Riv. Mat. Univ. Parma (7) **1** (2002), 19-53

ANGELA ALBANESE and FRANZISKA KÜHNEMUND (*)

Trotter-Kato approximation theorems for locally equicontinuous semigroups (**)

1 - Introduction

A systematical theory of equicontinuous semigroups has been developed on sequentially complete locally convex spaces by several authors (e.g. [**16**], [**6**], [**9**]). The assumption of equicontinuity permitted them to obtain generation and approximation theorems parallel to the case of Banach spaces. Indeed, this theory depends heavily on the fact that for any equicontinuous semigroup $(T(t))_{t \geq 0}$ with generator A on a sequentially complete locally convex space X, its Laplace transform defined as

$$\int_0^\infty e^{-\lambda t} T(t)\, x dt$$

exists for all $x \in X$ and $\lambda \in \mathbb{C}$ with $Re\lambda > 0$ and coincides with $(\lambda - A)^{-1} x$ (as for bounded strongly continuous semigroups on Banach spaces).

(*) Angela Albanese: Dipartimento di Matematica «E. De Giorgi», Università di Lecce, C. P. 193, Via Per Arnesano, I-73100 Lecce; e-mail: albanese@ilenic.unile.it; Franziska Kühnemund: Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen; e-mail: frku@michelangelo.mathematik.uni–tuebingen.de

(**) Received November 29th, 2001 and in revised form May 14th, 2002. AMS classification 47 D 03, 47 A 58, 35 K 65.

The authors want to thank Prof. R. Nagel for helpful discussions.

This work was done while the second author was visiting the Department of Mathematics of the University of Lecce in December 1999.

On the other hand, if equicontinuity fails, this parallelism may no longer hold. For instance, let $(T(t))_{t \geqslant 0}$ be the left translation semigroup on the space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$ endowed with the topology of uniform convergence on bounded subsets. Then $(T(t))_{t \geqslant 0}$ is not equicontinuous, its Laplace transform does not exist for any $\lambda \in \mathbb{C}$, and its generator given by $A = \dfrac{d}{dx}$ has empty resolvent set. To avoid these difficulties, T. Kōmura [**7**] dealt with locally equicontinuous semigroups and introduced the «generalized resolvent» defined as a suitable vector-valued distribution. She obtained a Hille-Yosida theorem for such locally equicontinuous semigroups in sequentially complete locally convex spaces by giving conditions on such a distributional resolvent (see [**7**], Thm. 3). Since this type of resolvent is hard to treat, S. $\overline{\text{O}}$uchi [**10**] used «asymptotic resolvents» to state a simplified generation theorem of Hille-Yosida type [**10**], Thm. 2.1. Following this approach, C. Grosu proved in [**5**] a version of the Trotter-Kato theorems for locally equicontinuous semigroups in Fréchet-Schwartz spaces for continuous (or, in some case, even bounded) generators.

The aim of this paper is to prove Trotter-Kato theorems and the Lie-Trotter product formula for locally equicontinuous semigroups in the general setting of sequentially complete locally convex spaces with no additional assumption on the generator, making only use of the notion of asymptotic resolvent due to S. $\overline{\text{O}}$uchi and of «pseudo asymptotic resolvent» (see Def. 8). Therefore, we substantially improve the results of C. Grosu [**5**] and extend the theory of locally equicontinuous semigroups. Moreover, we apply our results to prove that the Lie-Trotter product formula is also available for the Ornstein-Uhlenbeck semigroup on the space $C_b(\mathbb{R}^n)$ of bounded continuous functions endowed with a suitable locally convex topology τ agreeing with the compact-open topology on bounded subsets of $\mathbb{R}^n$. We point out that in [**13**], [**14**] the Lie-Trotter product formula has been proved for a class of transition Markov semigroups associated to stochastic differential equations, including Ornstein-Uhlenbeck semigroups. Hovewer it has been used there a probabilistic approach very different from the functional one which is considered in the present paper.

The paper is divided into 5 sections. After the introduction, we recall in Section 2 some basic facts and introduce pseudo asymptotic resolvents together with their main properties. In Section 3 we prove preliminary convergence results of sequences of locally equicontinuous semigroups and their asymptotic resolvents. Section 4 is devoted to the Trotter-Kato theorems. Finally, in Section 5 we prove the Lie-Trotter product formula for locally equicontinuous semigroups and apply it to the Ornstein-Uhlenbeck semigroup on $C_b(\mathbb{R}^n)$.

2 - Locally equicontinuous semigroups and asymptotic resolvents

Let X be a sequentially complete locally convex space. We denote by $\mathcal{L}(X)$ the space of all continuous linear operators on X and by P the set of all continuous seminorms on X. We consider locally equicontinuous semigroups and use the definition of S. Ōuchi [**10**], Def. 1.1.

