Riv. Mat. Univ. Parma (7) **1** (2002), 91-102

Giovanni Cimatti (*)

Forced oscillations in piezoelectric crystals (**)

1 - Introduction

In this paper we prove existence and uniqueness of a unique periodic solution for the hyperbolic-elliptic system of partial differential equations, which models the forced oscillations in a piezoelectric viscoelastic body. We refer to [4] for a detailed desciption of the basic equations.

Let Ω, the region occupied by the body, be an open and bounded subset of $\boldsymbol{R}^3$ with a boundary Γ of class C^2. We denote by n_k the unit vector normal to Γ and by $|\Omega|$ the volume of Ω. The forcing term $\boldsymbol{f}(x, t)$ is a function defined in $\Omega \times \boldsymbol{R}^1$, periodic in t with period T. To write the relevant equations we define the operators:

$$(\boldsymbol{A}\boldsymbol{u})_i = -(a_{ijlm} u_{l,m})_{,j} \tag{1.1}$$

$$C\phi = -(d_{kl}\phi_{,l})_{,k} \tag{1.2}$$

$$D_i(\phi, \boldsymbol{u}) = -d_{il}\phi_{,l} + e_{ilm} u_{l,m} \tag{1.3}$$

$$\boldsymbol{E}\boldsymbol{u} = -(e_{klm} u_{l,m})_{,k} \qquad (B\phi)_i = -(e_{ijl}\phi_{,l})_{,j} \quad i = 1, 2, 3. \tag{1.4}$$

(*) Department of Mathematics, University of Pisa, Via Buonarroti 2, 56100 Pisa, Italy.
(**) Received May 9th, 2002, and in revised form September 11th 2002. AMS classification 35 B 10, 35 M 10.

The index summation convention has been used and $\frac{\partial v_i}{\partial x_j} = v_{i,j}$. The vector D_i is the electric induction and a_{ijlm} the fourth order elastic tensor which is assumed to satisfy

$$a_{ijlm} = a_{lmij} = a_{jilm} = a_{ijml} \tag{1.5}$$

$$a_{ijlm} X_{ij} X_{lm} \geqslant \alpha X_{ij} X_{ij}, \quad \alpha > 0, \quad X_{ij} \in \boldsymbol{R}^1, \quad X_{ij} = X_{ji}. \tag{1.6}$$

The third order piezoelectric tensor e_{ijl} and the dielectric tensor d_{kl} obey the conditions:

$$e_{ijl} = e_{jil} = e_{ilj} \tag{1.7}$$

$$d_{kl} = d_{lk}, \quad d_{kl} \xi_k \xi_l \geqslant \delta |\xi|^2, \quad \xi \in \boldsymbol{R}^3. \tag{1.8}$$

The elastic displacement and the electric potential are denoted by $\boldsymbol{u}(x, t)$ and $\phi(x, t)$, respectively. We study the following problem: to find $\boldsymbol{u}(x, t)$ and $\phi(x, t)$ such that

$$\boldsymbol{u}'' + A\boldsymbol{u} + B\phi + \beta(\boldsymbol{u}') = \boldsymbol{f} \tag{1.9}$$

$$\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma \tag{1.10}$$

$$\boldsymbol{u}(x, t) = \boldsymbol{u}(x, t+T) \tag{1.11}$$

$$C\phi - E\boldsymbol{u} = 0 \tag{1.12}$$

$$D_k n_k = 0 \quad \text{on} \quad \Gamma \tag{1.13}$$

$$\phi(x, t) = \phi(x, t+T) \tag{1.14}$$

$$\int_\Omega \phi(x, t)\, dx = 0. \tag{1.15}$$

The body is supposed to be electrically insulated on the boundary and condition (1.13) reflects this fact. The electric potential ϕ is defined apart an arbitrary constant, which is normalized with condition (1.15). The viscosity of the medium is modelled by a continuous strictly monotonic map $\beta(\boldsymbol{\xi})$ from $\boldsymbol{R}^3$ to $\boldsymbol{R}^3$, on which we assume: (i) there exists $\delta > 0$ and $h > 0$ such that $\beta_i(\boldsymbol{\xi})\boldsymbol{\xi}_i \geqslant h|\boldsymbol{\xi}|^{\varrho+1}$ if $|\boldsymbol{\xi}| \geqslant \delta$, (ii) there exists $k > 0$ and $K > 0$ such that $|\beta(\boldsymbol{\xi})| \leqslant K + k|\boldsymbol{\xi}|^\varrho$ for all $\boldsymbol{\xi} \in \boldsymbol{R}^3$, $\varrho \geqslant 1$.

