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S. VIGNOLO and D. BRUNO (*)

Time-dependent vakonomic dynamics and presymplectic geometry (**)

1 - Introduction

The Dynamics of Lagrangian systems subject to non-holonomic constraints may be formulated through the so-called vakonomic approach [**1**], [**2**]: it consists in the study of variational problems where the variations are imposed to satisfy the constraints.

As a matter of fact, every vakonomic problem may be considered as a free variational one associated with an extended Lagrangian incorporating the constraints through a set of Lagrange multipliers.

By construction such an extended Lagrangian is always singular; all the known results regarding degenerate Lagrangians may be therefore applied to the theory of vakonomic systems.

A first step in this direction has been made by Cariñena and Rañada in [**3**], followed by some other recent papers [**4**], [**5**] where the presymplectic constraint algorithm developed by Gotay and Nester [**7**], [**8**] is used to study and solve vakonomic problems.

Using the time-dependent extension of the Gotay-Nester method proposed in [**11**], we generalize the previous works to the time-dependent case. This has been made possible by the adoption of the geometrical setting provided by the Lagrangian and Hamiltonian bundles [**9**], [**10**], [**11**], [**12**], [**13**].

(*) Dipartimento di Matematica dell'Università di Genova, Via Dodecaneso 35, 16146 Genova, Italia, e-mail: vignolo@dima.unige.it; D. Bruno: Dipartimento di Matematica dell'Università di Genova, Via Dodecaneso 35, 16146 Genova, Italia, e-mail: bruno@dima.unige.it

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We first present a brief review of the geometrical framework of Lagrangian and Hamiltonian bundles and of their use in the discussion of presymplectic time-dependent Lagrangian systems in Section 2.

In Section 3 we illustrate the application of the previous topic to vakonomic dynamics; the given approach is suitable to describe both regular and singular Lagrangian systems.

Nevertheless, the most significant results are obtained for systems with regular Lagrangian, subject to affine kinetic constraints; we focus on this particular case, showing that the transition to the Hamiltonian setting allows a sort of reduction of the problem to a free Hamiltonian one.

We conclude the paper with some illustrative examples: the first is an application to classical vakonomic Mechanics; the second shows the possible use of the proposed approach in economic models, where an explicit time-dependence is always present; the third and last one is an energy stationarity problem for a simple analog circuit.

2 - Geometrical preliminaries

2.1 - *The Lagrangian and Hamiltonian bundles*

Let us consider a Lagrangian system with n degrees of freedom and denote by $\mathcal{V}_{n+1}$ its configuration space-time. The axiom of absolute time makes $\mathcal{V}_{n+1}$ a fibered manifold over the real line (thought as an Euclidean space), with projection $t : \mathcal{V}_{n+1} \to \Re$ given by the absolute time function; the first jet-extension $j_1(\mathcal{V}_{n+1})$ may be naturally identified with the velocity space of the system. Let us denote by $t, q^1, \ldots, q^n$ a generic fibered local coordinate system on $\mathcal{V}_{n+1}$ and by $t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n$ the induced jet-coordinates on $j_1(\mathcal{V}_{n+1})$. The Euler-Lagrange equations associated with the Lagrangian system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = 0$$

are known to be invariant under arbitrary transformations of the form

$$L \to L' := L + \frac{df}{dt}$$

where f is a smooth function defined on $\mathcal{V}_{n+1}$ and $\frac{df}{dt}$ denotes its symbolic time derivative. For this reason, the Lagrangian L is not a geometrical object associated with the given system, being defined up to a gauge. In a recent paper [**9**], this situation has been analyzed and a gauge-invariant setup has been developed in the language of jet-bundle theory.

For later use, we present here a brief review of the subject, in order to provide the reader of the necessary definitions.

The idea is to associate to the given Lagrangian system a double fibration $P \to \mathcal{V}_{n+1} \to \Re$, where $\pi : P \to \mathcal{V}_{n+1}$ is a principal fiber bundle, with structural group $(\Re, +)$, called the bundle of *affine scalars* over $\mathcal{V}_{n+1}$.

The action of the structural group on P will be expressed through the additive notation

$$\psi_{\xi}(\nu) := \psi(\xi, \nu) : \Re \times P \to P \ : \ \psi_{\xi}(\nu) = \nu + \xi \,. \tag{2.1}$$

The properties of the bundle of affine scalars are now summarized:

- every function $u : P \to \Re$ satisfying $u(\nu + \xi) = u(\nu) + \xi$ provides a global trivialization of P, namely an identification of P with the Cartesian product $\mathcal{V}_{n+1} \times \Re$;
- the assignment of a trivialization u allows to lift every coordinate system t, q^i on $\mathcal{V}_{n+1}$ to a coordinate system $t, q^1, \dots, q^n, u$ over P, satisfying the following coordinate transformation rules

$$\bar{t} = t + c \,, \qquad \bar{q}^i = \bar{q}^i(t, q^1, \dots, q^n) \,, \qquad \bar{u} = u + f(t, q^1, \dots, q^n) \tag{2.2}$$

- the vector field $\dfrac{\partial}{\partial u}$ is the fundamental vector field associated to the principal fibration.

Let us now focus our attention of the fibration $P \to \Re$ and consider its first jet space $j_1(P, \Re)$, which we shall refer to local jet-coordinates $t, q^i, u, \dot{q}^i, \dot{u}$, with transformation laws

$$\bar{t} = t + c \,, \qquad \bar{q}^i = \bar{q}^i(t, q) \,, \qquad \bar{u} = u + f(t, q) \tag{2.3a}$$

$$\dot{\bar{q}}^i = \frac{\partial \bar{q}^i}{\partial q^k} \dot{q}^k + \frac{\partial \bar{q}^i}{\partial t} \,, \qquad \dot{\bar{u}} = \dot{u} + \frac{\partial f}{\partial q^k} \dot{q}^k + \frac{\partial f}{\partial t} := \dot{u} + \dot{f} \,. \tag{2.3b}$$

It is easy to verify that $j_1(P, \Re)$ may be considered a subspace of $T(P)$ through the identification

$$z \in j_1(P, \Re) \quad \Leftrightarrow \quad z = \left[\frac{\partial}{\partial t} + \dot{q}^i(z) \frac{\partial}{\partial q^i} + \dot{u}(z) \frac{\partial}{\partial u} \right]_{\pi(z)} . \tag{2.4}$$

The geometrical properties of $j_1(P, \Re)$ come from its jet-bundle structure first: in particular, it is endowed with

(i) a contact bundle, locally generated by the $(n+1)$ 1-forms $\omega_0 = du - \dot{u}\,dt$ and $\omega^i = dq^i - \dot{q}^i dt$;

(ii) a fiber differential d_v on the Grassman algebra of $j_1(P, \Re)$ (1), whose action on any function $f \in \mathcal{F}(j_1(P, \Re))$ is described as

$$d_v f = \frac{\partial f}{\partial \dot{u}}\,\omega_0 + \frac{\partial f}{\partial \dot{q}^i}\,\omega^i. \tag{2.5}$$

In addition to this, the bundle $j_1(P, \Re)$ carries two different actions of the structural group $(\Re, +)$, both arising from the principal bundle structure of P and based on the identification expressed by eq. (2.4). The first one may be defined as the push-forward $\psi_{\xi *}$ of the action (2.1); in local coordinates we have

$$\psi_{\xi *} : (t, q^i, u, \dot{q}^i, \dot{u}) \rightarrow (t, q^i, u + \xi, \dot{q}^i, \dot{u}). \tag{2.6}$$

The quotient of $j_1(P, \Re)$ under this action is a $(2n+2)$-dimensional manifold, which will be denoted by $\mathcal{L}(\mathcal{V}_{n+1})$. The quotient map makes $j_1(P, \Re)$ into a principal fiber bundle over $\mathcal{L}(\mathcal{V}_{n+1})$, with structural group $(\Re, +)$; $\mathcal{L}(\mathcal{V}_{n+1})$ results to be a fiber bundle over $\mathcal{V}_{n+1}$ referred to local coordinates $t, q^i, \dot{q}^i, \dot{u}$.

The second action is obtained adding to any $z \in j_1(P, \Re)$ a multiple of the fundamental vector field over P, namely $z \rightarrow z + \xi \dfrac{\partial}{\partial u}$; it is represented as

$$\phi_{\xi} : (t, q^i, u, \dot{q}^i, \dot{u}) \rightarrow (t, q^i, u, \dot{q}^i, \dot{u} + \xi). \tag{2.7}$$

Once again, the quotient of $j_1(P, \Re)$ under (2.7) is a $(2n+2)$-dimensional manifold, which will be denoted by $\mathcal{L}^c(\mathcal{V}_{n+1})$. The quotient map makes $j_1(P, \Re)$ into a principal fiber bundle over $\mathcal{L}^c(\mathcal{V}_{n+1})$ with structural group $(\Re, +)$; $\mathcal{L}^c(\mathcal{V}_{n+1})$ is a fiber bundle over $\mathcal{V}_{n+1}$ with local coordinates $t, q^i, u, \dot{q}^i$.