Definition 1. *A family $\{T(t): t \geq 0\}$ in $\mathcal{L}(X)$ is called a **locally equicontinuous semigroup** if it satisfies the following conditions.*

(a) $T(t)\,T(s) = T(t+s)$ *for all* $t, s \geq 0$, $T(0) = Id$.

(b) $\lim_{t \to s} T(t)\,x = T(s)\,x$ *for all* $s \geq 0$, $x \in X$.

(c) *For all* $a > 0$ *the subset* $\{T(t): 0 \leq t \leq a\}$ *is equicontinuous, i.e., for all* $p \in P$ *there exists* $q \in P$ *such that*

$$p(T(t)\,x) \leq q(x) \quad \text{for all} \quad 0 \leq t \leq a \quad \text{and} \quad x \in X\,.$$

*The **generator** $(A, D(A))$ of a locally equicontinuous semigroup $(T(t))_{t \geq 0}$ is defined as*

$$Ax := \lim_{t \to 0^+} \frac{T(t)\,x - x}{t} \tag{1}$$

with domain $D(A) := \left\{x \in X : \lim_{t \to 0^+} \frac{T(t)\,x - x}{t} \text{ exists}\right\}$.

Remark 2. If X is barreled, by [**12**], Chapt. III, Thm. 4.2, conditions (a) and (b) in Definition 1 imply the local equicontinuity of a family of continuous and linear operators on X. Moreover the continuity of the map $t \mapsto T(t)\,x$ at 0 and local equicontinuity imply the continuity at every point in $\mathbb{R}_+$. To see this, let $t_0 > 0$, $x \in X$ and $p \in P$. By Definition 1, (c), there exists $q \in P$ such that $p(T(t)\,y) \leq q(y)$ for all $0 \leq t \leq t_0$ and $y \in X$. Therefore, $p(T(t_0 + h)\,x - T(t_0)\,x) \leq q(T(h)\,x - x)$ if $h > 0$ and $p(T(t_0 + h)\,x - T(t_0)\,x) \leq q(x - T(-h)\,x)$ if $-t_0 \leq h < 0$ which implies the continuity at t_0.

We now put, for a fixed $a > 0$ and for each $\lambda \in \mathbb{C}$,

$$R(\lambda, A)\,x := \int_0^a e^{-\lambda t}\,T(t)\,x\,dt \quad \text{for } x \in X\,. \tag{2}$$

By the local equicontinuity of the semigroup $(T(t))_{t \geq 0}$ and the sequentially completeness of X, the integral exists in the sense of Riemann, and we obtain the following important properties of the operators $R(\lambda, A)$ and its relation to the generator of the semigroup $(T(t))_{t \geqslant 0}$ (see [**7**], Prop. 1.3, 1.4, Cor. p. 261, [**10**], Prop. 1.2).

Proposition 3. *Let $(T(t))_{t \geqslant 0}$ be a locally equicontinuous semigroup on X with generator $(A, D(A))$. Then $(A, D(A))$ is a closed and densely defined operator. Moreover, the operator $R(\lambda, A)$ defined as in* (2) *maps X into $D(A)$, and we have*

(i) $(\lambda - A) R(\lambda, A) x = x - e^{-\lambda a} T(a) x$ *for all* $x \in X$,

(ii) $R(\lambda, A) R(\mu, A) x = R(\mu, A) R(\lambda, A) x$ for all $x \in X$,

(iii) $R(\lambda, A) Ax = AR(\lambda, A) x$ for all $x \in D(A)$.

(iv) *The operator $R(\lambda, A) x$ is an X-valued holomorphic function in λ for all $x \in X$, $R(\lambda, A) \in \mathcal{L}(X)$ for all λ, and the family of operators*

$$\left\{ \frac{\lambda^{n+1}}{n!} \frac{d^n}{d\lambda^n} R(\lambda, A) : \lambda > 0,\ n = 0, 1, 2, \dots \right\}$$

is equicontinuous.

Based on these properties, S. Ōuchi [**10**] defines the so-called *asymptotic resolvent* and states a generalized Hille-Yosida Theorem (see [**10**], Def. 2.1, Thm. 2.1).

Definition 4. *A family $\{R(\lambda, A) : \lambda > \omega\}$, $\omega \in \mathbb{R}$, in $\mathcal{L}(X)$ is called an* ***asymptotic resolvent*** *of a closed operator $(A, D(A))$ if it satisfies the following conditions.*

(a) *The operator $R(\lambda, A) x$ is infinitely differentiable in λ $(\lambda > \omega)$ for all $x \in X$ and $R(\lambda, A)$ maps X into $D(A)$.*

(b) $AR(\lambda, A) = R(\lambda, A)A$ *on* $D(A)$.

