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# Anisotropic equations with measure data (**)

## Introduction

Variational equations with anisotropic operators have been studied by several Authors. In [**4**] the following problem is considered

$$-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)=\mu \quad \text{in } \Omega, \qquad u=0 \quad \text{on} \quad \partial\Omega$$

where $\mu$ is a bounded Radon measure. In such case it is proved the existence of at least a distributional solution $u$, such that $\partial_i u \in L^{q_i}(\Omega)$, with $q_i < \dfrac{n(\overline{p}-1)}{\overline{p}(n-1)}\,p_i$, where $\overline{p}=\left(\dfrac{1}{n}\sum_{i=1}^{n}\dfrac{1}{p_i}\right)^{-1}$, $p_i>1$. When $\mu$ is a regular datum, in [**7**] and in [**8**], are also considered operators like

$$-\sum_{i=1}^{n}\partial_i a_i(x, Du)$$

where $a=(a_1, \dots, a_n)$ is a Carathéodory function satisfying conditions of strict monotonicity, coerciveness and growth.

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In particular in [**8**] (see also [**9**]) the following growth condition is assumed:

$$|a_i(x, \xi)| \leqslant M\left(1 + \sum_{j=1}^{n} |\xi_j|^{p_j}\right)^{1-1/p_i}, \quad \forall i = 1, \ldots, n. \tag{0.1}$$

In [**9**] is proved that in the case where $a_i(x, \xi) = \partial_{\xi_i} f(x, \xi)$, with $f$ Carathéodory function, convex with respect to $\xi_i$, $i = 1, \ldots, n$ and such that

$$|f(x, \xi)| \leqslant C\left(1 + \sum_{j=1}^{n} |\xi_j|^{p_j}\right)$$

for some constant $C$, then $a$ satisfies condition (0.1).

In this paper we study the problem

$$-\sum_{i=1}^{n} \partial_i (b_i \partial_i u + a_i(\cdot, Du)) = \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega \tag{0.2}$$

when $\mu \in L^1(\Omega) + X^*$ and $X$ is the Banach space obtained as the closure of $C_0^1(\Omega)$ with respect to the norm:

$$|u|_X = \|u\|_p + \max_{1 \leqslant i \leqslant n} \left[ \|\partial_i u\|_{p_i} \vee \left( \int_\Omega b_i |\partial_i u|^2 \right)^{1/2} \right]$$

where $p = \min\{p_1, \ldots, p_n\}$.

We assume the coefficients $b_i$ to belong to $L^\infty(\Omega)$ and to be a.e. positive in $\Omega$, while we assume $a_i$ to satisfy the growth condition (0.1) and to be coercive and strict monotonous.

The definition adopted for the solutions of (0.2) is the one of renormalized kind, widely used to study existence and uniqueness for problems with measure data (see [**10**], [**6**]). Such definition is also strictly related to the one of «entropic solution» (see [**5**], [**2**]).

Moreover the idea of renormalized solutions has been introduced in [**3**] in the isotropic case for nonlinear elliptic equations involving lower order terms of the form $-\operatorname{div}(\Psi(u))$, with $\Psi : \boldsymbol{R} \to \boldsymbol{R}^n$ continuous.

In our paper the existence is obtained by means of the approximation method by equations with regular data, however without obtaining the strong convergence of the gradients of truncations, rather taking suitably advantage of their weak convergence in the introduced space.

The uniqueness is got thanks to the following asymptotic condition:

$$\lim_k \int_{\{k \leq |u| \leq k+1\}} \sum_{i=1}^{n} (b_i |\partial_i u_h|^2 + |\partial_i u_h|^{p_i}) = 0.$$

Moreover we point out that the hypothesis on the datum $\mu \in L^1(\Omega) + X^*$ has been assumed with analogy to the isotropic case, in which the assumption that a measure $\mu$ is absolutely continuous with respect to the $p$-capacity is equivalent to the fact that $\mu = \mu_0 + \Phi$, with $\mu_0 \in L^1(\Omega)$, $\Phi \in X^*$ (see [5]).

Besides we prove that if $\Phi \in X^*$, then there exist $F \in \times_{i=1}^{n} L^2(b_i)$, $G \in \times_{i=1}^{n} L^{p_i'}(\Omega)$ with $F = (F_1, \ldots, F_n)$, $G = (G_1, \ldots, G_n)$ for which

$$\Phi = -\sum_{i=1}^{n} \partial_i [b_i F_i + G_i].$$

## 1 - General hypotheses and formulation of the problem

Let $\Omega$ be a bounded open set in $\boldsymbol{R}^n$, $n \geq 2$. We assume

$$b_i \in L^\infty(\Omega),\ b_i(x) > 0 \quad \text{for a.e. } x \in \Omega,\ i = 1, \ldots, n.$$

Moreover we assume

$$p_1, \ldots, p_n \in ]1, \infty[, \qquad \sum_{i=1}^{n} \frac{1}{p_i} > 1$$

and $p = \min\{p_1, \ldots, p_n\}$. We define the space $X$ as the closure of $C_0^1(\Omega)$ with respect to the norm:

$$|u|_X = \|u\|_p + \max_{1 \leq i \leq n} \left[ \|\partial_i u\|_{p_i} \vee \left( \int_\Omega b_i |\partial_i u|^2 \right)^{1/2} \right].$$

We shall denote by $p_i'$ the conjugate exponent of $p_i$ and by $L^2(b_i)$ the Banach space

$$\left\{ v : \Omega \to \boldsymbol{R} : v \text{ measurable such that } \int_\Omega b_i |v|^2 < \infty \right\} \Big/ \mathcal{R}$$

endowed with the norm $\|v\|_{L^2(b_i)} = \left( \int_\Omega b_i |v|^2 \right)^{1/2}$, where $\mathcal{R}$ is the usual almost everywhere equivalence relation.

