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Holomorphic maps of Hartogs domains into complex space forms (**)

1 - Introduction and preliminaries

The study of holomorphic and isometric immersions of a Kähler manifold (M, g) into a finite or infinite dimensional complex space form started with Calabi [1] to whom we refer for details and further results (see also [2], [4], [5], [7], [8]). There are three types of complex space forms, depending on the sign of (the constant) holomorphic sectional curvature:

(i) the complex Euclidean space C^N , $N \leq \infty$ with the canonical metric denoted by G_{can} of zero holomorphic sectional curvature;

(ii) the complex projective space CP_b^N ($b > 0$ and $N \leq \infty$) with the Fubini-Study metric denoted by $G_{FS}(b)$ of positive holomorphic sectional curvature $4b$;

(iii) the complex hyperbolic space CP_b^N ($b < 0$ and $N \leq \infty$), namely the domain $B \subset C^N$ given by

$$B = \left\{ (z_1, z_2, \dots, z_n) \in C^N \mid \sum_{j=1}^N |z_j|^2 < -\frac{1}{b} \right\}$$

endowed with the hyperbolic metric denoted by $G_{hyp}(b)$ of negative holomorphic sectional curvature $4b$.

The first important result due to Calabi [1] is the following:

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Theorem 1.1. *If a Kähler manifold (M, g) admits a holomorphic and isometric immersion into a complex space form then g is real analytic.*

If a Kähler metric g on M is real analytic, then in a neighborhood of every point $p \in M$, one can introduce a very special Kähler potential D_p for the metric g , which Calabi christened *diastasis*. Recall that a Kähler potential is an analytic function Φ defined in a neighborhood of a point p such that $\omega = \frac{i}{2} \bar{\partial} \partial \Phi$, where ω is the Kähler form associated to g . In a complex coordinate system (z) around p one has:

$$g_{\alpha\bar{\beta}} = 2g \left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta} \right) = \frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta}.$$

A Kähler potential is not unique: it is defined up to the sum with the real part of a holomorphic function. By duplicating the variables z and \bar{z} a potential Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in N \times \bar{N}$ (here \bar{N} denotes the manifold conjugated of N). The *diastasis function* is the Kähler potential D_p around p defined by

$$D_p(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

Since D_p is real analytic one can consider its power series developments:

$$(1) \quad D_p(z, \bar{z}) = \sum_{j, k \geq 0} \alpha_{jk} z^{m_j} \bar{z}^{m_k}.$$

Here we are using the following convention: we arrange every n -tuple of non-negative integers as the sequence $m_j = (m_{1,j}, m_{2,j}, \dots, m_{n,j})_{j=0,1,\dots}$ such that $m_0 = (0, \dots, 0)$, $|m_j| \leq |m_{j+1}|$, with $|m_j| = \sum_{\alpha=1}^n m_{\alpha,j}$ and $z^{m_j} = \prod_{\alpha=1}^n (z_\alpha)^{m_{\alpha,j}}$.

Example 1.2. Let p be the origin in \mathbb{C}^N . Then the diastasis at p is given by:

$$D_p(q) = |p - q|^2, \quad \forall q \in \mathbb{C}^N.$$

Example 1.3. Let (Z_0, Z_1, \dots, Z_N) be the homogeneous coordinates in $\mathbb{C}P_b^N$, $b > 0$ and let $p = [1, 0, \dots, 0]$. In the affine chart $U_0 = \{Z_0 \neq 0\}$ endowed with coordinates (z_1, \dots, z_n) , $z_j = \frac{Z_j}{Z_0}$ the diastasis at p reads as:

$$(2) \quad D_p(z_j, \bar{z}_j) = \frac{1}{b} \log \left(1 + b \sum_{j=1}^n |z_j|^2 \right).$$

If one takes $b < 0$, formula (2) define the diastasis at p of the complex hyperbolic space CP_b^N , $b < 0$.