(c) $R(\lambda, A) R(\mu, A) = R(\mu, A) R(\lambda, A)$ *on X for all $\lambda, \mu > \omega$.*

(d) $(\lambda - A) R(\lambda, A) = Id + S(\lambda, A)$, *where $S(\lambda, A) \in \mathcal{L}(X)$ and $S(\lambda, A) x$ is infinitely differentiable in λ for all $x \in X$, and for all $p \in P$ there exists $q \in P$ such that*

$$p\left(\frac{d^n}{d\lambda^n} S(\lambda, A) x \right) \leqslant C_p^n e^{-C_p \lambda} q(x)$$

for all $x \in X$, $\lambda > \omega$, and some constant $C_p > 0$.

Consequently, if $R(\lambda, A)$ is an asymptotic resolvent of A, we have

$$(\lambda - A)\, R(\lambda, A) = Id + S(\lambda, A),$$

where $S(\lambda, A)\, x$ is infinitely differentiable in λ for every $x \in X$. Differentiating this equation $(k+1)$-times in λ, we obtain

$$(\lambda - A)\, R^{(k+1)}(\lambda, A)\, x + (k+1)\, R^{(k)}(\lambda, A)\, x = S^{(k+1)}(\lambda, A)\, x \tag{3}$$

for all $x \in X$ (cf. [**10**], p. 269). Moreover, if $(A, D(A))$ is the generator of a locally equicontinuous semigroup $(T(t))_{t \geqslant 0}$ on X, and $R(\lambda, A)$ is defined as in (2) for some fixed a and all $\lambda \in \mathbb{C}$, then $\{R(\lambda, A) : \lambda > 0\}$ is an asymptotic resolvent of A, which is called the **canonical asymptotic resolvent** of A, where we take

$$S(\lambda, A) := -e^{-\lambda a}\, T(a) \tag{4}$$

for all $\lambda > 0$, and hence

$$S^{(j)}(\lambda, A) = (-1)^j a^j e^{-\lambda a}\, T(a) \tag{5}$$

for all $\lambda > 0$ and $j \in \mathbb{N}$. The canonical asymptotic resolvent will play an essential role for convergence properties of sequences of locally equicontinuous semigroups in Section 3 and the Trotter-Kato approximation results in Section 4.

The notion of asymptotic resolvent leads to a generalized Hille-Yosida Theorem stated in [**10**], Thm. 2.1.

Theorem 5. *Let $(A, D(A))$ be a linear operator on X. Then the following conditions are equivalent.*

(i) *The operator $(A, D(A))$ is the generator of a locally equicontinuous semigroup $(T(t))_{t \geqslant 0}$.*

(ii) *The operator $(A, D(A))$ is closed and densely defined, and there exists an asymptotic resolvent $\{R(\lambda, A) : \lambda > \omega\}$ of A such that the family*

$$\left\{ \frac{\lambda^{n+1}}{n!} \frac{d^n}{d\lambda^n} R(\lambda, A) : \lambda > \omega,\ n = 0, 1, 2, \dots \right\}$$

is equicontinuous.

As in the case of strongly continuous semigroups on Banach spaces it is useful to introduce the notion of a core of a linear operator.

Definition 6. *A subspace D of the domain of a linear operator $A : D(A) \subseteq X \to X$ is called a* **core** *for A if D is dense in $D(A)$ for the set $\widetilde{P}$ of seminorms* *defined as*

$$\tilde{p}(x) := p(x) + p(Ax) \quad \textit{for } p \in P,\ x \in D(A).$$

We note that $D(A)$ endowed with the topology induced by $\tilde{P}$ is also sequentially complete if X is. Similarly to the Banach space case we can state a criterion for subspaces to be a core for the generator of a locally equicontinuous semigroup.

Proposition 7. *Let $(A, D(A))$ be the generator of a locally equicontinuous semigroup $(T(t))_{t \geq 0}$. A subspace D of $D(A)$ which is dense in X and invariant under the semigroup $(T(t))_{t \geq 0}$ is a core for A.*

Proof. For every $x \in D(A)$, we can find a net $(x_\alpha)_{\alpha \in A} \subseteq D$ such that $x_\alpha \to x$ in X because D is dense in X. Since for each α the map $\mathbb{R}_+ \ni s \mapsto T(s)\, x_\alpha \in D$ is continuous with respect to the family of seminorms $\tilde{P}$ (see [7], Prop. 1.2 (1)), it follows that

$$\int_0^t T(s)\, x_\alpha\, ds \in \overline{D}^{\tilde{P}}.$$

Similarly, the $\tilde{P}$-continuity of $\mathbb{R}_+ \ni s \mapsto T(s)\, x$ for $x \in D(A)$ and [7], Prop. 1.2 (2) implies that

$$\tilde{p}\left(\frac{1}{t}\int_0^t T(s)\, x ds - x\right) \leq p\left(\frac{1}{t}\int_0^t T(s)\, x ds - x\right) + p\left(\frac{1}{t}\int_0^t T(s)\, Ax\, ds - Ax\right)$$

for every $\tilde{p} \in \tilde{P}$. This converges to zero if t tends to zero, and also

$$\tilde{p}\left(\frac{1}{t}\int_0^t T(s)\, x_\alpha\, ds - \frac{1}{t}\int_0^t T(s)\, x ds\right) \to 0$$

as α varies in A for each $t > 0$. Therefore, for every $\varepsilon > 0$ there exists $t > 0$ and $\alpha \in A$ such that

$$\tilde{p}\left(\frac{1}{t}\int_0^t T(s)\, x_\alpha\, ds - x\right) < \varepsilon .$$