Now we introduce a Caratheodory vector function $a : \Omega \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ for which

(i) there exists $c \in \boldsymbol{R}_+$ such that

$$\langle a(x, \xi), \xi \rangle \geq c \sum_{i=1}^{n} |\xi_i|^{p_i}$$

for a.e. $x \in \Omega$ and every $\xi \in \boldsymbol{R}^n$;

(ii) there exists $c_1 \in \boldsymbol{R}_+$ such that for every $i = 1, \ldots, n$

$$\text{(ii0)} \qquad |a_i(x, \xi)| \leq c_1 \left(1 + \sum_{j=1}^{n} |\xi_j|^{p_j}\right)^{(1-1/p_i)}$$

$$\text{(ii1)} \qquad \langle a(x, \xi) - a(x, \eta), \xi - \eta \rangle > 0$$

for a.e. $x \in \Omega$, for every $\xi, \eta \in \boldsymbol{R}^n$, $\xi \neq \eta$.

If $\zeta : \Omega \to \boldsymbol{R}^n$ we introduce the notation $b(\zeta) = (b_1 \zeta_1, \ldots, b_n \zeta_n)$, so that it is well defined the operator $A : X \to X^*$ such that

$$\text{(1.1)} \qquad \langle Au, \varphi \rangle = \int_{\Omega} \langle b(Du) + a(\cdot, Du), D\varphi \rangle \qquad \forall \varphi \in X .$$

Our aim is to prove existence and uniqueness of a renormalized solution for the problem

$$\text{(I)} \quad \begin{cases} Au = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega , \end{cases}$$

where $\mu \in L^1(\Omega) + X^*$. Such a solution is understood in the sense of the next definition.

Definition 1.1. Let $\tau_k(s) = (s \wedge k) \vee (-k)$, $\sigma_k(s) = ((-|s| + k + 1) \vee 0) \wedge 1$, $s \in \boldsymbol{R}$, $k \in \boldsymbol{R}_+$. We say that $u : \Omega \to \boldsymbol{R}$ is a renormalized solution of problem (I), where $\mu = f + \phi$, $f \in L^1(\Omega)$, $\phi \in X^*$ if $\tau_k(u) \in X$ for all $k \in \boldsymbol{R}_+$ and moreover

$$\text{(1.2)} \qquad \langle Au, \sigma_k(u)\, \varphi \rangle = \int_{\Omega} f \sigma_k(u)\, \varphi + \langle \phi, \sigma_k(u)\, \varphi \rangle \qquad \forall \varphi \in X \cap L^\infty(\Omega),\ k \in \boldsymbol{R}_+$$

$$\text{(1.3)} \qquad \lim_{k \to \infty} \int_{\{k \leq |u| \leq k+1\}} \left( \langle b(Du), Du \rangle + \sum_{i=1}^{n} |\partial_i u|^{p_i} \right) = 0 .$$

We remark that in (1.2), thanks to the presence of $\sigma_k(u)$, the operator $Au$ is meaningful, because we can read it as $A\tau_{k+1}(u)$.

**2 - Operator and spaces properties.**

Theorem 2.1 ([**13**], theorem 1.2). *Let* $q_i \geq 1$, $i = 1, \dots, n$ *and* $u \in C_0^1(\Omega)$. *If* $\sum_{i=1}^{n} \frac{1}{q_i} > 1$, *it results*

$$\|u\|_{\bar{q}^*} \leq C \prod_{i=1}^{n} \|\partial_i u\|_{q_i}^{1/n} \tag{2.1}$$

*where* $\frac{1}{\bar{q}^*} = \frac{1}{n}\left(-1 + \sum_{i=1}^{n} \frac{1}{q_i}\right)$ *and* $C$ *depends only on* $q_i$ *and* $n$.

Theorem 2.2 ([**1**], theorem [8.III]). *Let* $V, W$ *be Banach spaces such that*

(i) $V \cap W$ *is dense in both* $V$ *and* $W$;

(ii) *if* $(u_h)_{h \in N}$ *is a sequence in* $V \cap W$ *such that* $\|u_h - v\|_V \to 0$, $\|u_h - w\|_W \to 0 \Rightarrow v = w \in V \cap W$.

*Then the map from* $V^* + W^*$ *to* $(V \cap W)^*$ *defined by*

$$\langle z, v^* + w^* \rangle_{(V \cap W)^*} := \langle z, v^* \rangle_{V^*} + \langle z, w^* \rangle_{W^*} \qquad \forall z \in V \cap W$$

*is an isometric isomorphism.*

For the reader's convenience we give an outline of the proof.

Proof. By condition (ii) $V \cap W$ and $V \times W$ are Banach spaces if endowed with the norms $\|z\|_{V \cap W} = \|z\|_V \vee \|z\|_W$, $z \in V \cap W$, $\|(v, w)\|_{V \times W} = \|v\|_V \vee \|w\|_W$, $v \in V$, $w \in W$.

Let $\phi : V \cap W \to V \times W$ defined by $\phi(z) = (z, z)$ and $\phi_0 : V \cap W \to \phi(V \cap W)$ given by $\phi_0(z) = \phi(z)$. Clearly $\phi_0$ is an isometrical isomorphism and so the same happens for the adjoint map $\phi_0^* : \phi(V \cap W)^* \to (V \cap W)^*$ defined by

$$\phi_0^*(\Lambda)(z) = \langle (z, z), \Lambda \rangle \qquad \forall \Lambda \in \phi(V \cap W)^*, \ z \in V \cap W.$$

We now introduce the annihilator $\phi(V \cap W)^0 := \{(v^*, w^*) \in V^* \times W^* : \langle z, v^* \rangle = \langle -z, w^* \rangle \forall z \in V \cap W\}$ and for each $v^* \in V^*$, $w^* \in W^*$ we define

$$\langle z, v^* + w^* \rangle_{(V \cap W)^*} := \langle v^*, z \rangle_{V^*} + \langle w^*, z \rangle_{W^*}.$$

Next we consider the space $V^* + W^* := \{v^* + w^* : v^* \in V^*, w^* \in W^*\}$ endowed with the norm $\|v^* + w^*\|_{V^* + W^*} := \inf\{\|v^* + u^*\|_{V^*} + \|w^* - u^*\|_{W^*} : u^* \in V^* \cap W^*\}$.