We are now ready to state the general criterium due to Calabi [1] for a Kähler manifold to admit a holomorphic and isometric immersion into a complex space form. This is expressed by Theorem 1.5 and Theorem 1.6 below. First we need the following:

Definition 1.4. *Let (M, g) be a real analytic Kähler manifold and let p be a point in M . We say that the Kähler metric g is resolvable of rank N at p if the $\infty \times \infty$ matrix a_{jk} given by formula (1) is positive semidefinite and of rank N . If $N = \infty$ we say that the Kähler metric g is resolvable of infinite rank.*

Theorem 1.5 (see Calabi [1]). *Let (M, g) be a real analytic Kähler manifold.*

(i) *if g is resolvable of rank N at $p \in M$ then it is resolvable of rank N at every point in M ;*

(ii) *suppose that M is simply-connected. Then (M, g) admits a holomorphic and isometric immersion into C^N if and only if g is resolvable of rank at most N ;*

(iii) *let $\varphi : M \rightarrow C^N$ be a holomorphic and isometric immersion which is full (i.e. the image $\varphi(M)$ is not contained in any hyperplane of C^N), then N is determined by the metric g and two such immersions are congruent under the unitary group $U(N)$.*

Now, we consider the case of holomorphic immersions into C_b^N . Let D_p be the diastasis relative to a point $p \in M$. Consider the «modified diastasis» $\frac{1}{b}(e^{bD_p} - 1)$ and its power series development:

$$(3) \quad \frac{1}{b}(e^{bD_p} - 1) = \sum_{j, k \geq 0} b_{jk} z^{m_j} \bar{z}^{m_k}.$$

We say that the metric g is b -resolvable of rank N at p , if the $\infty \times \infty$ matrix b_{jk} given by formula (3) is positive semidefinite and of rank N .

Theorem 1.6 (see Calabi [1]). *Let (M, g) be a real analytic Kähler manifold and let b a real number different from 0.*

(i) *if g is b -resolvable of rank N at $p \in M$ then it is resolvable of rank N at every point in M ;*

(ii) *suppose that M is simply-connected. Then (M, g) admits a holomorphic and isometric immersion into CP_b^N if and only if g is b -resolvable of rank at most N ;*

(iii) let $\varphi : M \rightarrow \mathbb{C}P_b^N$ be a holomorphic and isometric immersion which is full (i.e. the image $\varphi(M)$ is not contained in any hyperplane of $\mathbb{C}P_b^N$). Then N is determined by the metric g and the constant b and two such immersions are congruent under the isometry group of $\mathbb{C}P_b^N$.

In this paper we study the holomorphic and isometric immersions of a Hartogs domain (H_F, g_F) (see Sect. 2) into a complex space form. The main results of this paper are contained in Sect. 2 and 3. In Section 2 we give a necessary and sufficient condition for (H_F, g_F) to admit a holomorphic and isometric immersion into a complex space form (see Theorem 2.1.1 and Theorem 2.2.1). Moreover, we prove that (H_F, g_F) cannot be isometrically immersed either into \mathbb{C}^N or $\mathbb{C}P_b^N$ for $b > 0$ and N finite (see Corollaries 2.1.2 and 2.2.2). The previous result can be considered as an extension of a result of Calabi [1] (see Remark 2.2.5). In Section 3 we prove that if g_F satisfies the Einstein condition then (H_F, g_F) is biholomorphically isometric to $\mathbb{C}P_{-1}^2$.

2 - The main results

Let $F : [0, x_0] \rightarrow (0, +\infty]$ be a non increasing C^2 function from the interval $[0, x_0] \subset \mathbb{R}$ to the extended positive reals $(0, +\infty]$ (the case $x_0 = +\infty$ is not excluded). The Hartogs domain corresponding to the function F is the 2-complex dimensional manifold $H_F \subset \mathbb{C}^2$ defined as:

$$(4) \quad H_F = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 < x_0, |z_2|^2 < F(|z_1|^2)\}.$$

In the hypothesis that $F(0) < \infty$, one can define a real 2-form on H_F by

$$(5) \quad \omega_F = \frac{i}{2} \partial \bar{\partial} \log \frac{1}{F(|z_1|^2) - |z_2|^2}.$$

Theorem 2.1 (cf. [3]). *The following conditions are equivalent:*

- (i) ω_F is a Kähler form.
- (ii) $\left(\frac{x F'}{F}\right)' < 0, \forall x \in [0, x_0)$, (where F' denotes the first derivative of F).
- (iii) ∂H_F , the boundary of H_F , is strictly pseudoconvex.