Hence, $x \in \overline{D}^{\tilde{P}}$. ∎

In the remaining part of this section we introduce the so-called asymptotic pseudo resolvent which will be the main technical tool to obtain approximation results for locally equicontinuous semigroups.

Definition 8. *We consider* $\mathcal{R}(\lambda) \in \mathcal{L}(X)$ *for each* $\lambda > \omega$. *The family* $\{\mathcal{R}(\lambda) : \lambda > \omega\}$ *is called an* ***asymptotic pseudo resolvent*** *if for all* $\lambda > \omega$ *there exists* $\mathcal{S}(\lambda) \in \mathcal{L}(X)$ *such that the following conditions hold.*

(a) *The operator* $\mathcal{S}(\lambda)\, x$ *is infinitely differentiable in* λ *for all* $x \in X$, *and for all* $p \in P$ *there exists* $q \in P$ *such that*

$$p\left(\frac{d^k}{d\lambda^k}\, \mathcal{S}(\lambda)\, x\right) \leq a^k e^{-a\lambda} q(x)$$

for all $x \in X$, $\lambda > \omega$, $k = 0, 1, 2, \ldots$ *and some* $a > 0$.

(b) *For each* $\lambda, \mu > \omega$, *the equation*

$$(\lambda - \mu)\, \mathcal{R}(\lambda)\, \mathcal{R}(\mu) = \mathcal{R}(\mu) - \mathcal{R}(\lambda) + \mathcal{S}(\lambda)\, \mathcal{R}(\mu) - \mathcal{S}(\mu)\, \mathcal{R}(\lambda)$$

holds.

(c) *The operators* $\mathcal{S}(\lambda)$ *and* $\mathcal{R}(\mu)$ *commute for all* $\lambda, \mu > \omega$.

We remark that for an asymptotic pseudo resolvent $\mathcal{R}(\lambda)$ we have that

$$\mathcal{R}(\lambda)\, \mathcal{R}(\mu) = \mathcal{R}(\mu)\, \mathcal{R}(\lambda) \text{ for all } \lambda, \mu > \omega.$$

If we suppose that the asymptotic pseudo resolvent is injective with dense range at some point λ, we obtain the existence of a densely defined and closed linear operator having an asymptotic resolvent coinciding with the given asymptotic pseudo resolvent.

Proposition 9. *Let* $\{\mathcal{R}(\lambda) : \lambda > \omega\}$ *be an asymptotic pseudo resolvent on* X. *If* $Ker\, \mathcal{R}(\lambda_0) = \{0\}$ *and* $Rg\, \mathcal{R}(\lambda_0)$ *is dense in* X *for some* $\lambda_0 > \omega$, *then there exists a densely defined and closed linear operator* $(A, D(A))$ *such that* $\mathcal{R}(\lambda) = R(\lambda, A)$ *is an asymptotic resolvent of* A.

Proof. Since $Ker\, \mathcal{R}(\lambda_0) = \{0\}$, we can define a linear operator by

$$A := \lambda_0 Id - (Id + \mathcal{S}(\lambda_0))\, \mathcal{R}^{-1}(\lambda_0).$$

Since $\mathcal{R}^{-1}(\lambda_0)$ is closed and $(Id + \mathcal{S}(\lambda_0)) \in \mathcal{L}(X)$, the operator A is closed with domain $D(A) := Rg\ \mathcal{R}(\lambda_0)$ which is dense in X. From the definition of A it follows that

$$(\lambda_0 - A)\ \mathcal{R}(\lambda_0)\, x = x + \mathcal{S}(\lambda_0)\, x$$

for all $x \in X$ and

$$(\lambda_0 - A)\ \mathcal{R}(\lambda_0)\, x = \mathcal{R}(\lambda_0)(\lambda_0 - A)\, x = x + \mathcal{S}(\lambda_0)\, x$$

for all $x \in D(A)$. By the properties listed in Definition 8 we have

$$\begin{aligned} \mathcal{R}(\lambda_0)(Id + \mathcal{S}(\lambda)) &= (\lambda - \lambda_0)\ \mathcal{R}(\lambda)\,\mathcal{R}(\lambda_0) + \mathcal{R}(\lambda)[Id + \mathcal{S}(\lambda_0)] \\ &= \mathcal{R}(\lambda)[(\lambda - \lambda_0)\ \mathcal{R}(\lambda_0) + Id + \mathcal{S}(\lambda_0)] \\ &= \mathcal{R}(\lambda)[(\lambda - \lambda_0)\ \mathcal{R}(\lambda_0) + (\lambda_0 - A)\ \mathcal{R}(\lambda_0)] \\ &= \mathcal{R}(\lambda)(\lambda - A)\ \mathcal{R}(\lambda_0) \end{aligned} \tag{6}$$

for all $\lambda > \omega$. In the same way, we obtain

$$(\lambda - A)\ \mathcal{R}(\lambda_0)\ \mathcal{R}(\lambda) = \mathcal{R}(\lambda_0)[Id + \mathcal{S}(\lambda)]. \tag{7}$$