It is easy to verify that the linear map

$$P : V^* + W^* \to (V^* \times W^*)/\phi(V \cap W)^0, \qquad P(v^* + w^*) = (v^*, w^*) + \phi(V \cap W)^0$$

is an isometric isomorphism (recall that $\|(v^*, w^*) + \phi(V \cap W)^0\| := \inf\{\|(v^*, w^*) + \Lambda\|_{V^* \times W^*} : \Lambda \in \phi(V \cap W)^0\}$ and $\|(v^*, w^*)\|_{V^* \times W^*} = \|v^*\|_{V^*} + \|w^*\|_{W^*}$).

Another isometric isomorphism (see [**12**], theorem 3.3) is given by the map

$$\psi : (V^* \times W^*)/\phi(V \cap W)^0 \to \phi(V \cap W)^*,$$

$$\psi((v^*, w^*) + \phi(V \cap W)^0) = (v^*, w^*)_{|\phi(V \cap W)}.$$

Finally the following composition

$$\phi_0^* \circ \psi \circ P : V^* + W^* \to (V \cap W)^* \quad \text{such that } \forall z \in V \cap W$$

$$\langle z, \phi_0^* \circ \psi \circ P(v^* + w^*) \rangle_{(V \cap W)^*}$$

$$= \langle (z, z), (v^*, w^*)_{|\phi(V \cap W)} \rangle_{V^* \times W^*} = \langle z, v^* \rangle_{V^*} + \langle z, w^* \rangle_{W^*}$$

gives the desired isometric isomorphism. ■

Proposition 2.3. *The space $X$ introduced in section 1 is a reflexive Banach space. Moreover for any $\phi \in X^*$ there exist $F \in \bigtimes_{i=1}^{n} L^2(b_i)$, $G \in \bigtimes_{i=1}^{n} L^{p_i'}(\Omega)$ such that*

$$\langle \phi, \varphi \rangle = \int_\Omega (\langle b(D\varphi), F \rangle + \langle G, D\varphi \rangle), \qquad \forall \varphi \in X. \tag{2.2}$$

Proof. Clearly $X$ is isometrically isomorphic to a closed subspace of $L^p \times \left( \bigtimes_{i=1}^{n} (L^{p_i} \cap L^2(b_i)) \right)$. Let us prove, for $i \in \{1, \ldots, n\}$ the reflexivity of the space $L^{p_i} \cap L^2(b_i)$, endowed with the norm $\|\cdot\|_{p_i} \vee \|\cdot\|_{L^2(b_i)}$. Let $(u_h)$ be a bounded sequence in $L^{p_i} \cap L^2(b_i)$ and $\varphi \in C_c(\Omega)$, the space of continuous functions with compact support in $\Omega$. Then there exist $u \in L^{p_i}$, $v \in L^2(b_i)$ such that, by going to a subsequence if necessary, $u_h \underset{L^{p_i}}{\rightharpoonup} u$, $u_h \underset{L^2(b_i)}{\rightharpoonup} v$. Since $\varphi b_i \in L^{p_i'}$ it results $\int_\Omega u_h \varphi b_i \to \int_\Omega u \varphi b_i$, $\int_\Omega u_h \varphi b_i \to \int_\Omega v \varphi b_i$, so that $u = v$ by arbitrariness of $\varphi$ and assumptions on $b_i$.

Now by Theorem 2.2, thanks to the density of $L^{p_i} \cap L^2(b_i)$ in both $L^{p_i}$ and $L^2(b_i)$, it results $(L^{p_i} \cap L^2(b_i))^*$ isomorphic to $L^{p_i'} + L^2(b_i)$ according to the definition

$$\langle z, v^* + w^* \rangle_{(L^{p_i} \cap L^2(b_i))^*} = \langle z, v^* \rangle_{(L^{p_i})^*} + \langle z, w^* \rangle_{L^2(b_i)^*} \qquad \forall z \in L^{p_i} \cap L^2(b_i).$$

On the other hand, by density of $C_c(\Omega)$ in both $(L^{p_i})^*$ and $(L^2(b_i))^*$ it results $\langle u_h, v^* \rangle_{(L^{p_i})^*} \to \langle u, v^* \rangle_{(L^{p_i})^*}$ and $\langle u_h, w^* \rangle_{L^2(b_i)^*} \to \langle u, w^* \rangle_{L^2(b_i)^*}$ $\forall v^* \in (L^{p_i})^*$, $w^* \in (L^2(b_i))^*$. Hence $u_h \rightharpoonup u$ in $X$ and reflexivity is proved.

Let $P: X \to \times_{i=1}^{n} (L^2(b_i) \cap L^{p_i}(\Omega))$ such that $P(u) = Du$. If $\phi \in X^*$ is given, we define $\phi^*: P(X) \to \boldsymbol{R}$, $\phi^*(P(u)) = \phi(u)$. Clearly $\phi^* \in P(X)^*$, so, by Hahn-Banach theorem there exists a norm preserving extension $\tilde{\phi} \in \left( \times_{i=1}^{n} L^2(b_i) \cap L^{p_i}(\Omega) \right)^*$ of $\phi^*$. Therefore, by the isomorphism between $(L^2(b_i) \cap L^{p_i}(\Omega))^*$ and $L^2(b_i) + L^{p_i'}(\Omega)$, there exist $F = (F_1, \ldots, F_n)$ and $G = (G_1, \ldots, G_n)$ such that $F_i + G_i \in L^2(b_i) + L^{p_i'}(\Omega)$, $i = 1, \ldots, n$ and $\tilde{\phi}(w) = \sum_{i=1}^{n} \langle w_i, F_i \rangle_{L^2(b_i)} + \langle w_i, G_i \rangle_{L^{p_i'}}$ for each $w \in \times_{i=1}^{n} L^2(b_i) \cap L^{p_i}(\Omega)$. Now if $\varphi \in X$ we have $\phi(\varphi) = \phi^*(D\varphi) = \sum_{i=1}^{n} \langle \partial_i \varphi, F_i \rangle_{L^2(b_i)} + \langle \partial_i \varphi, G_i \rangle_{L^{p_i'}}$. ■

Remark 2.4. In the case where $\inf(b_i) > 0$ and $p_i \leq 2$ for every $i = 1, \ldots, n$, it is not difficult to verify that

$$X^* = H^{-1}(\Omega).$$

About the inclusion $H^{-1}(\Omega) \subset X^*$, it is sufficient to remind that, whenever we take $\Phi = \sum_{i=1}^{n} \partial_i \phi_i \in H^{-1}(\Omega)$, with $\phi_i \in L^2(\Omega)$, it is possible to decompose $\phi_i = \phi_i' + \phi_i''$, with $\phi_i' \in L^\infty(\Omega)$, $\phi_i'' \in L^2(\Omega)$. Therefore $(\phi_i - \phi_i')/b_i \in L^2(b_i)$.