Finally we observe that the two actions (2.6) and (2.7) commute and can be therefore used to induce a group action on the quotient space generated by the other. This makes both $\mathcal{L}(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1})$ into a principal fiber bundle over a «double quotient» space, which is easily identified with the first jet space

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(1) For a detailed description the reader is referred to [**9**].

$j_1(\mathcal{V}_{n+1})$. The situation is best represented by the following commutative diagram

$$\begin{array}{ccc} j_1(P, \mathfrak{R}) & \longrightarrow & \mathcal{L}^c(\mathcal{V}_{n+1}) \\ \downarrow & & \downarrow \\ \mathcal{L}(\mathcal{V}_{n+1}) & \longrightarrow & j_1(\mathcal{V}_{n+1}) \end{array} \tag{2.8}$$

where all arrows represent principal fibrations. The bundles $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ will be respectively denoted *Lagrangian* and *co-Lagrangian* bundles over $j_1(\mathcal{V}_{n+1})$.

The geometric environment introduced so far is sufficient to develop a gauge-invariant formulation of Lagrangian Mechanics. This is achieved replacing the concept of Lagrangian function by the one of *Lagrangian section*, namely a section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ of the Lagrangian bundle. The outcome is a scheme where the assignment of a trivialization of P determines a local description of l of the form

$$\dot{u} = L(t, q^i, \dot{q}^i) \tag{2.9}$$

namely in terms of a Lagrangian function $L(t, q^i, \dot{q}^i)$ over $j_1(\mathcal{V}_{n+1})$. As soon as the trivialization is changed, the representation (2.9) undergoes the transformation laws

$$\bar{\dot{u}} = \dot{u} + \dot{f} = L(t, q^i, \dot{q}^i) + \dot{f} := L'(t, q^i, \dot{q}^i) \tag{2.10}$$

which involves a gauge equivalent Lagrangian function. We also remark that any Lagrangian section l defines a trivialization $\widehat{\varphi}_l := \dot{u} - L(t, q, \dot{q})$ of the bundle $j_1(P, \mathfrak{R}) \to \mathcal{L}^c(\mathcal{V}_{n+1})$.

The importance of the concept of Lagrangian section consists in inducing a connection 1-form θ_l on the bundle $j_1(P, \mathfrak{R}) \to \mathcal{L}(\mathcal{V}_{n+1})$, which plays a crucial role in the transition from Lagrangian to Hamiltonian dynamics. The 1-form θ_l is defined through the action of the fiber differential on the trivialization $\widehat{\varphi}_l$, namely

$$\theta_l := d_v \widehat{\varphi}_l = \omega_0 - \frac{\partial L}{\partial \dot{q}^i}\, \omega^i . \tag{2.11}$$

A similar reasoning can be followed using the manifold $P \to \mathcal{V}_{n+1}$ as a starting point and taking its first jet space $j_1(P, \mathcal{V}_{n+1})$ into account. Given local coordinates t, q^i, u on P, we denote by t, q^i, u, p_0, p_i the induced fibered local coordinate system, subject to transformation laws

(2.12a) $\bar{t} = t + c\,, \qquad \bar{q}^i = \bar{q}^i(t, q^1, \ldots, q^n)\,, \qquad \bar{u} = u + f(t, q^1, \ldots, q^n)$

(2.12b) $\bar{p}_0 = p_0 + \dfrac{\partial f}{\partial t} + \left(p_k + \dfrac{\partial f}{\partial q^k}\right)\dfrac{\partial q^k}{\partial \bar{t}}\,, \qquad \bar{p}_i = \left(p_k + \dfrac{\partial f}{\partial q^k}\right)\dfrac{\partial q^k}{\partial \bar{q}^i}\,.$

The jet space structure endows $j_1(P, \mathcal{V}_{n+1})$ with the canonical contact 1-form Θ, locally written as

$$\Theta = du - p_0\,dt - p_i\,dq^i \tag{2.13}$$

and henceforth called the *Liouville* 1-*form*. The latter has the nature of a connection 1-form of the bundle $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$.

Moreover, $j_1(P, \mathcal{V}_{n+1})$ is a submanifold of the cotangent space $T^*(P)$, through the identification

$$\eta \in j_1(P, \mathcal{V}_{n+1}) \;\Leftrightarrow\; \eta = [du - p_0(\eta)\,dt - p_i(\eta)\,dq^i]_{\pi(\eta)}. \tag{2.14}$$

Finally, the manifold $j_1(P, \mathcal{V}_{n+1})$ carries two different actions of the structural group $(\mathfrak{R}, +)$, both defined on the basis of identification (2.14). The first is defined as the pull-back $(\psi_{-\xi})^*$ of (the inverse of) the action (2.1) and can be written as

$$\psi^*_\xi : (t, q^i, u, p_0, p_i) \to (t, q^i, u + \xi, p_i, p_0)\,. \tag{2.15}$$

The quotient of $j_1(P, \mathcal{V}_{n+1})$ under this action is a $(2n+2)$-dimensional manifold $\mathcal{H}(\mathcal{V}_{n+1})$ and $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ is a principal fiber bundle, with structural group $(\mathfrak{R}, +)$. $\mathcal{H}(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$ results to be an affine bundle, modelled on $T^*(\mathcal{V}_{n+1})$, locally described by the coordinate system t, q^i, p_0, p_i.

The second action is obtained subtracting to any $\eta \in j_1(P, \mathcal{V}_{n+1})$ a multiple of the invariant 1-form dt, namely $\eta \to \eta - \xi dt$, and can be expressed as

$$\phi_\xi : (t, q^i, u, p_0, p_i) \to (t, q^i, u, p_i, p_0 + \xi)\,. \tag{2.16}$$

The quotient of $j_1(P, \mathcal{V}_{n+1})$ under (2.16) will be denoted $\mathcal{H}^c(\mathcal{V}_{n+1})$. It results to be a fiber bundle over $\mathcal{V}_{n+1}$, with coordinates (t, q^i, u, p_0); moreover, $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}^c(\mathcal{V}_{n+1})$ is a principal fiber bundle with structural group $(\mathfrak{R}, +)$.

As in the Lagrangian case, the above described actions commute: thus we can define a «double quotient» space $\Pi(\mathcal{V}_{n+1})$, henceforth called the *phase space*, obtained by quotient of either $\mathcal{H}(\mathcal{V}_{n+1})$ or $\mathcal{H}^c(\mathcal{V}_{n+1})$ under the corresponding group action. The situation is summarized into the following commutative diagram

$$
\begin{array}{ccc}
j_1(P, \mathcal{V}_{n+1}) & \longrightarrow & \mathcal{H}^c(\mathcal{V}_{n+1}) \\
\downarrow & & \downarrow \\
\mathcal{H}(\mathcal{V}_{n+1}) & \longrightarrow & \Pi(\mathcal{V}_{n+1})
\end{array}
\tag{2.17}
$$

where all arrows denote principal fibrations. The bundles $\mathcal{H}(\mathcal{V}_{n+1}) \to \Pi(\mathcal{V}_{n+1})$ and $\mathcal{H}^c(\mathcal{V}_{n+1}) \to \Pi(\mathcal{V}_{n+1})$ will be respectively called the *Hamiltonian* and *co-Hamiltonian* bundles over $\Pi(\mathcal{V}_{n+1})$. Every section $h : \Pi(\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ will be called *Hamiltonian section*.

We remark that the differential $-d\Theta$ is the pull-back of a closed 2-form

$$
\Omega := dp_0 \wedge dt + dp_i \wedge dq^i
\tag{2.18}
$$

which endows $\mathcal{H}(\mathcal{V}_{n+1})$ with a symplectic structure. This is the framework where a time-dependent formulation of Hamiltonian dynamics may be developed [**10**].

Finally, a map between $j_1(P, \mathfrak{R})$ and $j_1(P, \mathcal{V}_{n+1})$ may be set up, observing that every Lagrangian section l induces the connection 1-form θ_l on the first, while the second is endowed with the canonical 1-form Θ. Then, it is easy to prove that there exists an unique map $\Lambda : j_1(P, \mathfrak{R}) \to j_1(P, \mathcal{V}_{n+1})$, fibered over P, such that $\Lambda^*(\Theta) = \theta_l$; this is called *Legendre transformation* [**9**], [**10**], [**13**].

2.2 - *Presymplectic time-dependent Lagrangian systems*

In [**11**], the geometrical properties of the Lagrangian bundle $\mathcal{L}(\mathcal{V}_{n+1})$ have been investigated; as a consequence, a mathematical setting suitable for the study of degenerate time-dependent Lagrangian systems has been developed. For later use, we briefly outline the basic aspects of the theory.