By (6) we conclude that

$$\mathcal{R}(\lambda)(\lambda - A) = Id + \mathcal{S}(\lambda) \quad \text{in } D(A) = Rg\ \mathcal{R}(\lambda_0).$$

Moreover, since $\mathcal{R}(\lambda)\, x \in D(A)$ for $x \in D(A)$, by (7), we have

$$(\lambda - A)\ \mathcal{R}(\lambda) = Id + \mathcal{S}(\lambda) \qquad \text{in } D(A). \tag{8}$$

To complete the proof, it is sufficient to show that $Rg\ \mathcal{R}(\lambda) \subseteq D(A)$ and that equation (8) holds on X. Let $x \in X$. Since $D(A) = Rg\ \mathcal{R}(\lambda_0)$ is dense in X, there exists a net $(x_\alpha)_{\alpha \in A} \subseteq D(A)$ such that $x_\alpha \to x$ and hence $\mathcal{R}(\lambda)\, x_\alpha \to \mathcal{R}(\lambda)\, x$ and $\mathcal{S}(\lambda)\, x_\alpha \to \mathcal{S}(\lambda)\, x$. By (8) we obtain

$$(\lambda - A)\ \mathcal{R}(\lambda)\, x_\alpha = x_\alpha + \mathcal{S}(\lambda)\, x_\alpha \to x + \mathcal{S}(\lambda)\, x\,.$$

Since $\lambda - A$ is a closed operator with domain $D(A)$, it follows that $\mathcal{R}(\lambda)\, x \in D(A)$ and $(\lambda - A)\ \mathcal{R}(\lambda)\, x = x + \mathcal{S}(\lambda)\, x$. ■

We remark that $Ker(\lambda_0 - A) = D(A) \cap Ker(Id + S(\lambda_0))$ and $(\lambda_0 - A)\ R(\lambda_0)\, X = (Id + S(\lambda_0))\, X$.

A particular case of Proposition 9 is stated in the following proposition.

Proposition 10. *Let* $\{\mathcal{R}(\lambda) : \lambda > \omega\}$ *be an asymptotic pseudo resolvent and* $(Id + \mathcal{S}(\lambda_0))$ *bijective for some* $\lambda_0 > \omega$, *where* $\mathcal{S}(\lambda_0)$ *denotes the operator corre-*

sponding to the asymptotic pseudo resolvent $\mathcal{R}(\lambda_0)$. *If for an unbounded sequence* $(\lambda_n)_{n\in\mathbb{N}}$, $\lambda_n > \omega$,

$$\lim_{n\to\infty} \lambda_n \mathcal{R}(\lambda_n)\, x = x \quad \textit{for all } x \in X, \tag{9}$$

then $\{\mathcal{R}(\lambda) : \lambda > \omega\}$ *is an asymptotic resolvent of a densely defined and closed operator. In particular,* (9) *holds if for all* $p \in P$ *there exists* $q \in P$ *such that*

$$p(\lambda_n \mathcal{R}(\lambda_n)\, x) \leqslant C_p\, q(x)$$

for all $x \in X$, $n \in \mathbb{N}$, *and some constant* $C_p \geqslant 0$, *and* $Rg\, \mathcal{R}(\lambda)$ *is dense for some* $\lambda > \omega$.

Proof. By the properties of the asymptotic pseudo resolvent (see Def. 8) we obtain

$$\mathcal{R}(\lambda_n) + \mathcal{S}(\lambda_0)\, \mathcal{R}(\lambda_n) = \mathcal{R}(\lambda_0) + \mathcal{S}(\lambda_n)\, \mathcal{R}(\lambda_0) - (\lambda_n - \lambda_0)\, \mathcal{R}(\lambda_n)\, \mathcal{R}(\lambda_0),$$

which is equivalent to

$$\mathcal{R}(\lambda_n)[Id + \mathcal{S}(\lambda_0)] = \mathcal{R}(\lambda_0)[Id + \mathcal{S}(\lambda_n) - (\lambda_n - \lambda_0)\, \mathcal{R}(\lambda_n)].$$