Proposition 2.5. *Under the assumptions in section* 1, *the operator A defined in* (1.1) *is coercive, strictly monotone, hemicontinue and bounded. Precisely*

(i) $\displaystyle\lim_{|u|_X \to \infty} \frac{\langle Au, u \rangle}{|u|_X} = \infty$

(ii) $\langle Au - Av, u - v \rangle > 0$ if $u \neq v$, $u, v \in X$.

(iii) *The map* $\lambda \in \boldsymbol{R} \mapsto \langle A(u + \lambda v), w \rangle$ *is continuous for each* $u, v, w \in X$.

(iv) *If* $Y \subset X$ *is bounded, then* $A(Y)$ *is bounded.*

Proof. (i) Let $\bar{i}$ be such that $\|\partial_{\bar{i}} u\|_{p_{\bar{i}}} = \max\{\|\partial_i u\|_{p_i} : i = 1, \ldots, n\}$. If $p$ and $\bar{p}^*$ are like in section 1 and in Theorem 2.1 respectively, we observe that $p \leq \bar{p}^*$

so, if $C \in \boldsymbol{R}_+$ is a suitable constant, by Theorem 2.1 we have

$$\|u\|_p \leq C \|\partial_{\bar{i}} u\|_{p_{\bar{i}}} .$$

Hence, thanks to assumption (i) on $a(x, \xi)$ we get:

$$\frac{\langle Au, u \rangle}{|u|_X} \geq \frac{\int_\Omega \left( \sum_{i=1}^n b_i |\partial_i u|^2 + c \sum_{i=1}^n |\partial_i u|^{p_i} \right)}{|u|_X}$$

$$\geq \frac{\int_\Omega \sum_{i=1}^n b_i |\partial_i u|^2 + \frac{c}{2} \sum_{i=1}^n \|\partial_i u\|_{p_i}^{p_i} + \frac{c}{2} \|\partial_{\bar{i}} u\|_{p_{\bar{i}}}^{p_{\bar{i}}}}{\|u\|_p + \|\partial_{\bar{i}} u\|_{p_{\bar{i}}} + \left( \int_\Omega \sum_{i=1}^n b_i |\partial_i u|^2 \right)^{1/2}} \geq K \frac{\int_\Omega \sum_{i=1}^n b_i |\partial_i u|^2 + \|\partial_{\bar{i}} u\|_{p_{\bar{i}}}^{p_{\bar{i}}} + \|u\|_p^{p_{\bar{i}}}}{\|u\|_p + \|\partial_{\bar{i}} u\|_{p_{\bar{i}}} + \left( \int_\Omega \sum_{i=1}^n b_i |\partial_i u|^2 \right)^{1/2}}$$

where $K$ is a suitable positive constant. Such inequality implies the coerciveness of $A$ by observing that the following assertion holds: $\lim\limits_{a+b+c \to \infty} \dfrac{a^2 + b^q + c^s}{a+b+c} = \infty$, when $a, b, c \in \boldsymbol{R}_+$ and $q, s \geq 1$.

(ii) The strict monotonicity follows easily from hypothesys (ii1) and definition of $A$, because, thanks to (2.1), if $u \neq v$, then $Du \neq Dv$.

(iii) The continuity of the map $\lambda \in \boldsymbol{R} \mapsto \langle A(u + \lambda v), w \rangle$ is an easy consequence of the Carathéodory assumption on $a$ and growth condition (ii0).

(iv) For the boundedness we observe that:

$$|\langle Au, v \rangle|$$

$$\leq \sum_{i=1}^n \left[ \left( \int_\Omega b_i |\partial_i u|^2 \right)^{1/2} \left( \int_\Omega b_i |\partial_i v|^2 \right)^{1/2} + c_1 \left( \int_\Omega \left( 1 + \sum_{j=1}^n |\partial_j u|^{p_j} \right) \right)^{1/p_i'} \left( \int_\Omega |\partial_i v|^{p_i} \right)^{1/p_i} \right]$$

$$\leq C |v|_X \sum_{i=1}^n \left[ \left( \int_\Omega b_i |\partial_i u|^2 \right)^{1/2} + \left( \int_\Omega \left( 1 + \sum_{j=1}^n |\partial_j u|^{p_j} \right) \right)^{1/p_i'} \right] \leq C |v|_X (1 + |u|_X)^\delta$$

where $\delta$ is a suitable positive exponent. So the boundedness of $A$ follows. ■

Thanks to Proposition 2.5 it is possible to apply Theorem 2.1, Chap. 2 of [**7**] in order to prove the following:

Theorem 2.6. *For every* $f \in X^*$, *under the hypotheses in section* 1, *there exists* $u \in X$ *solution of*

$$A(u) = f$$

## 3 - Approximating solutions and estimates

We now give some estimates which extend to the anisotropic case those proved in the literature for the isotropic case.