First of all, we recall that the assignment of a Lagrangian section l induces the following geometrical objects on the manifold $\mathcal{L}(\mathcal{V}_{n+1})$:

- a trivialization $\varphi_l := \dot{u} - L(t, q^i, \dot{q}^i)$ of the principal fiber bundle $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$;
- a smooth connection of $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$, whose connection 1-form is given by the differential $d\varphi_l = d\dot{u} - dL$; the related horizontal lift associates with every vector field $X = X^0 \dfrac{\partial}{\partial t} + X^i \dfrac{\partial}{\partial q^i} + \dot{X}^i \dfrac{\partial}{\partial \dot{q}^i} \in D^1(j_1(\mathcal{V}_{n+1}))$ a corresponding vector field X_l on $\mathcal{L}(\mathcal{V}_{n+1})$, invariant under the action of the structural group

$\left(\text{i.e., under the 1-parameter group of diffeomorphisms generated by } \frac{\partial}{\partial \dot{u}}\right)$ and expressed locally as

$$X_l = X^0 \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + \dot{X}^i \frac{\partial}{\partial \dot{q}^i} + X(L) \frac{\partial}{\partial \dot{u}} \; ; \tag{2.19}$$

- a $(1, 1)$-tensor field $\tilde{J}$ on $\mathcal{L}(\mathcal{V}_{n+1})$, having local expression

$$\tilde{J} = \omega^i \otimes \left(\frac{\partial L}{\partial \dot{q}^i} \frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{q}^i} \right) \tag{2.20}$$

where the notation $\omega^i := dq^i - \dot{q}^i dt \;\; i = 1, \dots, n$ for the pull-back to $\mathcal{L}(\mathcal{V}_{n+1})$ of the contact 1-forms on $j_1(\mathcal{V}_{n+1})$ has been preserved; it is immediate to see that $\tilde{J}$ is π-related to the fundamental tensor $J = \omega^i \otimes \frac{\partial}{\partial \dot{q}^i}$ of $j_1(\mathcal{V}_{n+1})$;

- an exact 2-form $\tilde{\Omega}_l$ on $\mathcal{L}(\mathcal{V}_{n+1})$, expressed in local fibered coordinates as

$$\tilde{\Omega}_l := d\dot{u} \wedge dt + d\left(\frac{\partial L}{\partial \dot{q}^i}\right) \wedge \omega^i - \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i \wedge dt \; ; \tag{2.21}$$

under the regularity assumption rank $\left\| \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right\| = n$, it is a straightforward matter to verify that the 2-form (2.21) has maximal rank, thus endowing the bundle $\mathcal{L}(\mathcal{V}_{n+1})$ with a symplectic structure; when this is the case, following the standard terminology, the section l is said to be a *regular* Lagrangian section; on the contrary, when the regularity hypothesis is violated, but $\tilde{\Omega}_l$ has constant rank everywhere, the 2-form (2.21) is *presymplectic*; in such a circumstance, we shall call l a *degenerate* (or *singular*) Lagrangian section; furthermore, denoting by $\Omega_l := d\left(L\,dt + \frac{\partial L}{\partial \dot{q}^i}\omega^i\right)$ the Poincaré-Cartan 2-form associated with the Lagrangian function $L(t, q^i, \dot{q}^i)$ on $j_1(\mathcal{V}_{n+1})$, we have $\Omega_l = l^*(\tilde{\Omega}_l)$.

By means of the 2-form $\tilde{\Omega}_l$ and of the above mentioned trivialization φ_l, we may construct the equations of motion directly on the Lagrangian bundle $\mathcal{L}(\mathcal{V}_{n+1})$. The algorithm is based on the search for vector fields $\tilde{Z} \in D^1(\mathcal{L}(\mathcal{V}_{n+1}))$ satisfying the requirement

$$\tilde{Z} \lrcorner \tilde{\Omega}_l = -d\varphi_l . \tag{2.22}$$

In [**11**] the problem (2.22) has been proved to be mathematically equivalent to the standard one formulated on $j_1(\mathcal{V}_{n+1})$, both in the regular and in the singular case, relying on the cosymplectic (precosymplectic) structure (Ω_l, dt) through the equations

$$Z \rfloor \Omega_l = 0\,, \qquad \langle Z, dt \rangle = 1 \tag{2.23}$$

with unknown $Z \in D^1(j_1(\mathcal{V}_{n+1}))$. Indeed, eq. (2.22) admits a solution if and only if eqs. (2.23) do and the solutions of both problems are related in a natural way. More precisely, if Z solves eqs. (2.23) on a submanifold $N \subset j_1(\mathcal{V}_{n+1})$, its horizontal lift Z_l (see eq. (2.19)) satisfies eq. (2.22) on the submanifold $\pi^{-1}(N) \subset \mathcal{L}(\mathcal{V}_{n+1})$. Conversely, any solution of eq. (2.22) on a submanifold $M \subset \mathcal{L}(\mathcal{V}_{n+1})$ is necessarily the horizontal lift Z_l of a field Z solving eqs. (2.23) on $\pi(M) \subset j_1(\mathcal{V}_{n+1})$.

The advantages of formulating the problem of motion on $\mathcal{L}(\mathcal{V}_{n+1})$ through eq. (2.22) consist in the possibility of implementing the presymplectic constraint algorithm developed by Gotay and co-workers [**6**], [**7**], also for *time-dependent singular* Lagrangians.

First of all, in the present geometrical context, the constraint algorithm generates a decreasing sequence of *constraint manifolds*

$$\mathcal{L}(\mathcal{V}_{n+1}) \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots\,, \tag{2.24}$$

each embedded in the previous one, where

$$M_0 = \{ z \in \mathcal{L}(\mathcal{V}_{n+1}) \mid \dot{u} = L(\pi(z)) \} \tag{2.25}$$

is the surface image of the Lagrangian section l and, for $k \geqslant 1$, we have

$$M_k = \{ z \in M_{k-1} \mid d\varphi_l(z) \in (T(M_{k-1}))^\flat \}, \tag{2.26}$$

or, equivalently,

$$M_k = \{ z \in M_{k-1} \mid \langle TM_{k-1}^\perp, d\varphi_l \rangle (z) = 0 \}; \tag{2.27}$$

here $\flat : T(\mathcal{L}(\mathcal{V}_{n+1})) \to T^*(\mathcal{L}(\mathcal{V}_{n+1}))$ denotes the map $\flat(X) = X^\flat := X \rfloor \widetilde{\Omega}_l$, and $TM_k^\perp$ indicates the presymplectic complement of TM_k with respect to $\widetilde{\Omega}_l$.

The constraint algorithm is said to *stabilize* if and only if there exists an integer $k \geqslant 0$ such that $M_{k+1} = M_k$ and $\dim M_k > 0$. This means that eq. (2.22) possesses at least one differential solution along the *final constraint manifold* $M := M_k$ and that such a manifold is automatically maximal.

Now, whenever the constraint algorithm stabilizes, thus ensuring the solvability of the equations of motion at least in the differential sense, there is still one question left: we should determine what solutions are *kinematically admissible*.

This is known in literature as the SODE problem and has been solved in [**11**] in the time-dependent case under the «so-called» assumption of *admissibility* of the Lagrangian section l; the reader is referred to it for the details of the below summary.

In order to make the definition of admissibility clear, let us consider the involutive distribution (2) $D := \ker \Omega_l \cap V(j_1(\mathcal{V}_{n+1})) \subset T(j_1(\mathcal{V}_{n+1}))$, Ω_l denoting the Poincaré-Cartan 2-form on $j_1(\mathcal{V}_{n+1})$ generated by l, and $V(j_1(\mathcal{V}_{n+1}))$ being the vertical bundle associated with the fibration $j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$. A Lagrangian section $l: j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ is called *admissible* if and only if the leaf space $\mathfrak{J} := j_1(\mathcal{V}_{n+1})/D$ of the foliation generated by D admits a manifold structure such that the canonical projection $\varrho : j_1(\mathcal{V}_{n+1}) \to \mathfrak{J}$ is a submersion.

Whenever this is the case, we can lift D to the horizontal distribution $D_l \subset T(\mathcal{L}(\mathcal{V}_{n+1}))$, obtaining a quotient space $\mathfrak{L} := \mathcal{L}(\mathcal{V}_{n+1})/D_l$ with a manifold structure and such that the canonical projection $\xi : \mathcal{L}(\mathcal{V}_{n+1}) \to \mathfrak{L}$ is a submersion. More specifically, $\mathfrak{L}$ is a principal fiber bundle over $\mathfrak{J}$, whose structural group is isomorphic to $(\mathfrak{R}, +)$.