This implies, by the surjectivity of $(Id + \mathcal{S}(\lambda_0))$, that

$$Rg\ \mathcal{R}(\lambda_n) \subseteq Rg\ \mathcal{R}(\lambda_0),$$

and therefore

$$\bigcup_{n\in\mathbb{N}} Rg\ \mathcal{R}(\lambda_n) \subseteq Rg\ \mathcal{R}(\lambda_0).$$

Consequently, by (9), we have

$$X = \overline{\bigcup_{n\in\mathbb{N}} Rg\ \mathcal{R}(\lambda_n)} = \overline{Rg\ \mathcal{R}(\lambda_0)},$$

hence $\mathcal{R}(\lambda_0)$ has dense range in X. If $x \in Ker\ \mathcal{R}(\lambda_0)$, we obtain, again by the properties of the pseudo asymptotic resolvent, that

$$\lambda_n \mathcal{R}(\lambda_n)[Id + \mathcal{S}(\lambda_0)]\, x = \lambda_n [Id + \mathcal{S}(\lambda_n) - (\lambda_n - \lambda_0)\, \mathcal{R}(\lambda_n)]\, \mathcal{R}(\lambda_0)\, x = 0$$

for all $n \in \mathbb{N}$. By passing to the limit we obtain $(Id + \mathcal{S}(\lambda_0))\, x = 0$, which implies $x = 0$ by assumption. Applying Proposition 9 we can conclude that there exists a densely defined and closed operator $(A, D(A))$ such that $\mathcal{R}(\lambda) = R(\lambda, A)$.

We now suppose that for all $p \in P$ there exists $q \in P$ such that

$$p(\lambda_n \mathcal{R}(\lambda_n)\, x) \leq C_p\, q(x)$$

for all $x \in X$, $n \in \mathbb{N}$, and some constant $C_p > 0$, and $Rg\ \mathcal{R}(\lambda)$ is dense for some $\lambda > \omega$. Fix $p \in P$ and take $q \in P$ such that the above inequality and condition (a) in Definition 8 hold. Then, by the asymptotic pseudo resolvent equation, we obtain that

$$\begin{aligned} p((\lambda_n \mathcal{R}(\lambda_n) - Id)\ \mathcal{R}(\lambda)\, x) &= p(\mathcal{S}(\lambda_n)\ \mathcal{R}(\lambda)\, x + \mathcal{R}(\lambda_n)[\lambda \mathcal{R}(\lambda) - (Id + \mathcal{S}(\lambda))]\, x) \\ &\leq p(\mathcal{S}(\lambda_n)\ \mathcal{R}(\lambda)\, x) + p(\mathcal{R}(\lambda_n)[\lambda \mathcal{R}(\lambda) - (Id + \mathcal{S}(\lambda))]\, x) \\ &\leq e^{-a\lambda_n} q(\mathcal{R}(\lambda)\, x) + \frac{C_p}{\lambda_n} q(\lambda \mathcal{R}(\lambda)\, x - (Id + \mathcal{S}(\lambda))\, x), \end{aligned}$$

which converges to zero as $n \to \infty$ for all $x \in X$. Since $Rg\ \mathcal{R}(\lambda)$ is dense in X for some $\lambda > \omega$, we can conclude that

$$\lambda_n \mathcal{R}(\lambda_n)\, x \to x$$

for all $x \in X$ as $n \to \infty$. ■

3 - Sequences of locally equicontinuous semigroups and asymptotic resolvents

In this section, we prove convergence properties of sequences of locally equicontinuous semigroups and of their asymptotic resolvents. To that purpose, we first introduce the notion of a uniformly locally equicontinuous sequence of semigroups.

Definition 11. *Let $\{(T_n(t))_{t \geq 0} : n \in \mathbb{N}\}$ be a sequence of locally equicontinuous semigroups on X. It is called* **uniformly locally equicontinuous**, *if for all $s > 0$ and for all $p \in P$ there exists $q \in P$ such that*

$$p(T_n(t)\, x) \leq q(x) \quad \textit{for all} \quad 0 \leq t \leq s,\ x \in X \textit{ and } n \in \mathbb{N}.$$

If A_n is the generator of $(T_n(t))_{t \geq 0}$ and $R(\lambda, A_n)$ its corresponding canonical asymptotic resolvent defined for the same $a > 0$, then

$$\frac{\lambda^{j+1}}{j!}\ \frac{d^j}{d\lambda^j} R(\lambda, A_n)\, x = (-1)^j \lambda^{j+1} \int_0^a e^{-\lambda t}\ \frac{t^j}{j!}\, T_n(t)\, x dt$$

for all $x \in X$, $n, j \in \mathbb{N}$ and $\lambda > 0$. Consequently, if $\{(T_n(t))_{t \geq 0} : n \in \mathbb{N}\}$ is uniformly locally equicontinuous, then for every $p \in P$ there exists $q \in P$ such that

$$p\left(\frac{\lambda^{j+1}}{j!}\frac{d^j}{d\lambda^j}R(\lambda, A_n)\,x\right) \leqslant q(x)\left|\frac{\lambda^{j+1}}{j!}\int_0^a e^{-\lambda t}t^j dt\right| \leqslant q(x) \tag{10}$$

for all $x \in X$, $j, n \in \mathbb{N}$ and $\lambda > 0$.

