ and $\nabla\varphi_n \to \nabla\varphi$ in $L^2(S)$.

Passing to the limit in (54) and (55), one establishes that

$$\varphi - h \in L^2(P;\, W_0^{1,2}(\Omega)), \qquad u \in L^2(P;\, W_0^{1,2}(\Omega))$$

and

$$\int_S (-\varphi\zeta_t + \sigma(u)\,\nabla\varphi\nabla\zeta)\, dx\, dt = 0,$$

$$\int_S (-u\zeta_t + \nabla u \nabla\zeta - (\sigma(u)\,|\nabla\varphi|^2 + f(x, t))\,\zeta)\, dx\, dt = 0,$$

for all $\zeta \in W^{1,2}(P; W_0^{1,2}(\Omega))$. i.e. φ, u are periodic solutions to (1a)-(2c) since

$$\sigma(u_n)\,|\nabla\varphi_n|^2 = \sigma(u_n)\,|\nabla(\varphi_n - \varphi)|^2 + 2\sigma(u_n)\,\nabla\varphi_n\nabla\varphi - \sigma(u_n)\,|\nabla\varphi|^2,$$

which gives us

$$\int_S \sigma(u_n)\,|\nabla\varphi_n|^2\,dx\,dt \to \int_S \sigma(u)\,|\nabla\varphi|^2\,dx\,dt.$$

This conclude the prove.

4 - Regularity

The results in [**19**] on the regularity of the weak solutions, for the time-dependent thermistor problem, allow to get $L^2(P\times\Omega)$ estimates for u, φ and ∇u, $\nabla\varphi$. We recall a few papers concerning the regularity of the solutions for problems analogous to (1a)-(2c) (see [**10**]-[**11**], [**15**]-[**16**] and [**19**]). In [**19**], one proves the regularity of the weak solutions, by means of a priori estimates in the Campanato spaces $L^{2,\mu}(Q)$, $(0\leq\mu<n+2)$ for which the DeGiorgi-Nash-Moser's estimate and a modified Poincaré's inequality, play an essential role. We mention the results of Yin without details.

We say that $u\in L^{2,\mu}(Q)$, $(0\leq\mu<n+2)$ if $u\in L^2(Q)$ and is such that

$$[u]_{2,\mu,Q_r} = \left(\sup_{z_0\in Q,\,0<\varrho<r} \varrho^{-\mu}\int\int_{Q_\varrho(z_0)} |u-u_{z_0,\varrho}|^2\,dz\right)^{1/2} < +\infty$$

where $z=(x,t)\in R^{n+1}$,

$$Q_r(z_0) := \{x\in R^n\colon\ |x-x_0|<r\}\times(t_0-r^2,\,t_0]$$

and

$$u_{z_0,r} := \frac{1}{|Q_r(z_0)|}\int\int_{Q_r(z_0)} u\,dz, \qquad |Q_r(z_0)| = \mathrm{meas}\,Q_r(z_0).$$

$L^{2,\mu}(Q)$ is a Banach space with the norm

$$\|u\|_{2,\mu,Q_r} = (\|u\|^2_{L^2(Q_r)} + [u]^2_{2,\mu,Q_r})^{1/2}.$$

The local regularity is obtained, proving that for any region Q with $dist(Q,\Sigma)>0$, one has $\nabla\varphi\in L^{2,\mu}(Q)$ where $\mu=n+2\alpha$, for some $\alpha\in(0,1)$. Since any function in

$L^\infty(Q)$ is a multiplier for $L^{2,\mu}(Q)$, $(0 \leq \mu < n+2)$ and φ is bounded by the maximum principle, set $U = u + \varphi^2/2$, where u is a weak solutions of (2a), we can rewrite this equation as follows

$$U_t - \Delta U = \operatorname{div}((\sigma(u) - 1)\,\varphi\nabla\varphi) + f(x, t).$$