Moreover, it is easily seen that there exist a presymplectic 2-form $\overline{\Omega}_l$ over $\mathfrak{L}$ and a trivialization $\overline{\varphi}_l$ of the principal fiber bundle $\pi : \mathfrak{L} \to \mathfrak{J}$, such that $\widetilde{\Omega}_l = \xi^*(\overline{\Omega}_l)$ and $\varphi_l = \xi^*(\overline{\varphi}_l)$.

Therefore, a reduced problem of motion on the quotient space $\mathfrak{L}$, consisting in the search for vector fields $\overline{Z} \in D^1(\mathfrak{L})$ satisfying the requirement

$$\overline{Z} \dashv \overline{\Omega}_l = -d\overline{\varphi}_l \tag{2.28}$$

may be set-up.

The reduced problem (2.28) is intimately related to the primary problem (2.22). Indeed, an Equivalence Theorem, stating that eq. (2.22) admits a differential solution along some final constraint manifold M if and only if eq. (2.28) do, has been proved.

The importance of the Equivalence Theorem consists in ensuring the existence of *semi-prolongable* solutions of (2.22), namely solutions projecting to $\mathfrak{L}$ modulo the vertical bundle $V(\mathcal{L}(\mathcal{V}_{n+1}))$ associated with the fibration $\mathcal{L}(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$. The latters play a crucial role in the solution of the SODE problem; the argument is based on the following facts:

i) the restriction $D_{l|M}$ is an involutive distribution in TM, foliating M; the cor-

(2) We suppose systematically that rank $\left\| \dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right\| = r < n$ const.. This ensures that the rank of D is constant everywhere.

responding leaf space $\mathfrak{M} := M/D_{l|M}$ is a submanifold embedded in $\mathfrak{L}$ and the induced projection $\xi_M : M \to \mathfrak{M}$ is a submersion;

ii) given a vector field $X \in D^1(M)$ solving eq. (2.22), then $\tilde{J}(X) \in D^1(M)$.

As a consequence, it has been proved that, for any semi-prolongable solution $X \in D^1(M)$ of eq. (2.22) there exists a unique point n_X in each leaf of the foliation of M generated by $D_{l|M}$ where X is (π-related to) a SODE.

The union S_X of all the points n_X results to be a submanifold of M, diffeomorphic to the leaf space $\mathfrak{M}$, along which X uniquely splits in the sum $X = \overline{X} + V$ with $\overline{X} \in TS_X$ and $V \in D_{l|S_{[X]}}$. By construction, $\overline{X}$ is the unique vector field solving eq. (2.22) along S_X and π-related to a SODE (i.e. kinematically admissible).

We finally recall that two semi-prolongable solutions X and Y of (2.22) are said to be *$\tilde{J}$-equivalent* whenever $\tilde{J}(X) = \tilde{J}(Y)$. In this case the associated submanifolds S_X and S_Y coincide and we have $\overline{X} = \overline{Y}$.

3 - Vakonomic dynamics

3.1 - *Vakonomic systems*

Let us now subject the Lagrangian system introduced in § 2.1 to a set of nonholonomic constraints, thought as a submanifold $\mathcal{A}$ of $j_1(\mathcal{V}_{n+1})$, fibered over $\mathcal{V}_{n+1}$ and described in terms of a set of r functions on $j_1(\mathcal{V}_{n+1})$, as

$$g^\sigma(t, q^i, \dot{q}^i) = 0\,, \qquad \text{with} \qquad \operatorname{rank}\left(\frac{\partial g^\sigma}{\partial \dot{q}^i}\right) = r \tag{3.1}$$

Vakonomic Dynamics is aimed at studying the evolution of the system making use of a *constrained* variational principle. Indeed, in the vakonomic formulation, admissible motions of the system are singled out as extremals of the functional

$$\gamma(t) \to \int_{t_0}^{t_1} L(\dot{\gamma}(t))\, dt \tag{3.2}$$

where $\gamma(t) : [t_0, t_1] \to \mathcal{V}_{n+1}$ are sections — joining two any fixed points — whose first-jet extensions $\dot{\gamma}(t) : [t_0, t_1] \to j_1(\mathcal{V}_{n+1})$ satisfy the constraints, and whose (first-jet extensions of the) allowed deformations are also requested to lie on $\mathcal{A}$.

It has been shown (see, for example, [**1**], [**2**], [**14**]) that a section $\gamma(t) = (t, q^i(t))$ is a vakonomic solution of motion if and only if there exist r functions

$\lambda_\sigma : [t_0, t_1] \rightarrow \Re$, satisfying together with $\gamma(t)$ the set of equations

$$\text{(3.3a)} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = -\lambda_\sigma \left(\frac{d}{dt}\left(\frac{\partial g^\sigma}{\partial \dot{q}^i}\right) - \frac{\partial g^\sigma}{\partial q^i}\right) - \frac{d\lambda_\sigma}{dt}\frac{\partial g^\sigma}{\partial \dot{q}^i} \qquad i = 1,\dots,n$$

$$\text{(3.3b)} \qquad g^\sigma(\dot{\gamma}(t)) = 0 \qquad \sigma = 1, \dots, r\,.$$

As a matter of fact, it is also known that vakonomic equations of motion (3.3) may be derived as Euler-Lagrange equations of the *free* variational problem based on the functional

$$\text{(3.4)} \qquad \int_{t_0}^{t_1} (L + \lambda_\sigma g^\sigma)\, dt\,.$$

The latter involves the *extended Lagrangian* $\mathbb{L}(t, q^i, \dot{q}^i, \lambda_\sigma, \dot{\lambda}_\sigma) = L(t, q^i, \dot{q}^i) + \lambda_\sigma g^\sigma(t, q^i, \dot{q}^i)$, depending on additional variables $\lambda_\sigma, \dot{\lambda}_\sigma$. In the following we shall focus our attention on this last point.

3.2 - *The extended presymplectic framework*

As already mentioned, eqs. (3.3) may be viewed as Euler-Lagrange equations generated by a unconditioned variational problem.

Clearly, the Lagrangian $\mathbb{L}$ appearing in the functional (3.4) is always singular, at least in the variables $\dot{\lambda}_\sigma$.

Therefore, the geometrical approach to degenerate time-dependent Lagrangians outlined in § 2.2 suitably applies to the situation.

In the next subsection, we shall deal with the topic in a particular but significant case for several applications.

Before doing this, we need to spend a few words to introduce the extended geometrical framework where the procedure will be worked up.

To this end, let us consider the Cartesian product $P \times \Re^r$ between the bundle of affine scalars P and $\Re^r$. The reason for this choice will appear clearer in the subsequent discussion.

We refer $P \times \Re^r$ to local coordinates $t, q^i, u, \lambda_\sigma$, obeying the transformation laws

$$\text{(3.5)} \qquad \bar{t} = t + c\,, \qquad \bar{q}^i = \bar{q}^i(t, q^k)\,, \qquad \bar{u} = u + f(t, q^k)\,, \qquad \bar{\lambda}_\sigma = \bar{\lambda}_\sigma(\lambda_\gamma)\,.$$

Obviously, $P \times \Re^r$ is still fibered over the real line through the absolute-time function t. Then, we may consider the associated first-jet bundle $j_1(P \times \Re^r, \Re)$, endowed with jet-coordinates t, q^i, u, λ_σ, $\dot{q}^i$, $\dot{u}$, $\dot{\lambda}_\sigma$, subject to the transformation laws expressed by eqs. (3.5) together with the relations

$$\dot{\bar{q}}^i = \frac{\partial \bar{q}^i}{\partial q^k}\dot{q}^k + \frac{\partial \bar{q}^i}{\partial t}, \qquad \dot{\bar{u}} = \dot{u} + \frac{\partial f}{\partial q^k}\dot{q}^k + \frac{\partial f}{\partial t}, \qquad \dot{\bar{\lambda}}_\sigma = \frac{\partial \bar{\lambda}_\sigma}{\partial \lambda_\gamma}\dot{\lambda}_\gamma . \tag{3.6}$$

In addition to this, the following natural identification holds:

$$j_1(P \times \Re^r, \Re) \simeq j_1(P, \Re) \times T\Re^r. \tag{3.7}$$

Moreover, we notice that the manifold $P \times \Re^r$ inherits from P a principal fiber bundle structure over the Cartesian product $\mathcal{V}_{n+1} \times \Re^r$; the group actions (2.6), (2.7) may be lifted to the bundle $j_1(P \times \Re^r, \Re)$ in a natural way.