In the theory of strongly continuous semigroups on Banach spaces the equation

$$\frac{d^n}{d\lambda^n}R(\lambda, A) = (-1)^n n!\, R(\lambda, A)^{n+1}, \qquad n \in \mathbb{N},\ \lambda \in \varrho(A), \tag{11}$$

is essential to obtain that the convergence of the resolvents at one point $\lambda_0 > 0$ implies already their convergence for all $\lambda > 0$. However, for pseudo asymptotic resolvents equation (11) fails, and we must work with the derivatives of the asymptotic resolvents instead of their powers. The following lemma is one of the basic steps towards Trotter-Kato theorems for locally equicontinuous semigroups.

Lemma 12. *Let $\{(T_n(t))_{t \geqslant 0} : n \in \mathbb{N}\}$ be a sequence of uniformly locally equicontinuous semigroups on X. Let $(A_n, D(A_n))$, $n \in \mathbb{N}$, be the generator of $(T_n(t))_{t \geqslant 0}$ and $R(\lambda, A_n)$ be the corresponding canonical asymptotic resolvent defined for the same $a > 0$. If*

$$\lim_{n \to \infty} \frac{d^j}{d\lambda^j} R(\lambda_0, A_n)\, x =: R^{(j)}(\lambda_0)\, x$$

exists for all $j \in \mathbb{N}$, $x \in X$ and some $\lambda_0 > 0$, then

$$\lim_{n \to \infty} \frac{d^j}{d\lambda^j} R(\lambda, A_n)\, x =: R^{(j)}(\lambda)\, x$$

exists for all $j \in \mathbb{N}$, $x \in X$ and $\lambda > 0$.

Proof. We set $R^{(j)}(\lambda, A_n) := \dfrac{d^j}{d\lambda^j} R(\lambda, A_n)$. By Proposition 3(iv) $R(\lambda, A_n)\,x$ is a vector-valued holomorphic function in λ for all $x \in X$ and $n \in \mathbb{N}$. Therefore, $R(\lambda, A_n)\,x$ has a power series expansion around λ_0 given by

$$R(\lambda, A_n)\,x = \sum_{j \geqslant 0} \frac{1}{j!}(\lambda - \lambda_0)^j R^{(j)}(\lambda_0, A_n)\, x\,,$$

where the series converges in X uniformly for $n \in \mathbb{N}$ if $\lambda > 0$. Indeed, by (10), for every $p \in P$ there exists $q \in P$ such that

$$p\left(\sum_{j \geqslant h} \frac{(\lambda-\lambda_0)^j}{j!} R^{(j)}(\lambda_0, A_n)\, x\right) \leqslant \sum_{j \geqslant h} \frac{|\lambda-\lambda_0|^j a^j}{j!} q(x) \int_0^a e^{-\lambda_0 t}\, dt$$

$$\leqslant q(x)\, \frac{1-e^{-\lambda_0 a}}{\lambda_0} \sum_{j \geqslant h} \frac{|\lambda-\lambda_0|^j a^j}{j!}$$

for all $x \in X$, and $h, n \in \mathbb{N}$. Let $\lambda > 0$ and $x \in X$. We now show that $\lim_{n \to \infty} R^{(k)}(\lambda, A_n)\, x$ exists for all $k \in \mathbb{N}$. Since X is sequentially complete, it is sufficient to prove that $\{R^{(k)}(\lambda, A_n)\, x\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X for all $k \in \mathbb{N}$. To that purpose, let $k \in \mathbb{N}$, $p \in P$ and $\varepsilon > 0$. There exists $h_0 \in \mathbb{N}$, $h_0 \geqslant k$, such that

$$\sum_{j \geqslant h} \frac{|\lambda-\lambda_0|^{j-k}}{(j-k)!} a^j q(x)\, \frac{1-e^{-\lambda_0 a}}{\lambda_0} < \frac{\varepsilon}{4}$$

for all $h \geqslant h_0$, where $q \in P$ is taken as in (10). Therefore, we obtain that

$$p(R^{(k)}(\lambda, A_n)\, x - R^{(k)}(\lambda, A_m)\, x)$$

$$\leqslant p\left(\sum_{j=k}^{h_0} \frac{(\lambda-\lambda_0)^{j-k}}{(j-k)!} (R^{(j)}(\lambda_0, A_n)\, x - R^{(j)}(\lambda_0, A_m)\, x)\right)$$

$$+ p\left(\sum_{j \geqslant h_0} \frac{(\lambda-\lambda_0)^{j-k}}{(j-k)!} (R^{(j)}(\lambda_0, A_n)\, x - R^{(j)}(\lambda_0, A_m)\, x)\right)$$