Proof. Let $\omega_F = \frac{i}{2} \sum_{j, k=1}^2 g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$ be the expression of the Kähler form

ω_F in the (global) coordinates (z_1, z_2) . A simple calculation shows that

$$\begin{aligned} g_{1\bar{1}} &= \frac{-HF' - H|z_1|^2 F'' + |z_1|^2 F'^2}{H^2} \Big|_{x=|z_1|^2}, \\ \bar{g}_{1\bar{2}} = g_{2\bar{1}} &= \frac{-F'}{H^2} z_1 \bar{z}_2 \Big|_{x=|z_1|^2}, \\ g_{2\bar{2}} &= \frac{F}{H^2} \Big|_{x=|z_1|^2}, \end{aligned}$$

where H is the real valued function on H_F defined by $H(z_1, z_2) = F(|z_1|^2) - |z_2|^2$. It follows that:

$$(6) \quad \det g_{\alpha\bar{\beta}} = g_{1\bar{1}}g_{2\bar{2}} - |g_{1\bar{2}}|^2 = -\frac{F^2}{H^3} \left(\frac{x F'}{F} \right)' \Big|_{x=|z_1|^2}.$$

The form ω_F is Kähler if and only if the matrix $g_{\alpha\bar{\beta}}$ is positive definite and, since $g_{2\bar{2}} > 0$, this is the case if and only if $\det g_{\alpha\bar{\beta}} > 0$. By (6) this condition turns out to be equivalent to condition (ii) in Proposition 2.1. This shows the equivalence between (i) and (ii). The equivalence between (ii) and (iii) can be found in [3]. ■

In the sequel we will suppose ω_F is a Kähler form and will denote by g_F the corresponding Kähler metric on H_F . Furthermore, we will suppose that g_F is real analytic. Let $p = (0, 0)$ be the origin in \mathbb{C}^2 . Then the diastasis at p , globally defined in $H_F \times \overline{H_F}$, is given by:

$$(7) \quad D_p(z, \bar{z}) = \log \frac{1}{F(|z_1|^2) - |z_2|^2}.$$

2.1. - Holomorphic immersions into \mathbb{C}^N

Define

$$(8) \quad C(\varrho_1, \varrho_2) = \log \frac{1}{F(\varrho_1) - \varrho_2}.$$

Since by hypothesis F is real analytic function it follows that the function C is real analytic in the open set

$$\{(\varrho_1, \varrho_2) \in \mathbb{R}^2 \mid \varrho_1 < \sqrt{x_0}, \varrho_2 < \sqrt{F(\varrho_1)}\}.$$

Hence (8) can be expanded in power series

$$(9) \quad C(\varrho_1, \varrho_2) = \sum_{j,k=0}^{+\infty} c_{jk} \varrho_1^j \varrho_2^k = \sum_{j,k=0}^{+\infty} \frac{\partial^{j+k} C}{\partial \varrho_1^j \partial \varrho_2^k} \varrho_1^j \varrho_2^k.$$

Therefore,

$$D_p(z, \bar{z}) = C(|z_1|^2, |z_2|^2) = \sum_{j,k=0}^{+\infty} c_{jk} |z_1|^{2j} |z_2|^{2k}.$$

Consequently, the $\infty \times \infty$ matrix a_{jk} given by formula (1) is diagonal, more precisely $a_{jk} = \delta_{jk} c_{m_j}$, where $m_j = (m_{1,j}, m_{2,j})$ (with the notation at page 2). Since H_F is simply-connected (even contractible) by Theorem 1.5 one easily gets:

Theorem 2.1.1. *The Hartogs domain H_F endowed with the Kähler metric g_F admits a holomorphic and isometric full immersion into \mathbb{C}^N , $N \leq \infty$ iff N among the c_{jk} 's, given by (9), are positive and all other are zero.*

Corollary 2.1.2. *The Hartogs domain (H_F, g_F) cannot admit a holomorphic and isometric map into \mathbb{C}^N for N finite.*

Proof. Suppose that there exists a holomorphic and isometric immersion of (H_F, g_F) into \mathbb{C}^N with N finite. Then, by Theorem 2.1.1 only finitely many c_{jk} 's would be strictly greater than zero. On the other hand,

$$c_{0k} = \frac{\partial^k C}{\partial \varrho_2^k} (p) = (F(0))^{-k} > 0 \quad \forall k,$$

which gives the desired contradiction. ■

Remark 2.1.3. Theorem 2.1.1 gives an infinite number of conditions which involve the derivatives of all orders of the function F at $x = 0$. For example $c_{10} \geq 0$ is equivalent to $\frac{\partial C}{\partial \varrho_1} (p) = -\frac{F'(0)}{F(0)} \geq 0$, which is automatically satisfied being $F(0) > 0$ and being F a non increasing function. The first non trivial condition comes from $c_{20} \geq 0$. In fact

$$c_{20} = \frac{\partial^2 C}{\partial \varrho_1^2} (p) = \frac{(F'(0))^2 - F''(0) F(0)}{F(0)^2} \geq 0,$$

i.e.

$$(10) \quad F''(0) \leq \frac{(F'(0))^2}{F(0)}.$$

Example 2.1.4. Let $F(x) = e^{-x}$, $x \in [0, +\infty)$. It is immediate to verify that condition (ii) in Proposition 2.1 is satisfied and hence ω_F is a Kähler form on H_F . This domain is considered also in [3], p. 451 and it is called the *Spring domain*.

The function C given by (8) reads, in this case, as:

$$C(\varrho_1, \varrho_2) = -\log(e^{-\varrho_1} - \varrho_2) = \varrho_1 + \sum_{j=0}^{+\infty} \sum_{k=1}^{+\infty} \frac{k^{j-1}}{j!} \varrho_1^j \varrho_2^k.$$

Then $c_{00} = c_{j0} = 0$, $\forall j > 2$, $c_{10} = 1$, and

$$c_{jk} > 0, \quad \forall j \geq 0, \quad \forall k > 1.$$

Therefore, by Theorem 2.1.1, the Spring domain admits a holomorphic and isometric immersion into \mathbb{C}^∞ .

Example 2.1.5. Consider the function $F(x) = e^{-x} + 2$, $x \in [0, 1)$. Since

$$\left(\frac{x F'}{F} \right)' = -\frac{1 + 2e^x(1-x)}{(1 + 2e^x)^2} < 0, \quad \forall x \in [0, 1),$$

it follows that the condition (ii) in Proposition 2.1 is satisfied and g_F is a Kähler metric on H_F . On the other hand,

$$F''(0) = 1 > \frac{1}{3} = \frac{(F'(0))^2}{F(0)}.$$

Therefore condition (10) is not satisfied, and so (H_F, g_F) cannot be holomorphically and isometrically immersed into \mathbb{C}^N for any $N \leq \infty$.

2.2. - Holomorphic immersions into CP_b^N

Define the function

$$(11) \quad C(\varrho_1, \varrho_2) = \frac{1}{b} (F(\varrho_1) - \varrho_2)^{-b} - 1,$$

which is real analytic on the open set

$$\{(\varrho_1, \varrho_2) \in \mathbb{R}^2 \mid \varrho_1 < \sqrt{x_0}, \varrho_2 < \sqrt{F(\varrho_1)}\}.$$