We use a method proposed by G. Prodi in [**6**] to find the apriori estimates on $\boldsymbol{u}(t)$ needed to prove existence of a periodic solution. The trick lies in definig new unknown functions $\boldsymbol{v}(t)$ and $\psi(t)$ with zero means with respect to t:

$$\boldsymbol{v}(t) = \boldsymbol{u}(t) - \overline{\boldsymbol{u}}, \qquad \psi(t) = \phi(t) - \overline{\phi}. \tag{1.16}$$

Here and hereafter an overbar denotes the mean over one period. Averaging the equations (1.9)-(1.15) over one period we find

$$A(\overline{\boldsymbol{u}}) + B(\overline{\phi}) + \overline{\beta}(\boldsymbol{v}') = \overline{\boldsymbol{f}} \tag{1.17}$$

$$\overline{\boldsymbol{u}} = 0 \quad \text{on} \quad \Gamma \tag{1.18}$$

$$C(\overline{\phi}) - E(\overline{\boldsymbol{u}}) = 0 \tag{1.19}$$

$$D_k(\overline{\phi}, \overline{\boldsymbol{u}})\, n_k = 0 \quad \text{on} \quad \Gamma \tag{1.20}$$

$$\int_\Omega \overline{\phi}(x)\, dx = 0. \tag{1.21}$$

It follows that $\boldsymbol{v}$ and ψ satisfy the problem

$$\boldsymbol{v}'' + A\boldsymbol{v} + B\psi + \beta(\boldsymbol{v}') - \overline{\beta}(\boldsymbol{v}') = \boldsymbol{g} \tag{1.22}$$

$$\boldsymbol{v} = 0 \quad \text{on} \quad \Gamma \tag{1.23}$$

$$C\psi - E\boldsymbol{v} = 0 \tag{1.24}$$

$$D_k(\psi, \boldsymbol{v})\, n_k = 0 \quad \text{on} \quad \Gamma \tag{1.25}$$

$$\int_\Omega \psi(x, t)\, dx = 0, \tag{1.26}$$

where $\boldsymbol{g} = \boldsymbol{f} - \overline{\boldsymbol{f}}$.

2 - Weak formulation and existence of periodic solution

The scalar product in $L^2(\Omega)$ and in $(L^2(\Omega))^3$ is denoted by $(,)$ and the corresponding norm by $\| \ \|$. Spaces of vector-valued functions are denoted by boldface. If B is a Banach space and $1 \leqslant p < \infty$, the set of functions $u(t)$ with values in B, periodic with period T, such that

$$\int_0^T \|u(t)\|_B^p\, dt < \infty$$

is denoted by $\boldsymbol{L}^p(T; B)$. We define the space

$$V = \left\{ \phi \in H^1(\Omega), \int_\Omega \phi(x, t)\, dx = 0 \right\}$$

and the bilinear forms: $a(\boldsymbol{u}, \boldsymbol{v})$, $b(\phi, \boldsymbol{u})$ and $c(\phi, \psi)$ on $\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega)$, $V \times \boldsymbol{H}_0^1(\Omega)$, and $V \times V$ respectively, by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega a_{ijlm} u_{l,m} v_{i,j}\, dx \tag{2.1}$$

$$b(\phi, \boldsymbol{u}) = \int_\Omega e_{ijl} \phi_{,l} u_{i,j}\, dx \tag{2.2}$$

$$c(\phi, \psi) = \int_\Omega d_{ij} \phi_{,i} \psi_{,j}\, dx. \tag{2.3}$$

They are all bounded and $a(\boldsymbol{u}, \boldsymbol{v})$, $c(\phi, \psi)$ are also symmetric and coercive by (1.5)-(1.8). The operators

$$A : \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega) \to \boldsymbol{L}^2(\Omega) \tag{2.4}$$