Theorem 3.1. *Let* $(f_h)$ *be a sequence in* $C_0^1(\Omega)$ *bounded in the* $L^1$ *norm,* $F \in \bigtimes_{i=1}^n L^2(b_i)$, $G \in \bigtimes_{i=1}^n L^{p_i'}(\Omega)$ *and* $\phi \in X^*$ *defined by* (2.2). *Moreover let* $u_h \in X$ *be a solution of*

$$\langle Au_h, \varphi\rangle = \int_\Omega f_h \varphi + \langle \phi, \varphi\rangle \qquad \forall \varphi \in X . \tag{3.1}$$

*Then there exist positive constants* $C, M$ *such that for each* $h \in \boldsymbol{Z}_+$, $k \in \boldsymbol{R}_+$ *the following estimates hold*:

$$\int_{\{|u_h| \leq k\}} \sum_{i=1}^n (b_i |\partial_i u_h|^2 + |\partial_i u_h|^{p_i}) \leq Ck + M \tag{3.2}$$

$$\int_{\{k \leq |u_h| \leq k+1\}} \sum_{i=1}^n (b_i |\partial_i u_h|^2 + |\partial_i u_h|^{p_i}) \leq C \int_{\{k \leq |u_h|\}} \Big(|f_h| + \sum_{i=1}^n (b_i |F_i|^2 + |G_i|^{p_i'})\Big). \tag{3.3}$$

Proof. By (3.1) with $\varphi = \tau_k(u_h)$ we get:

$$\int_{\{|u_h| \leq k\}} \sum_{i=1}^n (b_i |\partial_i u_h|^2 + c |\partial_i u_h|^{p_i}) \leq \int_\Omega \langle b(Du_h), D\tau_k(u_h)\rangle + \langle a(\cdot, Du_h), D\tau_k(u_h)\rangle$$

$$\leq k \sup \|f_h\|_1 + \frac{1}{2} \int_{\{|u_h| \leq k\}} \sum_{i=1}^n b_i |\partial_i u_h|^2 + C \sum_{i=1}^n (\|F_i\|^2_{L^2(b_i)} + \|G_i\|^{p_i'}_{L^{p_i'}})$$

$$+ \frac{c}{2} \int_{\{|u_h| \leq k\}} \sum_{i=1}^n |\partial_i u_h|^{p_i}.$$

Then (3.2) clearly follows.

Now from (3.1) with $\gamma_k(u_h) = \tau_{k+1}(u_h) - \tau_k(u_h)$ as test function:

$$\int_{\{k \leq |u_h| \leq k+1\}} \sum_{i=1}^{n} (b_i |\partial_i u_h|^2 + c|\partial_i u_h|^{p_i}) \leq \int_{\{|u_h| \geq k\}} |f_h| + \frac{1}{2} \int_{\{k \leq |u_h| \leq k+1\}} \sum_{i=1}^{n} b_i |\partial_i u_h|^2$$

$$+ C \int_{\{k \leq |u_h|)} \sum_{i=1}^{n} (b_i |F_i|^2 + |G_i|^{p_i'}) + \frac{c}{2} \int_{\{k \leq |u_h| \leq k+1\}} \sum_{i=1}^{n} |\partial_i u_h|^{p_i}.$$

Such inequality gives (3.3).

Proposition 3.2. *Let* $(f_h), (u_h)$, $F$, $G$ *be like in the previous theorem. Then there exist* $g_i \colon \Omega \to \boldsymbol{R}$, $u : \Omega \to \boldsymbol{R}$, $i = 1, \ldots, n$, *measurable and a subsequence of* $(u_h)$, *still denoted by* $(u_h)$, *such that* $\tau_k(u) \in X$ *for every* $k \in \boldsymbol{R}_+$, *and moreover*:

(i) $\tau_k(u_h) \rightharpoonup \tau_k(u)$ *in* $X$ *for every* $k \in \boldsymbol{R}_+$;

(ii) $u_h \to u$ *a.e. in* $\Omega$;

(iii) $a_i(\cdot, Du_h)\, \mathbf{1}_{\{|u_h| \leq k\}} \rightharpoonup g_i \mathbf{1}_{\{|u| \leq k\}}$ *in* $L^{p_i'}(\Omega)$, $i = 1, \ldots, n$, $k \in \boldsymbol{R}_+$.

Proof. Thanks to estimate (3.2), for each $k \in \boldsymbol{Z}_+$ there exists $u^k \in X$ such that, by going to a subsequence if necessary, $\tau_k(u_h) \underset{X}{\rightharpoonup} u^k$. Then we have

$$\partial_i \tau_k(u_h) \rightharpoonup \partial_i u^k \text{ in } L^{p_i'} \text{ and in } L^2(b_i),\ i = 1, \ldots, n.$$

Moreover $\tau_k(u_h) \rightharpoonup u^k$ in $H_0^{1,p}(\Omega)$ ($p = \min\{p_1, \ldots, p_n\}$), so, like in Theorem 2.2 of [**11**], there exists $u : \Omega \to \boldsymbol{R}$ measurable such that $\tau_k(u) = u^k$ for each $k \in \boldsymbol{Z}_+$ and $u_h \to u$ a.e. in $\Omega$. This proves (i) and (ii).

By growth condition (ii0) and (3.2) it follows that for any $k \in \boldsymbol{R}_+$, $i = 1, \ldots, n$ the sequence $(a_i(\cdot, Du_h)\, \mathbf{1}_{\{|u_h| \leq k\}})_h$ is bounded in $L^{p_i'}$. Hence, by going to a subsequence if necessary, there exist $g_i^k \in L^{p_i'}$ such that $a_i(\cdot, Du_h)\, \mathbf{1}_{\{|u_h| \leq k\}} \rightharpoonup g_i^k$ in $L^{p_i'}$. Clearly there exist $g_i \colon \Omega \to \boldsymbol{R}$ measurables such that (see also the proof of Theorem 2.2 iii) of [**11**])

$$g_i = g_i^k \text{ a.e. on } \{|u| \leq k\},\ k \in \boldsymbol{R}_+,\ i = 1, \ldots, n. \quad \blacksquare$$

We don't study the problem of the strong convergence for the sequence $(Du_h)_{h \in \boldsymbol{N}}$, but we prove the following proposition which gives a sort of almost everywhere convergence result for the gradients of $u_h$.