It follows that

$$D_p(z, \bar{z}) = \sum_{j, k=1}^{+\infty} c_{jk} |z_1|^{2j} |\bar{z}_1|^{2k}$$

where $c_{jk} = \frac{\partial C^{j+k}}{\partial \varrho_1^j \partial \varrho_2^k}(p)$. Consequently the $\infty \times \infty$ matrix b_{jk} given by formula (3) is diagonal, more precisely $b_{jk} = \delta_{jk} c_{m_j}$ where $m_j = (m_{1j}, m_{2j})$. Since H_F is simply-connected by Theorem 1.6 one gets:

Theorem 2.2.1. *The Hartogs domain H_F endowed with the Kähler metric g_F admits a holomorphic and isometric full immersion into $\mathbb{C}P_b^N$, $N \leq \infty$ iff N among the c_{jk} 's, given by formula (3), are positive and all other are zero.*

Corollary 2.2.2. *The Hartogs domain (H_F, g_F) cannot admit a holomorphic and isometric immersion into the finite dimensional complex projective space, $\mathbb{C}P_b^N$ ($b > 0$ and N finite).*

Proof. Suppose that there exists a holomorphic and isometric immersion of (H_F, g_F) into the complex projective space $\mathbb{C}P_b^N$ with N finite. Then, by Theorem 2.2.1 only finitely many c_{jk} 's would be strictly greater than zero. On the other hand, it is immediate to verify that $c_{0k} = \frac{\partial^k C}{\partial \varrho_2^k}(p) > 0, \forall k$, the desired contradiction. ■

Example 2.2.3. Let $b = 1$ and $F(x) = e^{-x}$, $x \in [0, +\infty)$. The function C given by (11) reads as:

$$C(\varrho_1, \varrho_2) = \frac{1}{e^{-\varrho_1} - \varrho_2} - 1 = \sum_{j, k=0}^{+\infty} \frac{(k+1)^j}{j!} \varrho_1^j \varrho_2^k.$$

Thus $c_{jk} > 0, \forall j, k$ and, by Theorem 2.2.1, the Spring domain admits a holomorphic and isometric map in $\mathbb{C}P_1^\infty$.

Remark 2.2.4. Let $b = -1$. The function C given by (11) reads as:

$$C(\varrho_1, \varrho_2) = 1 + \varrho_2 - F(\varrho_1) = 1 + \varrho_2 - \sum_{j=0}^{+\infty} F_j \varrho_1^j,$$

where

$$F_j = \frac{\partial^j F}{\partial x^j}(0).$$

Then the matrix b_{jk} given by formula (3) is positive semidefinite iff $F_j \leq 0$. So, for example the Spring domain cannot admit a holomorphic and isometric immersion into the hyperbolic space $\mathbb{C}P_{b < 0}^N$ for any $N \leq \infty$, since the second derivative of e^{-x} at 0 is negative.

Remark 2.2.5. Observe that if $F(x) = 1 - x$ then (H_F, g_F) is equal to the 2-dimensional hyperbolic space $\mathbb{C}P_{-1}^2$. Thus, Corollaries 2.1.2 and 2.2.2 can be considered as a generalization of a result due to Calabi [1] Theorem 13, which asserts that $\mathbb{C}P_{-1}^2$ cannot admit a holomorphic and isometric immersion into \mathbb{C}^N and \mathbb{C}_b^N for $b > 0$ and N finite.

3 - The Einstein condition

Theorem 3.1. *Let H_F be a Hartogs domain with strictly pseudoconvex boundary endowed with its Kähler metric g_F given by Theorem 2.1. Suppose that g_F is Einstein. Then (H_F, g_F) is biholomorphically isometric to the 2-complex hyperbolic space $\mathbb{C}P_{-1}^2$.*

We first prove an elementary lemma

Lemma 3.2. *Let ϕ be a holomorphic function on an open set $U \subset \mathbb{C}$ containing the origin. Suppose that there exists a real analytic function $f: (-x_0, x_0) \rightarrow \mathbb{R}$ such that $|\phi(z)|^2 = f(|z|^2)$. Then $\phi(z)$ reduces to the constant $\phi(0)$.*