Thus,

$$f_i(x, t) = (\sigma(u) - 1)\,\varphi\varphi_{x_i} \in L^{2,\mu}(Q), \qquad i = 1, \dots, n$$

and assuming $f \in L^{2,(\mu-2)^+}$, we get $\nabla u \in L^{2,\mu}(Q)$, $(2 < \mu < n+2)$ (see [**19**]), which implies a local estimate in $L^2(\Omega \times P)$ for ∇u. Once we have this result, we can proceed as in ([**19**], Lemmas 2.4 and 2.6), to show that $u, \varphi \in L^{2,\mu+2}(Q)$, $(n < \mu < n+2)$. This gives us the local regularity of the solutions in the Hölder class, that is u, $\varphi \in C^{\alpha,\alpha/2}(\Omega \times P)$ for $\alpha = (\mu - n)/2$, (see also [**11**], Thm. 6.29).

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References

[**1**] W. Allegretto and H. Xie, *Existence of solutions for the time-dependent thermistor equations*, IMA J. Appl. Math. **48** (1992), 271-281.

[**2**] F. E. Browder, *Nonlinear maximal monotone operators in a Banach space*, Math. Ann. **175** (1968), 89-113.

[**3**] G. Cimatti and G. Prodi, *Existence results for a nonlinear elliptic system modelling a temperature dependent electrical resistor*, Ann. Mat. Pura Appl. (IV) **CLII** (1988), 227-236.

[**4**] G. Cimatti, *Existence of weak solution for a nonstationary problem of the Joule heating of a conductor*, Ann. Mat. Pura Appl. (IV) **CLXII** (1992), 33-42.

[**5**] G. Cimatti, *A bound for the temperature in the thermistor problem*, IMA J. Appl. Math. **49** (1988), 15-22.

[**6**] G. Cimatti, *On two problems of electrical heating of conductors*, Quart. Appl. Math. **XLIX** (1991), 729-740.

[**7**] G. Cimatti, *Existence of periodic solutions for a thermally dependent resistore in a R-C circuit*, Internat. J. Engrg. Sci. **12** (1994), 69-78.

[**8**] S. D. Howson, J. F. Rodrigues and M. Shillor, *Stationary solutions to the thermistor problem*, J. Math. Anal. Appl. **174** (1993), 573-588.

[**9**] M. A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, Pergamon Press, New York 1964.

[10] G. M. LIEBERMAN, *Time-periodic solutions of linear parabolic differential equations*, Comm. Partial Differential Equations **24** (1999), 631-663.

[11] G. M. LIEBERMAN, *Second order parabolic differential equations*, World Scientific, Singapore 1996.

[12] J. L. LIONS, *Quelques méthodes de résolution de problemes aux limites non-linéaires*, Dunod, Paris 1968.

[13] J. L. RODRIGUES, *A nonlinear parabolic system arising in the thermomechanics and in thermomagnetism*, Math. Models Methods Appl. Sci. **2** (1992), 271-281.

[14] J. SIMON, *Écoulement d'un fluide nonhomogène avec une densité initiale s'annulant*, C. R. Acad. Sci. Paris Sér. A-B **287** (1978), 1009-1012.

[15] HONG XIE, $L^{2,\mu}(\Omega)$ *estimate to the mixed boundary value problem for second order elliptic equations and its application in the thermistor problem*, Nonlinear Anal. **24** (1995), 9-27.

[16] WEI QING XIE, *A nonlinear elliptic system modelling the thermistor problem*, Appl. Anal. **50** (1993), 263-276.

[17] X. XU, *Existence theory for a strongly degenerate parabolic system*, Rocky Mountain J. Math. **23** (1993), 1555-1580.

[18] X. XU, *A unidirectional flow with temperature dependent viscosity*, Nonlinear Anal. **23** (1994), 369-386.

[19] HONGMING YIN, $L^{2,\mu}(Q)$*-estimates for parabolic equations and applications*, J. Partial Differential Equations **10** (1997), 31-44.

Summary

This paper studies the existence of the periodic solutions for a time-dependent thermistor model, seeking a solution in a suitable space of t-periodic functions. Finally, we comment the regularity of the periodic solutions by means of estimates in the Campanato spaces.

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