Proposition 3.3. *If* $g = (g_1, \ldots, g_n)$ *and* $u$ *are given as in Proposition* 3.2, *then it results*

$$g = a(\cdot, Du).$$

Proof. Let $\vartheta \in C_0^1(\Omega)$, $\vartheta \geqslant 0$, $\xi \in \boldsymbol{R}^n$. Moreover let $(f_h), (u_h)$, $F$, $G$ be like in theorem 3.1. By monotonicity it results:

$$0 \leqslant \int_\Omega \langle b(Du_h) + a(\cdot, Du_h) - b(\xi) - a(\cdot, \xi), Du_h - \xi \rangle \tau'_\varepsilon(u_h - u)\, \sigma_\lambda(u_h)\, \sigma_\lambda(u)\, \vartheta$$

$$= I_1(h) + I_2(h) - I_3(h) - I_4(h),$$

where

$$I_1(h) = \int_\Omega \langle b(Du_h) + a(\cdot, Du_h), Du_h - Du \rangle \tau'_\varepsilon(u_h - u)\, \sigma_\lambda(u_h)\, \sigma_\lambda(u)\, \vartheta\,,$$

$$I_2(h) = \int_\Omega \langle b(Du_h) + a(\cdot, Du_h), Du \rangle \tau'_\varepsilon(u_h - u)\, \sigma_\lambda(u_h)\, \sigma_\lambda(u)\, \vartheta\,,$$

$$I_3(h) = \int_\Omega \langle b(Du_h) + a(\cdot, Du_h), \xi \rangle \tau'_\varepsilon(u_h - u)\, \sigma_\lambda(u_h)\, \sigma_\lambda(u)\, \vartheta\,,$$

$$I_4(h) = \int_\Omega \langle b(\xi) + a(\cdot, \xi), Du_h - \xi \rangle \tau'_\varepsilon(u_h - u)\, \sigma_\lambda(u_h)\, \sigma_\lambda(u)\, \vartheta\,.$$

We denote $w_{h,\varepsilon} = \tau_\varepsilon(u_h - u)\, \sigma_\lambda(u_h)\, \sigma_\lambda(u)\, \vartheta$. By (3.1) and growth conditions we have:

$$I_1(h) = \langle Au_h, w_{h,\varepsilon} \rangle - \int_\Omega \langle b(Du_h) + a(\cdot, Du_h), D(\sigma_\lambda(u_h)\, \sigma_\lambda(u)\, \vartheta) \rangle\, \tau_\varepsilon(u_h - u) = \int_\Omega f_h w_{h,\varepsilon}$$

$$+ \int_\Omega \big( \langle b(Dw_{h,\varepsilon}), F \rangle + \langle G, Dw_{h,\varepsilon} \rangle \big) - \int_\Omega \langle b(Du_h) + a(\cdot, Du_h), D(\sigma_\lambda(u_h)\, \sigma_\lambda(u)\, \vartheta) \rangle\, \tau_\varepsilon(u_h - u)$$

$$\leqslant \varepsilon \|\vartheta\|_\infty \sup_h \|f_h\|_1 + \int_\Omega \sum_{i=1}^n b_i(\partial_i \tau_\varepsilon(u_h - u))\, F_i\, \sigma_\lambda(u_h)\, \sigma_\lambda(u)\, \vartheta$$

$$+ \int_\Omega \sum_{i=1}^n b_i\, \partial_i(\sigma_\lambda(u_h)\, \sigma_\lambda(u)\, \vartheta)\, F_i\, \tau_\varepsilon(u_h - u) + \int_\Omega \langle G, D\tau_\varepsilon(u_h - u) \rangle\, \sigma_\lambda(u_h)\, \sigma_\lambda(u)\, \vartheta$$

$$+\int_\Omega \langle G, D(\sigma_\lambda(u_h)\,\sigma_\lambda(u)\,\vartheta)\rangle\,\tau_\varepsilon(u_h-u)$$

$$+\varepsilon\|\vartheta\|_\infty \int_{\{|u_h|<\lambda+1\}} \left(\sum_{i=1}^n b_i(\partial_i u_h)^2 + C\left(1+\sum_{j=1}^n |\partial_j u_h|^{p_j}\right) + C|\partial_i u_h|^{p_i}\right)$$

$$+\varepsilon \int_{\{|u_h|\leqslant\lambda+1\}} \left(\sum_{i=1}^n b_i\,|\partial_i u_h| + c_1\left(1+\sum_{j=1}^n |\partial_j u_h|^{p_j}\right)^{1-1/p_i}\right)|D(\sigma_\lambda(u)\,\vartheta)|\,.$$

By Proposition 3.2 it results $\partial_i\tau_{\lambda+1}(u_h)\rightharpoonup\partial_i\tau_{\lambda+1}(u)$ in $L^{p_i'}$ and in $L^2(b_i)$, $i=1,\ldots,n$ which implies that

$$\lim_{h\to\infty}\int_\Omega \sum_{i=1}^n b_i(\partial_i\tau_\varepsilon(u_h-u))\,F_i\,\sigma_\lambda(u_h)\,\sigma_\lambda(u)\,\vartheta=0$$

$$\lim_{h\to\infty}\int_\Omega \langle G, D\tau_\varepsilon(u_h-u)\rangle\,\sigma_\lambda(u_h)\,\sigma_\lambda(u)\,\vartheta=0\,.$$

Then, by estimate (3.2), by using Young or Hölder inequalities we get the existence of $M_\lambda\in\boldsymbol{R}_+$ such that:

$$\limsup_h I_1(h)\leqslant\varepsilon M_\lambda.$$

Now we consider the following strong convergences in $L^{p_i'}$ and in $L^2(b_i)$, $i=1,\ldots,n$

$$\partial_i u\tau'_\varepsilon(u_h-u)\,\sigma_\lambda(u_h)\,\sigma_\lambda(u)\,\vartheta\to\partial_i u\sigma_\lambda(u)^2\vartheta,$$

$$\xi_i\tau'_\varepsilon(u_h-u)\,\sigma_\lambda(u_h)\,\sigma_\lambda(u)\,\vartheta\to\xi_i\sigma_\lambda(u)^2\vartheta$$

which together the convergence given by Proposition 3.2 give:

$$\lim_h(I_2(h)-I_3(h))=\int_\Omega\sum_{i=1}^n(b_i\partial_i u+g_i)(\partial_i u-\xi_i)\,\sigma_\lambda(u)^2\vartheta\,.$$