$$C : V \cap H^2(\Omega) \to L^2(\Omega) \tag{2.5}$$

defined in (1.1) and (1.2) correspond to $a(\boldsymbol{u}, \boldsymbol{v})$ and $c(\phi, \psi)$. The operators

$$B : V \cap H^2(\Omega) \to \boldsymbol{L}^2(\Omega), \qquad E : \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega) \to L^2(\Omega), \tag{2.6}$$

defined in (1.4), correspond to $b(\phi, \boldsymbol{u})$ via

$$(B\phi, \boldsymbol{u}) = b(\phi, \boldsymbol{u}) = (E\boldsymbol{u}, \phi). \tag{2.7}$$

We assume:

$$\boldsymbol{f}(t) \in L^{\frac{\varrho+1}{\varrho}}(T; \boldsymbol{L}^{\frac{\varrho+1}{\varrho}}(\Omega)) \tag{2.8}$$

$$a_{ijlm} \in L^\infty(\Omega), \quad e_{ijl} \in L^\infty(\Omega), \quad d_{ij} \in L^\infty(\Omega) \tag{2.9}$$

and intend (1.6) and (1.8) to hold a.e. in Ω. The weak formulation of problem

(1.17)-(1.26) is the following: to find $\boldsymbol{v}$, ψ and $\overline{\boldsymbol{u}}$, $\overline{\phi}$ such that

$$\boldsymbol{v}(t) \in L^{\infty}(T; \boldsymbol{H}_0^1(\Omega)) \tag{2.10}$$

$$\boldsymbol{v}'(t) \in L^{\varrho+1}(T; \boldsymbol{L}^{\varrho+1}(\Omega)) \cap L^{\infty}(T; \boldsymbol{L}^2(\Omega)) \tag{2.11}$$

$$\overline{\boldsymbol{v}} = 0 \tag{2.12}$$

$$(\boldsymbol{v}''(t), \boldsymbol{\nu}) + a(\mathbf{v}(t), \boldsymbol{\nu}) + b(\psi(t), \boldsymbol{\nu}) + (\beta(\mathbf{v}') - \overline{\beta}(\boldsymbol{v}'), \boldsymbol{\nu}) = (\boldsymbol{g}(t), \boldsymbol{\nu}) \tag{2.13}$$

for all $\boldsymbol{\nu} \in \boldsymbol{L}^{\varrho+1}(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$

$$\psi(t) \in L^{\infty}(T; V) \tag{2.14}$$

$$c(\psi(t), \eta) = b(\eta, \boldsymbol{v}(t)) \tag{2.15}$$

for all $\eta \in V$

$$\overline{\boldsymbol{u}} \in \boldsymbol{H}_0^{1, \frac{\varrho+1}{\varrho}}(\Omega) \tag{2.16}$$

$$\overline{\phi} \in H^{1, \frac{\varrho+1}{\varrho}}(\Omega), \qquad \int_{\Omega} \overline{\phi}\, dx = 0 \tag{2.17}$$

$$a(\overline{\boldsymbol{u}}, \boldsymbol{w}) + b(\overline{\phi}, \boldsymbol{w}) + (\overline{\beta}(\boldsymbol{v}'), \boldsymbol{w}) = (\overline{\boldsymbol{f}}, \boldsymbol{w}) \tag{2.18}$$

for all $\boldsymbol{w} \in \boldsymbol{L}^{\varrho+1}(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$

$$c(\overline{\phi}, \eta) = b(\eta, \overline{\boldsymbol{u}}) \tag{2.19}$$

for all $\eta \in H^{1,\varrho+1}(\Omega)$ such that $\int_{\Omega} \eta dx = 0$.