The quotient spaces of $j_1(P \times \Re^r, \Re)$ with respect to (the lift of) the actions (2.6) and (2.7) will be denoted by $\mathcal{L}(\mathcal{V}_{n+1} \times \Re^r)$ and $\mathcal{L}^c(\mathcal{V}_{n+1} \times \Re^r)$ respectively. Once again, the following identifications are straightforward

$$\mathcal{L}(\mathcal{V}_{n+1} \times \Re^r) \simeq \mathcal{L}(\mathcal{V}_{n+1}) \times T\Re^r, \qquad \mathcal{L}^c(\mathcal{V}_{n+1} \times \Re^r) \simeq \mathcal{L}^c(\mathcal{V}_{n+1}) \times T\Re^r. \tag{3.8}$$

Actions (2.6) and (2.7) commute. Therefore, we may follow a procedure similar to the one adopted in § 2.1 and make both $\mathcal{L}(\mathcal{V}_{n+1} \times \Re^r)$ and $\mathcal{L}^c(\mathcal{V}_{n+1} \times \Re^r)$ into principal fiber bundles over a double quotient space, easily identified with the first-jet bundle $j_1(\mathcal{V}_{n+1} \times \Re^r, \Re) \simeq j_1(\mathcal{V}_{n+1}) \times T\Re^r$.

The situation is summarized into the following commutative diagram

$$\begin{array}{ccc} j_1(P \times \Re^r, \Re) & \longrightarrow & \mathcal{L}^c(\mathcal{V}_{n+1} \times \Re^r) \\ \downarrow & & \downarrow \\ \mathcal{L}(\mathcal{V}_{n+1} \times \Re^r) & \longrightarrow & j_1(\mathcal{V}_{n+1} \times \Re^r, \Re) \end{array} \tag{3.9}$$

representing the analogous of (2.8) in the present «extended» geometrical context.

In a totally similar way, the Hamiltonian counterpart of this framework is achieved starting from the fibration $P \times \Re^r \to \mathcal{V}_{n+1} \times \Re^r$, taking its first-jet bundle $j_1(P \times \Re^r, \mathcal{V}_{n+1} \times \Re^r)$ into account and repeating the arguments pointed out in § 2.1.

Omitting the straightforward details, we let the reader verify that the resulting situation is summarized into the commutative diagram

$$\begin{array}{ccc} j_1(P \times \Re^r, \mathcal{V}_{n+1} \times \Re^r) & \longrightarrow & \mathcal{H}^c(\mathcal{V}_{n+1} \times \Re^r) \\ \downarrow & & \downarrow \\ \mathcal{H}(\mathcal{V}_{n+1} \times \Re^r) & \longrightarrow & \Pi(\mathcal{V}_{n+1} \times \Re^r) \end{array} \tag{3.10}$$

where, once again, all arrows indicate principal fibrations, whose structural groups are isomorphic to $(\Re, +)$.

We refer $j_1(P \times \Re^r, \mathcal{V}_{n+1} \times \Re^r)$ to jet-coordinates $t, q^i, u, \lambda_\sigma, p_0, p_i, p^\sigma$, subject to the transformations laws (3.5) together with

$$\overline{p}_0 = p_0 + \frac{\partial f}{\partial t} + \left(p_k + \frac{\partial f}{\partial q^k}\right)\frac{\partial q^k}{\partial \bar{t}}, \quad \overline{p}_i = \left(p_k + \frac{\partial f}{\partial q^k}\right)\frac{\partial q^k}{\partial \overline{q}^i}, \quad \overline{p}^\sigma = p_\gamma \frac{\partial \lambda_\gamma}{\partial \overline{\lambda}_\sigma}. \tag{3.11}$$

Then, we obtain the straightforward identifications

$$j_1(P \times \Re^r, \mathcal{V}_{n+1} \times \Re^r) \simeq j_1(P, \mathcal{V}_{n+1}) \times T^*\Re^r \tag{3.12a}$$

and

$$\mathcal{H}(\mathcal{V}_{n+1} \times \Re^r) \simeq \mathcal{H}(\mathcal{V}_{n+1}) \times T^*\Re^r, \qquad \mathcal{H}^c(\mathcal{V}_{n+1} \times \Re^r) \simeq \mathcal{H}^c(\mathcal{V}_{n+1}) \times T^*\Re^r. \tag{3.12b}$$

Let us now return to the *extended Lagrangian bundle* $\mathcal{L}(\mathcal{V}_{n+1} \times \Re^r)$. Any section $l : j_1(\mathcal{V}_{n+1} \times \Re^r, \Re) \to \mathcal{L}(\mathcal{V}_{n+1} \times \Re^r)$ will be called *extended Lagrangian section.*

In particular, we may consider l of the form

$$\dot{u} = \mathbb{L}(t, q^i, \dot{q}^i, \lambda_\sigma, \dot{\lambda}_\sigma) = L(t, q^i, \dot{q}^i) + \lambda_\sigma g^\sigma(t, q^i, \dot{q}^i) \tag{3.13}$$

involving a function $\mathbb{L}$ like the one appearing in the functional (3.4).

As pointed out in § 2.2, every such l bears:

- a trivialization $\varphi_l := \dot{u} - \mathbb{L}(t, q^i, \dot{q}^i, \lambda_\sigma, \dot{\lambda}_\sigma)$ of the principal fiber bundle $\mathcal{L}(\mathcal{V}_{n+1} \times \Re^r) \to j_1(\mathcal{V}_{n+1} \times \Re^r, \Re)$;
- a horizontal lift associating with every vector field $X = X^0 \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + X_\sigma \frac{\partial}{\partial \lambda_\sigma} + \dot{X}^i \frac{\partial}{\partial \dot{q}^i} + \dot{X}_\sigma \frac{\partial}{\partial \dot{\lambda}_\sigma} \in D^1(j_1(\mathcal{V}_{n+1} \times \Re^r, \Re))$ a corresponding vector field X_l on $\mathcal{L}(\mathcal{V}_{n+1} \times \Re^r)$, expressed locally as

$$X_l = X^0 \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + X_\sigma \frac{\partial}{\partial \lambda_\sigma} + \dot{X}^i \frac{\partial}{\partial \dot{q}^i} + \dot{X}_\sigma \frac{\partial}{\partial \dot{\lambda}_\sigma} + X(\mathbb{L}) \frac{\partial}{\partial \dot{u}} \tag{3.14}$$

- a (1, 1)-tensor field $\tilde{J}$ on $\mathcal{L}(\mathcal{V}_{n+1} \times \Re^r)$, having local expression

$$\tilde{J} = \omega^i \otimes \left(\left(\frac{\partial L}{\partial \dot{q}^i} + \lambda_\sigma \frac{\partial g^\sigma}{\partial \dot{q}^i}\right)\frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{q}^i}\right) + \omega_\sigma \otimes \frac{\partial}{\partial \dot{\lambda}_\sigma} \tag{3.15}$$

where $\omega_\sigma := d\lambda_\sigma - \dot{\lambda}_\sigma dt$, $\sigma = 1, \dots, r$;

- an exact 2-form $\widetilde{\Omega}_l$ on $\mathcal{L}(\mathcal{V}_{n+1}\times\Re^r)$, expressed in local fibered coordinates as

$$\widetilde{\Omega}_l := d\dot{u}\wedge dt + d\left(\frac{\partial L}{\partial \dot{q}^i} + \lambda_\sigma \frac{\partial g^\sigma}{\partial \dot{q}^i}\right)\wedge \omega^i - \left(\frac{\partial L}{\partial \dot{q}^i} + \lambda_\sigma \frac{\partial g^\sigma}{\partial \dot{q}^i}\right) d\dot{q}^i \wedge dt\,. \tag{3.16}$$

Once again, we may implement an extended problem of motion consisting in the search for vector fields $\widetilde{Z}\in D^1(\mathcal{L}(\mathcal{V}_{n+1}\times\Re^r))$ satisfying the requirement

$$\widetilde{Z}\,\lrcorner\,\widetilde{\Omega}_l = -d\varphi_l. \tag{3.17}$$

It is a straightforward matter to see that solving eqs (3.3) is mathematically equivalent to find kinematically admissible (SODE) solutions $\widetilde{Z}$ of (3.17).

3.3 - *The case of affine constraints*

The constraint algorithm outlined in § 2.2 applies to the study of problem (3.17), whatever the choice of L and g^σ is.

In this subsection we shall examine systems described by a *regular* Lagrangian $L(t, q, \dot{q})$ and subject to *affine* non-holonomic constraints, expressed as

$$g^\sigma(t, q, \dot{q}) = g^\sigma_i(t, q)\,\dot{q}^i + g^\sigma_0(t, q)\,. \tag{3.18}$$

As we shall see, these are common requirements in many applications. An explicit example of a more general system with a degenerate Lagrangian is proposed in § 4.

Lagrangian formalism. Under the stated assumptions, it is a straightforward matter to see that the 2-form (3.16), associated with the extended Lagrangian section (3.13), is presymplectic. In fact, its kernel is locally generated by the $2r$ vector fields

$$\frac{\partial}{\partial\lambda_\sigma} - a^{ij} g^\sigma_i \frac{\partial}{\partial\dot{q}^j} - a^{ij} g^\sigma_i\left(\frac{\partial L}{\partial\dot{q}^j} + \lambda_\gamma g^\gamma_j\right)\frac{\partial}{\partial\dot{u}}\,, \qquad \frac{\partial}{\partial\dot{\lambda}_\sigma} \tag{3.19}$$

where a^{ij} indicates the inverse matrix of $a_{ij} := \dfrac{\partial^2 L}{\partial\dot{q}^i\,\partial\dot{q}^j}$. Then, we may apply the presymplectic algorithm outlined in § 2.2.