Proof. Let $\phi(z) = \sum_{j=0}^{+\infty} a_j z^j$ be the power series expansion of ϕ at the origin, and $f(x) = \sum_{l=0}^{+\infty} b_l x^l$ be the Taylor expansion of f at the origin. By hypothesis,

$$\sum_{j, k=0}^{+\infty} a_j \bar{a}_k z^j \bar{z}^k = \sum_{l=0}^{+\infty} b_l |z|^{2l},$$

which implies that all the terms of the form $a_j \bar{a}_k z^j \bar{z}^k$ with $j \neq k$, are zero. It follows that $a_j = 0$ for $j > 0$, and so the result. ■

Proof of Theorem 3.1. If g_F is Kähler-Einstein, then

$$(12) \quad \varrho_{\omega_F} = -i\partial\bar{\partial} \log \det g_{\alpha\bar{\beta}} = \lambda \omega_F = \lambda \frac{i}{2} \partial\bar{\partial} \log \frac{1}{H} = -\frac{i}{2} \partial\bar{\partial} \log H^\lambda,$$

where λ is the scalar curvature and ϱ_{ω_F} is the Ricci form (see [6]). Thus

$$\partial\bar{\partial} \log(H^{-\frac{\lambda}{2}} \det g_{j\bar{k}}) = 0.$$

Since the domain H_F is simply connected there exists a holomorphic function ϕ on H_F such that

$$H^{-\frac{\lambda}{2}} \det g_{j\bar{k}} = |\phi|^2.$$

Therefore, by formula (6) above, one gets:

$$|\phi|^2 = -\frac{F^2}{H^{\frac{\lambda}{2}+3}} \left(\frac{x F'}{F} \right)' \Big|_{x=|z_1|^2} = -\frac{(F' + |z_1|^2 F'') F - |z_1|^2 F'^2}{H^{\frac{\lambda}{2}+3}} \Big|_{x=|z_1|^2}.$$

Since the Kähler metric g_F is Einstein it is also real analytic and hence the function F is real analytic in $(-x_0, x_0)$. By Lemma 3.2, being ϕ holomorphic, one can deduce that the function ϕ equals a constant, say C . Therefore

$$(13) \quad \frac{(F' + |z_1|^2 F'') F - |z_1|^2 F'^2}{H^{\frac{\lambda}{2}+3}} = -C^2.$$

Observe that the numerator of (13) depends only on $|z_1|^2$, while the denominator depends also on $|z_2|^2$. Then formula (13) makes sense if and only if $\lambda = -6$ and

$$(14) \quad (F' + x F'') F - x F'^2 = -C^2, \quad \forall x \in (-x_0, x_0).$$

Taking the first derivative of (14) at zero one gets $2F(0)F''(0) = 0$. Since $F(0) \neq 0$, it follows that $F''(0) = 0$. Taking the higher order derivatives of (14) at zero one obtains

$$0 = \frac{\partial^k ((F' + x F'') F - x F'^2)}{\partial x^k} (0) = (k+1) F(0) \frac{\partial^k F}{\partial x^k} (0), \quad k \geq 1,$$

and so $\frac{\partial^k F}{\partial x^k} (0) = 0$. Using again the analyticity of F one immediately obtains

that $F(x) = \alpha - \beta x$, where α and β are positive constants. Then the map

$$\varphi : H_F \rightarrow \mathbb{C}P_{-1}^2 : (z_1, z_2) \mapsto \left(\sqrt{\frac{\beta}{\alpha}} z_1, \sqrt{\frac{1}{\alpha}} z_2 \right)$$

is the desired biholomorphism satisfying

$$\varphi^*(G_{hyp}(-1)) = g_F. \quad \blacksquare$$

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Abstract

Let H_F be a Hartogs domain with strictly pseudoconvex boundary endowed with its natural Kähler metric g_F (see Sect. 2). Following Calabi [1] we give necessary and sufficient conditions for (H_F, g_F) to admit a holomorphic and isometric map into a finite or infinite dimensional complex space form. Moreover we prove that, if g_F is Einstein, then (H_F, g_F) is biholomorphically isometric to the unit ball endowed with the hyperbolic metric.