Instead of (2.13) and (2.15) we can take, as an equivalent weak formulation,

$$\int_0^T \{ -(\boldsymbol{v}'(t), \boldsymbol{\gamma}'(t)) + a(\boldsymbol{v}(t), \boldsymbol{\gamma}(t)) + b(\psi(t), \boldsymbol{\gamma}(t)) + (\beta(\boldsymbol{v}') - \overline{\beta}(\boldsymbol{v}'), \boldsymbol{\gamma}(t)) \}\, dt = \int_0^T (\boldsymbol{g}(t), \boldsymbol{\gamma}(t))\, dt \tag{2.20}$$

for all $\boldsymbol{\gamma}(t) \in L^{\infty}(T; \boldsymbol{H}_0^1(\Omega)) \cap L^{\varrho+1}(T; \boldsymbol{L}^{\varrho+1}(\Omega))$, $\boldsymbol{\gamma}'(t) \in L^{\varrho+1}(T; \boldsymbol{L}^{\varrho+1}(\Omega)) \cap L^{\infty}(T; \boldsymbol{L}^2(\Omega))$

$$\int_0^T \{c(\psi(t), \zeta(t)) - b(\zeta(t), \boldsymbol{v}(t))\}\, dt = 0 \tag{2.21}$$

for all $\zeta(t) \in L^2(T, V)$.

Theorem. *There exists one and only one solution to problem* (2.10)-(2.21).

Proof. We apply the Faedo-Galerkin method. Let $\{\boldsymbol{w}_k\}$ be a sequence of functions of class $C_0^{\infty}(\Omega)$ free and total in $\boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^{\varrho+1}(\Omega)$. For each m we define an approximate solution

$$\boldsymbol{v}_m(t) = \sum_{k=1}^{m} c_{mk}(t)\, \boldsymbol{w}_k$$

and determine $\psi_m(t) \in V$ as the unique solution to the following problem

$$\psi_m(t) \in V \quad c(\psi_m(t), \eta) = b(\eta, \boldsymbol{v}_m(t)) \tag{2.22}$$

for all $\eta \in V$. The coefficients $c_{mk}(t)$ are computed with the system of ordinary differential equations

$$(\boldsymbol{v}''_m(t), \boldsymbol{w}_k) + a(\boldsymbol{v}_m(t), \boldsymbol{w}_k) + b(\psi_m(t), \boldsymbol{w}_k) + (\beta(\boldsymbol{v}'_m) - \overline{\beta}(\boldsymbol{v}'_m), \boldsymbol{w}_k) = (\boldsymbol{g}(t), \boldsymbol{w}_k). \tag{2.23}$$

To prove that there exists one and only one solution to the system (2.22), (2.23) we apply the Leray-Schauder method, considering the following auxiliary problem P_λ containing the parameter $\lambda \in [0, 1]$:

$$(\boldsymbol{v}''_m(t), \boldsymbol{w}_k) + a(\boldsymbol{v}_m(t), \boldsymbol{w}_k) + b(\psi_m(t), \boldsymbol{w}_k) + (\boldsymbol{v}'_m(t), \boldsymbol{w}_k)$$

$$= \lambda(\boldsymbol{v}'_m(t), \boldsymbol{w}_k) - \lambda(\beta(\boldsymbol{v}'_m) - \overline{\beta}(\boldsymbol{v}'_m), \boldsymbol{w}_k) + (\boldsymbol{g}(t), \boldsymbol{w}_k),\ k = 1, 2, \dots, m. \tag{2.24}$$

$$\psi_m(t) \in V \quad c(\psi_m(t), \eta) = b(\eta, \boldsymbol{v}_m(t)).$$

If $\lambda = 0$ the linear system P_o has one and only one solution periodic of period T. Resonance phenomena are in fact excluded by the presence of the dissipative term $(\boldsymbol{v}'_m(t), \boldsymbol{w}_k)$. Since all possible periodic solutions of period T of problems P_λ are «a priori» bounded indipendently of λ, we conclude that problem P_1, i.e. (2.22), (2.23), admits one and only one periodic solution of period T. Moreover, integrating (2.23) over one period and taking into account that $\bar{\boldsymbol{g}} = 0$, we obtain

$$a(\bar{\boldsymbol{v}}_m, \boldsymbol{w}_k) + b(\bar{\psi}_m, \boldsymbol{w}_k) = 0.$$

By (2.22) this implies

$$a(\bar{\boldsymbol{v}}_m, \bar{\boldsymbol{v}}_m) + c(\bar{\psi}_m, \bar{\psi}_m) = 0,$$

hence $\bar{\boldsymbol{v}}_m = 0$. We proceed to find an apriori estimate for $\{\boldsymbol{v}'_m(t)\}$. Taking the time derivative in (2.22) and choosing $\eta = \psi_m(t)$ in the resulting equation, we have

$$\frac{1}{2}\frac{d}{dt} c(\psi_m(t), \psi_m(t)) = b(\psi_m(t), \boldsymbol{v}'_m(t)). \tag{2.25}$$