In this connection, as pointed out in [**11**], there is no loss in generality in looking for solutions of (3.17) only on the surface $M_0 := \{z\in\mathcal{L}(\mathcal{V}_{n+1}\times\Re^r)\,|\,\dot{u} = \mathbb{L}(\pi(z))\}$, image of the section (3.13); in fact, every solution is automatically tangent to M_0 and also invariant under transport along the fibers of $\mathcal{L}(\mathcal{V}_{n+1} \times \Re^r)$.

A direct calculation shows that the constraint algorithm stabilizes at the first step, singling out the final constraint manifold $M := M_1 = \{z \in M_0 \mid g^\sigma(z) = 0\}$, where we denoted again by g^σ the pull-back of the functions (3.18) on $\mathcal{L}(\mathcal{V}_{n+1} \times \Re^r)$. Taking the first identification (3.8) into account, it is easily seen that M is (locally) diffeomorphic to the Cartesian product $\mathcal{A} \times T\Re^r$.

A direct application of the contents of §2.2 shows that the solution of the equations of motion (3.17) is provided by the vector fields $\tilde{Z} \in D^1(M)$ of the form

$$\tilde{Z} = Z + Z(\mathbb{L}) \frac{\partial}{\partial \dot{u}} \tag{3.20a}$$

with

$$Z = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + Z_\sigma \frac{\partial}{\partial \lambda_\sigma} + \dot{Z}^i \frac{\partial}{\partial \dot{q}^i} + \dot{Z}_\sigma \frac{\partial}{\partial \dot{\lambda}_\sigma} \tag{3.20b}$$

the components Z_σ and $\dot{Z}^i$ having explicit expression

$$Z_\sigma(t, q^i, \lambda_\sigma, \dot{q}^i) = b_{\sigma\gamma}\left(\frac{\partial g_k^\gamma}{\partial t} \dot{q}^k + \frac{\partial g_0^\gamma}{\partial t} + \frac{\partial g_k^\gamma}{\partial q^i} \dot{q}^k \dot{q}^i + \frac{\partial g_0^\gamma}{\partial q^i} \dot{q}^i + F^i g_i^\gamma \right) \tag{3.20c}$$

and

$$\dot{Z}^i(t, q^i, \lambda_\sigma, \dot{q}^i) = -a^{ij} Z_\sigma g_j^\sigma + F^i \tag{3.20d}$$

where

$$\begin{aligned} &F^i(t, q^i, \lambda_\sigma, \dot{q}^i) \\ &= a^{ij}\left(\frac{\partial L}{\partial q^j} - \frac{\partial^2 L}{\partial t \partial \dot{q}^j} - \frac{\partial^2 L}{\partial q^k \partial \dot{q}^j} \dot{q}^k - \lambda_\sigma \left(\frac{\partial g_j^\sigma}{\partial t} + \frac{\partial g_j^\sigma}{\partial q^k} \dot{q}^k - \frac{\partial g^\sigma}{\partial q^j} \right) \right) \end{aligned} \tag{3.20e}$$

and $b_{\gamma\sigma}$ is the inverse matrix of $b^{\gamma\sigma} = a^{ij} g_i^\sigma g_j^\gamma$.

In conclusion, there exists a whole family of solutions for eq. (2.22) along M, provided by the vector fields $\tilde{Z} \in D^1(M)$ satisfying requirements (3.20); as a consequence, the only arbitrariness in the description of any such $\tilde{Z}$ is placed in the components $\dot{Z}^\sigma(t, q, \lambda, \dot{q}, \dot{\lambda})$.

Moreover, under the stated assumptions, the Lagrangian section (3.13) is always admissible in the sense of § 2.2. More precisely, the regularity condition

$\left\| \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right\| \neq 0$ implies that both involutive distributions D and D_l are locally spanned by the vector fields $\frac{\partial}{\partial \dot{\lambda}_\sigma}$, $\sigma = 1, \ldots, r$ (considered as fields on $j_1(\mathcal{V}_{n+1} \times \Re^r, \Re)$ and $\mathcal{L}(\mathcal{V}_{n+1} \times \Re^r)$ respectively). What is more, taking identifications (3.7) and (3.8) into account again, the leaf spaces $\mathfrak{J} := j_1(\mathcal{V}_{n+1} \times \Re^r, \Re)/D$, $\mathfrak{L} := \mathcal{L}(\mathcal{V}_{n+1} \times \Re^r)/D_l$ and $\mathfrak{M} := M/D_{l|M}$ are respectively diffeomorphic to $j_1(\mathcal{V}_{n+1}) \times \Re^r$, $\mathcal{L}(\mathcal{V}_{n+1}) \times \Re^r$ and $\mathcal{A} \times \Re^r$. We refer $\mathfrak{L}$ to natural local coordinates $t, q, \dot{q}, \dot{u}, \lambda$.

We remark that all solutions (3.20) are *prolongable* (i.e. they project to the quotient space $\mathfrak{L}$ [8]) and are $\tilde{J}$-equivalent. Then, following § 2.2 (see [**11**] for more details), we may associate with the whole family of solutions (3.20) a unique submanifold $S \subset M$ on which each of them is a SODE. More in detail, it is easily seen that the submanifold S is the image space of the map (section) $\alpha : \mathfrak{M} \to M$ locally expressed as $(t, q^i, \dot{q}^i, \lambda_\sigma) \to (t, q^i, \dot{q}^i, \lambda_\sigma, \dot{\lambda}_\sigma = Z_\sigma(t, q, \dot{q}, \lambda))$.

Finally, still referring to [**11**], it is a straightforward matter to see that eq. (2.22) admits a unique SODE solution $\tilde{Z}$ along S; the latter is of the form (3.20a) with $Z = \overline{Z} + \overline{Z}(Z_\sigma) \frac{\partial}{\partial \dot{\lambda}_\sigma}$, being

$$\overline{Z} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + Z_\sigma \frac{\partial}{\partial \lambda_\sigma} + \dot{Z}^i \frac{\partial}{\partial \dot{q}^i} \tag{3.21}$$

and the components Z_σ and $\dot{Z}^i$ obeying eqs. (3.20c, d).

Hamiltonian formalism. A Hamiltonian description of the given system may be set up implementing the Legendre transformation induced by the extended Lagrangian section (3.13). To this end, we observe that the connection 1-form θ_l (see eq. (2.11)) assumes the local expression

$$\theta_l = \omega_0 - \left(\frac{\partial L}{\partial \dot{q}^i} + \lambda_\sigma g_i^\sigma \right) \omega^i \tag{3.22}$$

while $j_1(P \times \Re^r, \mathcal{V}_{n+1} \times \Re^r)$ is endowed with the canonical contact 1-form Θ, locally described as

$$\Theta = du - p_0\, dt - p_i\, dq^i - p^\sigma d\lambda_\sigma . \tag{3.23}$$

As briefly reminded at the end of § 2.1, the Legendre transformation is defined as the unique map $\Lambda : j_1(P \times \Re^r, \Re) \to j_1(P \times \Re^r, \mathcal{V}_{n+1} \times \Re^r)$, fibered over $P \times \Re^r$, satisfying $\Lambda^*(\Theta) = \theta_l$. A direct calculation shows that Λ is locally represented as

$$p_0 = \dot{u} - \left(\frac{\partial L}{\partial \dot{q}^i} + \lambda_\sigma g_i^\sigma \right) \dot{q}^i, \qquad p_i = \left(\frac{\partial L}{\partial \dot{q}^i} + \lambda_\sigma g_i^\sigma \right), \qquad p^\sigma = 0 \,. \tag{3.24}$$

It is straightforward to check that Λ induces a corresponding map $\widetilde{\Lambda} : \mathcal{L}(\mathcal{V}_{n+1} \times \Re^r) \to \mathcal{H}(\mathcal{V}_{n+1} \times \Re^r)$, still described by eqs. (3.24).

As a matter of fact, the reader may easily verify that:

a) the image space $P := \widetilde{\Lambda}(\mathcal{L}(\mathcal{V}_{n+1} \times \Re^r))$ is a submanifold of $\mathcal{H}(\mathcal{V}_{n+1} \times \Re^r)$, locally defined by the equations $p^\sigma = 0$;

b) $\widetilde{\Lambda}$ is a submersion on its image and its fibers are connected submanifolds of $\mathcal{L}(\mathcal{V}_{n+1} \times \Re^r)$.

Being the above properties verified, following the standard terminology, the extended Lagrangian section l is said *almost regular* [**7**].