Moreover it is clear that:

$$\lim_h I_4(h)=\int_\Omega\langle b(\xi)+a(\cdot,\xi), Du-\xi\rangle\,\sigma_\lambda(u)^2\vartheta\,.$$

Therefore we conclude that:

$$0 \leq \varepsilon M_\lambda + \int_\Omega \langle b(Du) + g - b(\xi) - a(\cdot, \xi), Du - \xi \rangle \, \sigma_\lambda(u)^2 \vartheta \, .$$

By arbitrariness of $\varepsilon$ and of $\vartheta$ it follows:

$$0 \leq \langle b(Du) + g - b(\xi) - a(\cdot, \xi), Du - \xi \rangle \quad \text{a.e. on } \{ |u| \leq \lambda \} \, .$$

By continuity of the map $\xi \in \boldsymbol{R}^n \mapsto b(\xi) + a(\cdot, \xi) \in \boldsymbol{R}^n$ it is easy to obtain, choosing $\xi = Du(x) + t\eta$, with $\eta \in \boldsymbol{R}^n$ and letting $t$ tend to zero, that:

$$b(Du) + g = b(Du) + a(\cdot, Du) \quad \text{a.e. on } \Omega$$

which is the assertion of the proposition. ■

#### 4 - Existence and uniqueness

Theorem 4.1. *Let $\mu \in L^1(\Omega) + X^*$. Then there exists a unique renormalized solution of problem* (I).

Proof. **Existence**

Let $\mu = f + \phi$ and $(f_h)$ be a sequence in $C_0^1(\Omega)$ such that $f_h \to f$ in $L^1(\Omega)$. Moreover let us consider, for every $h \in \boldsymbol{Z}_+$, a solution $u_h \in X$ of (3.1). Then, as given by Proposition 3.2, there exist $u : \Omega \to \boldsymbol{R}$, $g : \Omega \to \boldsymbol{R}^n$ for which convergences in (i), (ii), (iii) of the same proposition hold.

Let $\varphi \in X \cap L^\infty(\Omega)$, it is clear that we can choose $\varphi \sigma_t(u) \, \sigma_\lambda(u_h)$ as test function in (3.1). Since $\sigma_t(u) \, \sigma_\lambda(u_h) \, \varphi \underset{X}{\to} \sigma_t(u) \, \sigma_\lambda(u) \, \varphi$, we obtain:

$$\int_\Omega f \sigma_t(u) \, \sigma_\lambda(u) \, \varphi + \langle \phi, \sigma_t(u) \, \sigma_\lambda(u) \, \varphi \rangle = \lim_h \int_\Omega f_h \sigma_t(u) \, \sigma_\lambda(u_h) \, \varphi + \langle \phi, \sigma_t(u) \, \sigma_\lambda(u_h) \, \varphi \rangle$$

$$= \lim_h \langle Au_h, \, \sigma_t(u) \, \sigma_\lambda(u_h) \, \varphi \rangle \leq \limsup_h \int_\Omega \sum_{i=1}^n [ b_i \, \partial_i u_h \, \partial_i (\sigma_t(u) \, \varphi) \, \sigma_\lambda(u_h)$$

$$+ a_i(\cdot, Du_h) \, \partial_i (\sigma_t(u) \, \varphi) \, \sigma_\lambda(u_h)]$$

$$+ \limsup_h \int_{\{\lambda \leq |u_h| \leq \lambda + 1\}} \sum_{i=1}^n (b_i (\partial_i u_h)^2 + a_i(\cdot, Du_h) \, \partial_i u_h) \, \sigma_t(u) \, \varphi \, .$$

Now, by growth assumption (ii0) and (3.3), we get:

$$\int_{\{\lambda \leq |u_h| \leq \lambda+1\}} \sum_{i=1}^{n} \big(b_i(\partial_i u_h)^2 + a_i(\cdot, Du_h)\,\partial_i u_h\big)\,\sigma_t(u)\,\varphi$$

$$\leq \|\varphi\|_\infty \int_{\{\lambda \leq |u_h| \leq \lambda+1\}} \sum_{i=1}^{n} \Big(b_i(\partial_i u_h)^2 + c_1\Big(1 + \sum_{j=1}^{n} |\partial_j u_h|^{p_j}\Big)^{1/p_i'} |\partial_i u_h|\Big)$$

$$\leq C\|\varphi\|_\infty \int_{\{\lambda \leq |u_h| \leq \lambda+1\}} \Big(1 + \sum_{i=1}^{n} (b_i(\partial_i u_h)^2 + |\partial_i u_h|^{p_i})\Big)$$

$$\leq C\|\varphi\|_\infty \int_{\{\lambda \leq |u_h|)} \Big(1 + |f_h| + \sum_{i=1}^{n} (b_i|F_i|^2 + |G_i|^{p_i'})\Big).$$

Then, by taking Proposition 3.3 into account, it follows that:

$$\int_\Omega f\sigma_t(u)\,\sigma_\lambda(u)\,\varphi + \langle \phi, \sigma_t(u|\,\sigma_\lambda(u)\,\varphi\rangle \leq C\|\varphi\|_\infty \int_{\{\lambda \leq |u|\}} \Big(1 + |f| + \sum_{i=1}^{n} (b_i|F_i|^2 + |G_i|^{p_i'})\Big)$$

$$+ \int_\Omega \sum_{i=1}^{n} (b_i\,\partial_i u \partial_i(\sigma_t(u)\,\varphi)\,\sigma_\lambda(u) + a_i(\cdot, Du)\,\partial_i(\sigma_t(u)\,\varphi)\,\sigma_\lambda(u)).$$

By letting $\lambda \to \infty$, we get

$$\int_\Omega f\sigma_t(u)\,\varphi + \langle \phi, \sigma_t(u)\,\varphi\rangle \leq \langle Au, \sigma_t(u)\,\varphi\rangle,$$

so that (1.2) follows by arbitrariness of $\varphi$. Finally (1.3) follows from (3.3) by lower weak semicontinuity of the norm.