Let us multiply (2.23) by $c'_{mk}(t)$, add for $k = 1, \dots m$ and recall (2.25). Defining

$$\xi_m(t) = \frac{1}{2}\|\boldsymbol{v}'_m(t)\|^2 + \frac{1}{2} a(\boldsymbol{v}_m(t), \boldsymbol{v}_m(t)) + \frac{1}{2} c(\psi_m(t), \psi_m(t)) \tag{2.26}$$

we obtain

$$\xi'_m(t) + (\beta(\boldsymbol{v}'_m(t)) - \bar{\beta}(\boldsymbol{v}'_m), \boldsymbol{v}'_m(t)) = (\boldsymbol{g}(t), \boldsymbol{v}'_m(t)) \tag{2.27}$$

and, integrating over one period,

$$\int_0^T (\beta(\boldsymbol{v}'_m(t)), \boldsymbol{v}'_m(t))\, dt = \int_0^T (\boldsymbol{g}(t), \boldsymbol{v}'_m(t))\, dt. \tag{2.28}$$

By assumption (i) we find easily that there exists a constant C_1 such that

$$\int_0^T \|\boldsymbol{v}'_m(t)\|^{\varrho+1}_{\boldsymbol{L}^{\varrho+1}(\Omega)}\, dt \leqslant C_1. \tag{2.29}$$

Since $\bar{\boldsymbol{v}}_m = 0$ we have also

$$\max_{t \in [0,T]} \|\boldsymbol{v}_m(t)\|_{\boldsymbol{L}^{\varrho+1}(\Omega)} \leqslant C_2. \tag{2.30}$$

The easy proof of estimate (2.30) is the main reason for adopting the present method. Let us multiply (2.23) by $c_{mk}(t)$, sum over k and integrate over one period. We obtain

$$\int_0^T \{-\|\boldsymbol{v}_m'(t)\|^2 + a(\boldsymbol{v}_m(t), \boldsymbol{v}_m(t)) + c(\psi_m(t), \psi_m(t)) + (\beta(\boldsymbol{v}_m'(t), \boldsymbol{v}_m(t))\}\, dt = \int_0^T (\boldsymbol{g}(t), \boldsymbol{v}_m(t))\, dt. \tag{2.31}$$

Use has been made of (2.22) with $\eta = \psi_m(t)$. Then (1.7), (1.9) and assumption (ii) yield, by the Hoelder inequality,

$$\alpha\|\boldsymbol{v}_m\|^2_{L^2(T;\boldsymbol{H}_0^1(\Omega))} + d\|\psi_m(t)\|^2_{L^2(T;V)} \leqslant \|\boldsymbol{v}_m'\|^2_{L^2(T;\boldsymbol{L}^2(\Omega))} + K|\Omega|^{\frac{1}{2}}T^{\frac{1}{2}}\|\boldsymbol{v}_m\|_{L^2(T;\boldsymbol{L}^2(\Omega))} + k\|v_m'\|^{\varrho}_{L^{\varrho+1}(T;\boldsymbol{L}^{\varrho+1}(\Omega))}\|\boldsymbol{v}_m\|_{L^{\varrho+1}(T;\boldsymbol{L}^{\varrho+1}(\Omega))} + \|\boldsymbol{g}\|_{L^{\frac{\varrho+1}{\varrho}}(T;\boldsymbol{L}^{\frac{\varrho+1}{\varrho}}(\Omega))}\|\boldsymbol{v}_m\|_{L^{\varrho+1}(T;\boldsymbol{L}^{\varrho+1}(\Omega))}. \tag{2.32}$$