The restriction of the 2-form $-d\Theta$ to the submanifold P gives rise to a pre-symplectic structure whose local expression is

$$\widehat{\Omega} = dp_0 \wedge dt + dp_i \wedge dq^i . \tag{3.25}$$

Moreover:

(i) the first relation (3.12b) shows that the submanifold P is locally identified with the Cartesian product $\mathcal{H}(\mathcal{V}_{n+1}) \times \Re^r$; the latter inherits a principal fiber bundle structure over $\Pi(\mathcal{V}_{n+1}) \times \Re^r$ in a natural way;

(ii) the regularity condition $\det \left\| \dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right\| \neq 0$ ensures the local invertibility of the relations $p_i = \left(\dfrac{\partial L}{\partial \dot{q}^i} + \lambda_\sigma g_i^\sigma \right)$ in terms of $\dot{q}^i = \dot{q}^i(t, q^k, p_k, \lambda_\sigma)$;

(iii) the image of M_0 under the map $\widetilde{\Lambda}$ may be locally viewed as the image space P_0 of a corresponding section $\widehat{h} : \Pi(\mathcal{V}_{n+1}) \times \Re^r \to \mathcal{H}(\mathcal{V}_{n+1}) \times \Re^r$, expressed as

$$p_0 = -\mathbb{H}(t, q^k, p_k, \lambda_\gamma) := \mathbb{L} - \left(\frac{\partial L}{\partial \dot{q}^i} + \lambda_\sigma g_i^\sigma \right) \dot{q}^i(t, q^k, p_k, \lambda_\gamma) \tag{3.26}$$

(iv) strictly associated with the section (3.26) there is the trivialization $\widehat{\sigma}_h := p_0 + \mathbb{H}$ of the bundle $\mathcal{H}(\mathcal{V}_{n+1}) \times \Re^r \to \Pi(\mathcal{V}_{n+1}) \times \Re^r$.

On the basis of the stated results, we may construct a problem of motion on P

through the equation

$$\widehat{Z} \dashv \widehat{\Omega} = -d\widehat{\sigma}_h \tag{3.27}$$

with unknown $\widehat{Z} \in D^1(P)$.

Once again, the problem (3.27) may be studied by means of the presymplectic constraint algorithm. As in the Lagrangian case, we may look for solutions of (3.27) only on the surface $P_0 := \{z \in P \,|\, \sigma_h(z) = 0\}$, image of the section $\widehat{h}$.

The Equivalence Theorem [8] — obviously adapted to the present context — implies that the constraint algorithm stops at the first step again, singling out the final constraint manifold $P_1 := \{z \in P_0 \,|\, \langle TP_0^{\perp}, d\sigma_h \rangle(z) = 0\}$ (see eq. (2.27)).

A direct calculation shows that locally $TP_0^{\perp} = Span\left\{ \dfrac{\partial}{\partial \lambda_\gamma} \right\}$ and, as a consequence, the submanifold P_1 is locally described as

$$\chi^\gamma := \frac{\partial \sigma_h}{\partial \lambda_\gamma} = -g_i^\gamma \dot{q}^i(t, q^k, p_k, \lambda_\sigma) - g_0^\gamma = 0\,, \tag{3.28}$$

Assuming that the condition $\det \left\| \dfrac{\partial \chi^\gamma}{\partial \lambda_\sigma} \right\| \neq 0$ is satisfied everywhere (3), eqs. (3.28) may be locally solved with respect to the λ_σ, i.e. $\lambda_\sigma = \lambda_\sigma(t, q^i, p_i)$. This allows to give a local representation of the submanifold P_1 as the image space of a map $\tilde{h} : \Pi(\mathcal{V}_{n+1}) \to P$, described by the relations

$$\lambda_\sigma = \lambda_\sigma(t, q^i, p_i)\,, \qquad p_0 = -H(t, q^i, p_i) := -\mathbb{H}(t, q^i, p_i, \lambda_\sigma(t, q^i, p_i)). \tag{3.29}$$

As a consequence, the problem (3.27) may be pulled-back on $\Pi(\mathcal{V}_{n+1})$, since eq. (3.27) implies

$$\widehat{Z} \dashv \tilde{h}^*(\widehat{\Omega}) = 0\,, \qquad \langle \widehat{Z}, dt \rangle = 1 \tag{3.30}$$

on P_1 ($\simeq \Pi(\mathcal{V}_{n+1})$). Noticing that $\tilde{h}^*(\widehat{\Omega}) = -dH \wedge dt + dp_i \wedge dq^i$, eqs. (3.30) describe a standard Hamiltonian system associated with the Hamiltonian H on the phase space $\Pi(\mathcal{V}_{n+1})$.

It follows that eqs. (3.30) admit as unique solution the Hamiltonian flow

$$\widehat{Z} = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}\,. \tag{3.31}$$

(3) Such a condition is always verified when the Lagrangian is quadratic in the variables $\dot{q}^i$.

We notice that, passing to the Hamiltonian context, a sort of reduction of the problem is possible: indeed, the degrees of freedom of the system have been reduced from $n + r$ to n. A striking aspect of the above result is the possibility of applying all the standard methods of free Hamiltonian systems such as Hamilton-Jacobi theory, Marsden-Weinstein reduction etc.

Remark. Following [4] and adapting the argument to the present context, it is easy to prove that P_1 defines a set of second class constraints and that the canonical Poisson Bracket on P_1 is exactly the restriction of the Dirac bracket defined on $\mathcal{H}(\mathcal{V}_{n+1} \times \mathfrak{R}^r)$.

4 - Examples

Example 1. Consider a homogeneous disk of radius R and mass m rolling without sliding on a horizontal plane and constrained to remain vertical. Introduce coordinates x, y, φ and θ where x and y indicate the position of the center of mass, φ denotes the angle between the tangent of the disk at the point of contact and the x-axis and θ the angle of rotation of the disk around its center. A time-dependent viscous resistance force $\boldsymbol{F} = -\dfrac{1}{2}\beta(t)\, \boldsymbol{v}_p$ acts on each point p of the disk.

We are interested in finding the evolutions of the disk which make the work of the resistance force between any two instants t_0 and t_1

$$\int_{t_0}^{t_1} \frac{1}{2}\beta(t)\big(\dot{x}^2 + \dot{y}^2 + I_1\dot{\varphi}^2 + I_2\dot{\theta}^2\big)\, dt$$

stationary. I_1 and I_2 denote the moments of inertia (normalized by the constant density of the disk).

The constraints due to the rolling condition are

$$g_1 := \dot{x} - R\,\dot{\theta}\cos\varphi = 0\,, \qquad g_2 := \dot{y} - R\,\dot{\theta}\sin\varphi = 0\,.$$

The above constrained variational problem may be handled as a free one associated with the extended Lagrangian

$$\mathbb{L} = \frac{1}{2}\beta(t)\big(\dot{x}^2 + \dot{y}^2 + I_1\dot{\varphi}^2 + I_2\dot{\theta}^2\big) + \lambda_1 g_1 + \lambda_2 g_2.$$

Following the procedure described in § 3.3, the constraint algorithm stabilizes on the final constraint manifold $g_1 = g_2 = 0$ where there exists a whole family of solutions of the form (3.20) with

$$\dot{Z}^1 = -\frac{1}{\beta} Z_1 - \frac{\beta'}{\beta}\dot{x} \tag{4.1a}$$

$$\dot{Z}^2 = -\frac{1}{\beta} Z_2 - \frac{\beta'}{\beta}\dot{y} \tag{4.1b}$$

$$\dot{Z}^3 = \frac{1}{\beta I_1}\left(-\beta' I_1 \dot{\varphi} + \lambda_1 R\dot{\theta}\sin\varphi - \lambda_2 R\dot{\theta}\cos\varphi\right) \tag{4.1c}$$

$$\dot{Z}^4 = \frac{1}{\beta I_2}\left(R\cos\varphi Z_1 + R\sin\varphi Z_2 - \beta' I_2\dot{\theta} - \lambda_1 R\dot{\varphi}\sin\varphi + \lambda_2 R\dot{\varphi}\cos\varphi\right) \tag{4.1d}$$

$$\begin{aligned} Z_1 = \beta R\dot{\varphi}\dot{\theta}\sin\varphi + \frac{1}{I_2+R^2}[&-\beta'\dot{x}(I_2+R^2\sin^2\varphi) + RI_2\beta'\dot{\theta}\cos\varphi \\ &+\beta' R^2\dot{y}\cos\varphi\sin\varphi + \lambda_1 R^2\dot{\varphi}\cos\varphi\sin\varphi - \lambda_2 R^2\dot{\varphi}\cos^2\varphi] \end{aligned} \tag{4.1e}$$

$$\begin{aligned} Z_2 = -\beta R\dot{\varphi}\dot{\theta}\cos\varphi + \frac{1}{I_2+R^2}[&-\beta'\dot{y}(I_2+R^2\cos^2\varphi) + RI_2\beta'\dot{\theta}\sin\varphi \\ &+\beta' R^2\dot{x}\cos\varphi\sin\varphi - \lambda_2 R^2\dot{\varphi}\cos\varphi\sin\varphi + \lambda_1 R^2\dot{\varphi}\sin^2\varphi] \end{aligned} \tag{4.1f}$$

where everything must be evaluated on the final constraint manifold $g_1 = g_2 = 0$.