**Uniqueness**

The following uniqueness proof is closely related to that of [**2**] and to that of [**5**].

Let $u, v \in X$ be solutions of problem (I). We choose $\sigma_t(v)\,\tau_k(u-v)$ as test function in the equation related to $u$ and $-\sigma_t(u)\,\tau_k(u-v)$ in the one related to $v$.

By adding such equations we obtain

$$
\begin{aligned}
&\int_\Omega \langle b(D(u-v)),\, D(\sigma_t(u)\,\sigma_t(v))\rangle\, \tau_k(u-v)\\
&+\langle b(D\tau_k(u-v)),\, D\tau_k(u-v)\rangle\, \sigma_t(u)\,\sigma_t(v)\\
&+\int_\Omega \langle a(\cdot,\, Du)-a(\cdot,\, Dv),\, \tau_k(u-v)\, D(\sigma_t(u)\,\sigma_t(v))\\
&+\sigma_t(u)\,\sigma_t(v)\, D\tau_k(u-v)\rangle = 0\,.
\end{aligned}
\tag{4.1}
$$

We prove that

$$
\begin{aligned}
&\limsup_t \Big\{ \int_\Omega \langle b(D\tau_k(u-v)),\, D\tau_k(u-v)\rangle\, \sigma_t(u)\,\sigma_t(v)\\
&+\int_\Omega \langle a(\cdot,\, Du)-a(\cdot,\, Dv),\, D\tau_k(u-v)\rangle\, \sigma_t(u)\,\sigma_t(v)\Big\} \leqslant 0\,.
\end{aligned}
\tag{4.2}
$$

In fact the other terms in (4.1) go to zero as $t \to \infty$. To show this, let us consider for instance:

$$
-\int_\Omega \big[\langle b(Du),\, D\sigma_t(v)\rangle + \langle a(\cdot,\, Du),\, D\sigma_t(v)\rangle\big]\, \tau_k(u-v)\,\sigma_t(u)
$$

$$
= \langle Au, (1-\sigma_t(v))\,\sigma_t(u)\,\tau_k(u-v)\rangle - \int_\Omega \langle b(Du),\, D(\sigma_t(u)\,\tau_k(u-v))\rangle(1-\sigma_t(v))
$$

$$
-\int_\Omega \langle a(\cdot, Du), D(\sigma_t(u)\,\tau_k(u-v))\rangle(1-\sigma_t(v)) = \int_\Omega f(1-\sigma_t(v))\,\sigma_t(u)\,\tau_k(u-v)
$$

$$
+\int_\Omega \langle b(D(1-\sigma_t(v))\,\sigma_t(u)\,\tau_k(u-v)), F\rangle + \langle G, D((1-\sigma_t(v))\,\sigma_t(u)\,\tau_k(u-v))\rangle - I_1(t) - I_2(t)
$$

where $I_1(t) = \int_\Omega \langle b(Du),\, D(\sigma_t(u)\,\tau_k(u-v))\rangle(1-\sigma_t(v))$ and $I_2(t) = \int_\Omega \langle a(\cdot,\, Du),$ $D(\sigma_t(u)\,\tau_k(u-v))\rangle(1-\sigma_t(v))$.

The first two integrals in the last member of the above decomposition go to zero as $t \to \infty$. We see for instance the behaviour of:

$$\int_\Omega \langle G, D((1-\sigma_t(v))\,\sigma_t(u)\,\tau_k(u-v))\rangle = -\int_\Omega \langle G, D\sigma_t(v)\rangle\,\sigma_t(u)\,\tau_k(u-v)$$

$$+\int_\Omega \langle G, D\sigma_t(u)\rangle(1-\sigma_t(v))\,\tau_k(u-v) + \int_\Omega \langle G, D\tau_k(u-v)\rangle\,\sigma_t(u)(1-\sigma_t(v))$$

$$\leq k\sum_{i=1}^n \left(\int_\Omega |G_i|^{p_i'}\right)^{1/p_i'}\left(\int_{\{t\leq|v|\leq t+1\}} |\partial_i v|^{p_i}\right)^{1/p_i} + k\sum_{i=1}^n \|G_i\|_{p_i'}\left(\int_{\{t\leq|u|\leq t+1\}} |\partial_i u|^{p_i}\right)^{1/p_i}$$

$$+\sum_{i=1}^n\left(\int_{\{t-k\leq|u|\leq t+1\}} |G_i|\,|\partial_i u| + \int_{\{t\leq|v|\leq t+1+k\}} |G_i|\,|\partial_i v|\right).$$

Then the limit as $t \to \infty$ is zero thanks to (1.3).

Moreover

$$I_1(t) \leq k\int_{\{t\leq|u|\leq t+1\}} \sum_{i=1}^n b_i\,|\partial_i u|^2 + \int_{\{t-k\leq|u|\leq t+1\}} \sum_{i=1}^n b_i\,|\partial_i u|^2$$

$$+\int_{\substack{\{t-k\leq|u|\leq t+1,\\ t\leq|v|\leq t+1+k\}}} \sum_{i=1}^n b_i\,|\partial_i u|\,|\partial_i v|$$

and by (1.3) again it follows that $\lim_{t\to\infty} I_1(t) = 0$.

Analogously thanks to growth assumption (ii0), $\lim_{t\to\infty} I_2(t) = 0$.

In the same way it is straightforward to obtain the convergence to zero of the other integrals of (4.1) except for the ones in (4.2).

On the other hand the integrand in (4.2) is nonnegative and increasing with respect to $t$. Hence (4.2) gives $D\tau_{t+1}(u) = D\tau_{t+1}(v)$ for any $t \in \boldsymbol{R}_+$, so that by (2.1), which holds in $X$, we conclude that $u = v$. ■

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## Abstract

*We give a result about existence and uniqueness of the renormalized solution for an equation with measure data, in the case where the left side is given by the sum of a linear second order operator plus a nonlinear second order operator with anisotropic growth.*

* * *