Recalling (2.29) and (2.30) we find

$$\int_0^T \|\boldsymbol{v}_m(t)\|^2_{\boldsymbol{H}_0^1(\Omega)} \leqslant C_3 \tag{2.33}$$

and

$$\int_0^T \|\psi_m(t)\|^2_V\, dt \leqslant C_4. \tag{2.34}$$

Let us integrate (2.27) over an arbitrary interval $[\tau, t]$ and then integrate again the resulting equation, with respect to τ, over an interval of periodicity. We obtain

$$T\xi_m(t) + \int_0^T \int_\tau^t (\beta(\boldsymbol{v}_m'(\lambda)), \boldsymbol{v}_m'(\lambda))\, d\lambda\, d\tau = \int_0^T \left\{\xi_m(\tau) + \int_\tau^t (\overline{\beta}(\boldsymbol{v}_m'), \boldsymbol{v}_m'(\lambda))\, d\lambda + \int_\tau^t (\boldsymbol{g}(\lambda), \boldsymbol{v}_m'(\lambda))\, d\lambda\right\} d\tau. \tag{2.35}$$

The left hand side in (2.35) is estimated from below using (ii), whereas the right hand side can be majorized using (2.29), (2.30) and (2.33). Therefore, there exists a constant C_5 such that

$$\max_{t\in[0,T]}\{\|\boldsymbol{v}_m'(t)\|^2 + \|\boldsymbol{v}_m(t)\|^2_{\boldsymbol{H}_0^1(\Omega)} + \|\psi_m(t)\|^2_V\} \leqslant C_5. \tag{2.36}$$

All constants C_i depends only on the data. It follows that from $\{\boldsymbol{v}_m\}$ and $\{\psi_m\}$ it is possible to extract two subsequences, not relabelled, such that

(2.37) $$\boldsymbol{v}_m \to \boldsymbol{v} \quad \text{weak} * \text{ in } L^\infty(T; \boldsymbol{H}_0^1(\Omega))$$

(2.38) $$\boldsymbol{v}'_m \to \boldsymbol{v}' \text{ weak} * \text{ in } L^\infty(T; \boldsymbol{L}^2(\Omega)) \text{ and weakly in } L^{\varrho+1}(T; \boldsymbol{L}^{\varrho+1}(\Omega))$$

(2.39) $$\psi_m \to \psi \text{ in } L^2(T; V) \text{ weakly}.$$

In addition, by (ii), there exists $\boldsymbol{h}$ such that:

(2.40) $$\beta(\boldsymbol{v}'_m) \to \boldsymbol{h} \text{ weakly in } L^{\frac{\varrho+1}{\varrho}}(T; \boldsymbol{L}^{\frac{\varrho+1}{\varrho}}(\Omega))$$

and

(2.41) $$\overline{\beta}(\boldsymbol{v}'_m) \to \overline{\boldsymbol{h}} \text{ weakly in } \boldsymbol{L}^{\frac{\varrho+1}{\varrho}}(\Omega).$$

By the assumed properties of $\{\boldsymbol{w}_k\}$ we have, recalling (2.23):

(2.42) $$\int_0^T \{-(\boldsymbol{v}'(t), \boldsymbol{\gamma}'(t)) + a(\boldsymbol{v}(t), \boldsymbol{\gamma}(t)) + b(\psi(t), \boldsymbol{\gamma}(t)) + (\boldsymbol{h}(t) - \overline{\boldsymbol{h}}, \boldsymbol{\gamma}(t))\}\, dt = \int_0^T (\boldsymbol{g}(t), \boldsymbol{\gamma}(t))\, dt$$

for all $\boldsymbol{\gamma}(t) \in L^\infty(T; \boldsymbol{H}_0^1(\Omega))$ and $\boldsymbol{\gamma}'(t) \in L^{\varrho+1}(T; \boldsymbol{L}^{\varrho+1}(\Omega)) \cap L^\infty(T; \boldsymbol{L}^2(\Omega))$. From (2.23) we obtain, by (2.39),

(2.43) $$c(\psi(t), \eta) = b(\eta, \boldsymbol{v}(t))$$

for all $\eta \in V$. It remains to prove that

$$\boldsymbol{h}(t) = \beta(\boldsymbol{v}'(t)).$$

Taking formally the time derivative in (2.43), setting $\eta = \psi(t)$ in the resulting equation and $\boldsymbol{\gamma}(t) = \boldsymbol{v}'(t)$ in (2.42), we obtain, by periodicity,

(2.44) $$\int_0^T (\boldsymbol{h}(t) - \overline{\boldsymbol{h}}, \boldsymbol{v}'(t))\, dt = \int_0^T (\boldsymbol{g}(t), \boldsymbol{v}'(t))\, dt.$$