According to § 3.3 there exists a unique submanifold S: $\dot{\lambda}_1 - Z_1 = 0$, $\dot{\lambda}_2 - Z_2 = 0$ of the constraint manifold where the SODE solution given by (the corresponding of) eq. (3.21) is unique. Eqs. (4.1) yield the evolution laws for the unknowns $x(t)$, $y(t)$, $\varphi(t)$, $\theta(t)$, $\lambda_1(t)$ and $\lambda_2(t)$.

Example 2. In the following example we shall apply the proposed geometrical construction to a simple economic model. Given two firms, we denote by q^1, q^2 the amount of goods produced by each of them and by $\dot{q}^1$, $\dot{q}^2$ their rate of production.

Under the hypothesis that the law ruling the unitary cost of production is $e^{\beta t}\frac{1}{2}A^i\dot{q}^i$, $i = 1, 2$ (A^i = const., β = constant discount rate), the functional expressing the total cost of production between two instants t_0, t_1 is

$$C = \int_{t_0}^{t_1} e^{\beta t} \frac{1}{2} [A^1 (\dot{q}^1)^2 + A^2 (\dot{q}^2)^2] \, dt \,. \tag{4.2}$$

We are interested in finding the stationary points of the total cost functional (4.2) subject to the following constraint

$$g := \dot{q}^2 - D + B(q^1 + q^2) \tag{4.3}$$

where D and B are suitable constants.

The constraint (4.3) could be meant as a law imposition aimed at reducing the rate of production of the second firm in order to stop the production when the percentage B of the total production equals the assigned quantity D.

The extended Lagrangian associated with the problem is now

$$\mathbb{L} = e^{\beta t} \frac{1}{2} [A^1 (\dot{q}^1)^2 + A^2 (\dot{q}^2)^2] + \lambda g \,.$$

The final constraint manifold is described by the equation $g = 0$, where there exists a whole family of solutions of the form (3.20) with

$$Z = e^{-\beta t} A^2 [B(\dot{q}^1 + \dot{q}^2) + \beta \dot{q}^2] + \lambda B \tag{4.4a}$$

$$\dot{Z}^1 = \beta \dot{q}^1 + \frac{\lambda B e^{\beta t}}{A^1} \tag{4.4b}$$

$$\dot{Z}^2 = -B(\dot{q}^1 + \dot{q}^2) \tag{4.4c}$$

where, once again, everything must be evaluated on the final constraint manifold $g = 0$.

As above, $S : \dot{\lambda} - Z = 0$ is the unique submanifold of the final constraint manifold where the SODE solution given by (the corresponding of) eq. (3.21) is unique.

The Hamiltonian description of the problem may be achieved by implementing the Legendre transformation (3.24) and following the subsequent discussion. Leaving the details to the reader, it is a straightforward matter to see that the «redu-

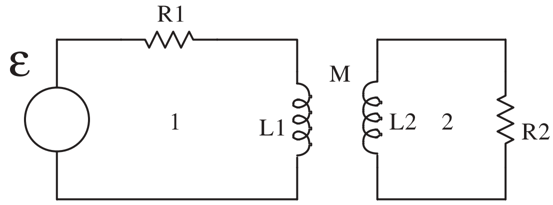

Figure 1. – Circuit of Example 3.

ced» Hamiltonian (3.29) is expressed as

$$H(t, q^1, q^2, p_1, p_2) = \frac{p_1^2 e^{\beta t}}{2A^1} + p_2[D - B(q^1 + q^2)] - \frac{e^{-\beta t} A^2}{2}[D - B(q^1 + q^2)]^2$$

yielding the corresponding Hamilton equations

$$\dot{q}^1 = \frac{p_1 e^{\beta t}}{A^1}, \quad \dot{q}^2 = D - B(q^1 + q^2), \quad \dot{p}_1 = \dot{p}_2 = B[p_2 - e^{-\beta t} A^2 (D - B(q^1 + q^2))].$$

The latters are simpler to solve than their Lagrangian counterpart (4.4), since they can be decoupled more easily.

Example 3. In this last example we present a system whose Lagrangian is degenerate, in order to show how the machinery works in its full generality; for simplicity, we deal with a time-independent problem.

Consider an analog circuit composed by a voltage generator coupled to a load through a transformer as illustrated in Fig. 1 where: R_1 and R_2 denote resistors, L_1 and L_2 the self-inductances of the two solenoids, M the mutual-inductance between them and ε the e.m.f. supplied by a generator.

The circuit is ruled by the Kirchoff laws

(4.5a) $$g_1 := \varepsilon - R_1 I_1 - L_1 \dot{I}_1 - M \dot{I}_2 = 0$$

(4.5b) $$g_2 := R_2 I_2 + L_2 \dot{I}_2 + M \dot{I}_1 = 0$$

where I_1 and I_2 are the currents circulating in the loops 1 and 2 respectively.

Considering eqs. (4.5) as imposed kinetic constraints, our problem is to find the stationary points of the total thermal energy

$$E = \int_{t_0}^{t_1} \left[\frac{(\varepsilon - L_1 \dot{I}_1 - M \dot{I}_2)^2}{R_1} + \frac{(L_2 \dot{I}_2 + M \dot{I}_1)^2}{R_2} \right] dt$$

dissipated by the resistors between any two instants t_0, t_1 (4).

The associated extended Lagrangian is

$$\mathbb{L} = \frac{(\varepsilon - L_1 \dot{I}_1 - M \dot{I}_2)^2}{R_1} + \frac{(L_2 \dot{I}_2 + M \dot{I}_1)^2}{R_2} + \lambda_1 g_1 + \lambda_2 g_2 .$$

Following the procedure briefly outlined in § 2.2, we obtain:

- the constraint algorithm stops at the third step singling out a final constraint manifold $M := M_3$, described by the Cartesian equations

$$g_1 = 0, \quad g_2 = 0, \quad \lambda_1 = -2 I_1, \quad M R_2 \lambda_2 + R_1 L_2 \lambda_1 = 0,$$

$$M R_2 \dot{I}_2 + R_1 R_2 I_1 - L_2 R_1 \dot{I}_1 = 0$$

- the discussion of the SODE problem identifies a unique submanifold $S \subset M$, defined by the additional equations

$$\dot{\varepsilon} = R_1 \dot{I}_1 - \frac{M R_2}{L_2} \dot{I}_2 + \left(L_1 - \frac{M^2}{L_2} \right) \alpha I_1, \quad \dot{\lambda}_1 = -2 \dot{I}_1, \quad \dot{\lambda}_2 = 2 \dot{I}_2 + 2 \frac{R_1}{M} I_1$$

where there exists a unique SODE solution of the form (3.20a) with $Z = \overline{Z} + \overline{Z}(\dot{\varepsilon}) \frac{\partial}{\partial \dot{\varepsilon}} + \overline{Z}(\dot{\lambda}_1) \frac{\partial}{\partial \dot{\lambda}_1} + \overline{Z}(\dot{\lambda}_2) \frac{\partial}{\partial \dot{\lambda}_2}$, being

$$\overline{Z} = \frac{\partial}{\partial t} + \dot{I}_1 \frac{\partial}{\partial I_1} + \dot{I}_2 \frac{\partial}{\partial I_2} + \dot{\varepsilon} \frac{\partial}{\partial \varepsilon}$$

$$+ \dot{\lambda}_1 \frac{\partial}{\partial \lambda_1} + \dot{\lambda}_2 \frac{\partial}{\partial \lambda_2} + \alpha I_1 \frac{\partial}{\partial \dot{I}_1} - \frac{1}{L_2} (R_2 \dot{I}_2 + M \alpha I_1) \frac{\partial}{\partial \dot{I}_2}$$

(4) The choice of using the expression $\frac{V^2}{R}$ instead of RI^2 for the power dissipated by the resistors is made in order to have a non-linear (and thus totally singular) extended Lagrangian.

with

$$\alpha := \frac{R_1 R_2^2}{R_2 M^2 + R_1 L_2^2}\,.$$

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Abstract

Vakonomic systems may be considered singular Lagrangian ones, using the multipliers as additional variables. The recent geometrical approach to degenerate Lagrangians, developed in the framework of Lagrangian bundles, is here applied to the study of time-dependent vakonomic dynamics. Some illustrative applications to Mechanics, Economy and analog circuits are given.

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