To prove rigorously (2.44) we note that we have, in the distributional sense,

(2.45) $$\boldsymbol{v}'' + A\boldsymbol{v} + B\psi + \boldsymbol{h} - \overline{\boldsymbol{h}} = \boldsymbol{g}$$

and

$$C\psi = E\boldsymbol{v}\ . \tag{2.46}$$

If $\varrho_n(t)$ is a regularizing sequence of even periodic functions of period T and $*$ denotes the corresponding convolution on the circle, we find

$$\boldsymbol{v}' * \varrho_n * \varrho_n \in C^\infty(T;\, L^{\varrho+1}(\Omega)).$$

Hence

$$\int_0^T (\boldsymbol{v}'', \boldsymbol{v}' * \varrho_n * \varrho_n)\, dt = 0, \qquad \int_0^T (A\boldsymbol{v}, \boldsymbol{v}' * \varrho_n * \varrho_n)\, dt = 0.$$

From (2.46) we obtain

$$\int_0^T (B\psi, \boldsymbol{v}' * \varrho_n * \varrho_n)\, dt = \int_0^T (E\boldsymbol{v}' * \varrho_n * \varrho_n, \psi)\, dt = \int_0^T (C\psi' * \varrho_n * \varrho_n, \psi)\, dt = 0.$$

Consequently, by (2.45)

$$\int_0^T (\boldsymbol{h}(t) - \overline{\boldsymbol{h}}, \boldsymbol{v}' * \varrho_n * \varrho_n)\, dt = \int_0^T (\boldsymbol{g}(t), \boldsymbol{v}' * \varrho_n * \varrho_n)\, dt. \tag{2.47}$$

Letting $n \to \infty$ we arrive at (2.44) and, since

$$\int_0^T (\overline{\boldsymbol{h}}, \boldsymbol{v}'(t))\, dt = 0,$$

we get also

$$\int_0^T (\boldsymbol{h}(t), \boldsymbol{v}'(t))\, dt = \int_0^T (\boldsymbol{g}(t), \boldsymbol{v}'(t))\, dt. \tag{2.48}$$

From (2.28) we obtain

$$\lim_{m\to\infty} \int_0^T (\beta(\boldsymbol{v}'_m(t)), \boldsymbol{v}'_m(t))\, dt = \int_0^T (\boldsymbol{h}(t), \boldsymbol{v}'(t))\, dt. \tag{2.49}$$

Let $\boldsymbol{w}(t) \in L^{\varrho+1}(T; \boldsymbol{L}^{\varrho+1}(\Omega))$. By monotonicity we have

$$\int_0^T (\beta(\boldsymbol{v}'_m(t)) - \beta(\boldsymbol{w}(t)), \boldsymbol{v}'_m(t) - \boldsymbol{w}(t))\, dt \geq 0$$

and, by (2.49),

$$\int_0^T (\boldsymbol{h}(t) - \beta(\boldsymbol{w}(t)), \boldsymbol{v}'(t) - \boldsymbol{w}(t))\, dt \geq 0 .$$

Setting $\boldsymbol{w}(t) = \boldsymbol{v}'(t) - \lambda \boldsymbol{w}_1(t)$, with $\lambda \geq 0$, and letting $\lambda \to 0+$, we obtain

$$\int_0^T (\boldsymbol{h}(t) - \beta(\boldsymbol{v}'(t)), \boldsymbol{w}_1(t))\, dt \geq 0$$

for all $\boldsymbol{w}_1(t) \in L^{\varrho+1}(T; \boldsymbol{L}^{\varrho+1}(\Omega))$. Hence

$$\int_0^T (\boldsymbol{h}(t) - \beta(\boldsymbol{v}'(t)), \boldsymbol{w}_1(t))\, dt = 0$$

and we conclude that $\boldsymbol{h}(t) = \beta(\boldsymbol{v}'(t))$ as required.

We prove uniqueness. Let $\boldsymbol{v}_1$, ψ_1 and $\boldsymbol{v}_2$, ψ_2 be two solutions and define

$$\boldsymbol{w} = \boldsymbol{v}_1 - \boldsymbol{v}_2, \qquad \zeta = \psi_1 - \psi_2 .$$

By difference we have

$$\int_0^T \{ -(\boldsymbol{w}'(t), \boldsymbol{\gamma}'(t)) + a(\boldsymbol{w}(t), \boldsymbol{\gamma}(t))$$
$$+ b(\zeta(t), \boldsymbol{\gamma}(t)) + (\beta(\boldsymbol{v}'_1(t)) - \beta(\boldsymbol{v}'_2(t)) - \overline{\beta}(\boldsymbol{v}'_1) + \overline{\beta}(\boldsymbol{v}'_2), \boldsymbol{\gamma}(t))\}\, dt = 0$$

for all $\boldsymbol{\gamma}(t) \in L^\infty(T; \boldsymbol{H}^1_0(\Omega))$, and $\boldsymbol{\gamma}'(t) \in L^{\varrho+1}(T; \boldsymbol{L}^{\varrho+1}(\Omega)) \cap L^\infty(T; \boldsymbol{L}^2(\Omega))$ and, again by difference from (2.43),

$$c(\zeta(t), \eta) = b(\eta, \boldsymbol{w}(t)) \tag{2.50}$$

for all $\eta \in V$. Reasoning as in the proof of (2.44) we obtain

$$\int_0^T (\beta(\boldsymbol{v}'_1(t)) - \beta(\boldsymbol{v}'_2(t)), \boldsymbol{v}'_1(t) - \boldsymbol{v}'_2(t))\, dt = 0 . \tag{2.51}$$

By the strict monotonicity of β it follows $\boldsymbol{v}'_1(t) = \boldsymbol{v}'_2(t)$. On the other hand, $\overline{\boldsymbol{v}}_1 = \overline{\boldsymbol{v}}_2 = 0$, hence $\boldsymbol{v}_1(t) = \boldsymbol{v}_2(t)$. From (2.50) we have

$$c(\zeta(t), \zeta(t)) = 0 . \tag{2.52}$$

Thus $\psi_1(t) = \psi_2(t)$. It remains to prove that problem (2.16)-(2.19) has one and only one solution. We use the L^p-theory for elliptic system with $p \in (1, 2)$, referring for more details to [*1*] and [*8*] page 201. This theory can be applied to (2.16)-(2.19) if we recall that Γ is of class C^2 and that

$$\bar{\boldsymbol{f}} - \bar{\beta}(\boldsymbol{v}') \in L^{\frac{\varrho+1}{\varrho}}(\Omega).$$

The need to solve an elliptic problem with the left hand side in L^p with $p \in (1, 2)$ is inherent to the present method and has relevant consequences. If, for example, Γ is not of class C^2 but only lipschitzian, uniqueness fails (see the example given in [**6**]); this in turn implies cases of nonuniqueness for the problem as a whole.

References

[1] S. Agmon, *The L^p approach to the Dirichlet problem I. Regularity theorems*, Ann. Sc. Norm. Sup. Pisa **13** (1959), 405-448.

[2] L. Amerio, *Soluzioni quasi-periodiche, o limitate, di sistemi differenziali non lineari quasi periodici o limitati*, Ann. Mat. Pura Appl. **39** (1955), 97-119.

[3] L. Amerio and G. Prouse, *Almost-periodic functions and functional equations*, Van Nostrand, New York-Toronto, Ont.-Melbourne 1971.

[4] A. C. Eringen and G. A. Maugin, *Electrodynamics of continua I. Foundations and solid media*, Springer-Verlag, New York 1990.

[5] J. L. Lions, *Quelques méthodes des résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris 1969.

[6] G. Prodi, *Soluzioni periodiche dell'equazione delle onde con termine dissipativo non-lineare*, Rend. Sem. Mat. Univ. Padova **36** (1966), 37-49.

[7] G. Prouse, *Soluzioni periodiche dell'equazione delle onde non omogenea con termine dissipativo quadratico*, Ricerche Mat. **13** (1964), 261-280.

[8] G. T. Troianiello, *Elliptic differential equations and obstacle problems*, Plenum Press, New York 1987.

Abstract

A theorem of existence and uniqueness of forced periodic solutions in a piezoelectric viscoelastic body is proved.

* * *