$$\leqslant p\left(\sum_{j=k}^{h_0} \frac{(\lambda-\lambda_0)^{j-k}}{(j-k)!} (R^{(j)}(\lambda_0, A_n)\, x - R^{(j)}(\lambda_0, A_m)\, x)\right) + \frac{\varepsilon}{2}$$

$$=: C + \frac{\varepsilon}{2}$$

for all $n, m \in \mathbb{N}$. By assumption $\{R^{(j)}(\lambda_0, A_n)\, x\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X for all $j \in \mathbb{N}$. Hence, there exists $n_0 \in \mathbb{N}$ such that $C < \frac{\varepsilon}{2}$ for all $n, m \geqslant n_0$. This concludes the proof. ■

Remark 13. Under the assumptions of Lemma we obtain that

$$\frac{d^j}{d\lambda^j} R(\lambda)\, x = \lim_{n\to\infty} R^{(j)}(\lambda, A_n)\, x = R^{(j)}(\lambda)\, x\,,$$

and hence, by using (10), for each $p \in P$ there exists $q \in P$ such that

$$p\left(\frac{\lambda^{j+1}}{j!}\,\frac{d^j}{d\lambda^j} R(\lambda)\, x\right) \leqslant q(x)$$

for all $x \in X, j \in \mathbb{N}$ and $\lambda > 0$.

4 - Trotter-Kato approximation theorems

In this section we characterize the convergence of uniformly locally equicontinuous semigroups by the convergence of their canonical asymptotic resolvents.

Theorem 14. *Let $\{(T_n(t))_{t \geqslant 0} : n \in \mathbb{N}\}$ and $(T(t))_{t \geqslant 0}$ be uniformly locally equicontinuous semigroups on X with generators A_n and A, respectively. For each $n \in \mathbb{N}$ and some $\lambda_0 > 0$ let $R(\lambda_0, A_n)$ and $R(\lambda_0, A)$ be the corresponding canonical asymptotic resolvents defined for the same $a > 0$ and $S(\lambda_0, A_n)$ and $S(\lambda_0, A)$ be defined by* (4), *respectively. Suppose that $S(\lambda_0, A_n)\, x \to S(\lambda_0, A)\, x$ for all $x \in X$ and $Rg\, R(\lambda_0, A)$ is dense in X. Then the following assertions are equivalent.*

(a) $R(\lambda_0, A_n)\, x \to R(\lambda_0, A)\, x$ *for all* $x \in X$.

(b) $T_n(t)\, x \to T(t)\, x$ *for all $x \in X$ and uniformly for t in compact intervals in $\mathbb{R}_+$.*

Proof. The proof is inspired by the one given in [**11**], Thm. 4.2.
$(a) \Rightarrow (b)$: Let $x \in X$, $0 \leqslant t \leqslant T$ and $p \in P$. Then

$$\begin{aligned} &p((T_n(t) - T(t))\, R(\lambda_0, A)\, x) \\ &\qquad \leqslant p(T_n(t)(R(\lambda_0, A) - R(\lambda_0, A_n))\, x) \\ &\qquad\quad + p(R(\lambda_0, A_n)(T_n(t) - T(t))\, x) + p((R(\lambda_0, A_n) - R(\lambda_0, A))\, T(t)\, x) \\ &\qquad =: a_n(t) + b_n(t) + c_n(t)\,. \end{aligned} \tag{12}$$

Since $p(T_n(t)\, x) \leqslant q(x)$ for all $0 \leqslant t \leqslant T$ and some suitable $q \in P$, it follows that $a_n(t)$ converges to zero uniformly on $[0, T]$ as n tends to infinity. Also since $t \mapsto T(t)\, x$ is continuous, the set $\{T(t)\, x : 0 \leqslant t \leqslant T\}$ is compact in X and therefore, by [**12**], Chapt. III, Thm. 4.3, $c_n(t) \to 0$ uniformly on $[0, T]$ as $n \to \infty$.

It remains to prove that $b_n(t) \to 0$ uniformly on $[0, T]$ as $n \to \infty$. To show this, we consider, for each $t \in [0, T]$ and $y \in X$, the map

$$[0, t] \ni s \mapsto T_n(t-s)\, R(\lambda_0, A_n)\, T(s)\, R(\lambda_0, A)\, y \in X$$

which is differentiable in $[0, t]$, and its derivative is given by

$$\begin{aligned}[0, t] \ni s \mapsto & -T_n(t-s)\, A_n R(\lambda_0, A_n)\, T(s)\, R(\lambda_0, A)\, y \\ & + T_n(t-s)\, R(\lambda_0, A_n)\, T(s)\, AR(\lambda_0, A)\, y \in X .\end{aligned}$$

Consequently, for each $t \in [0, T]$, $y \in X$, and $p' \in P$ we have

$$\begin{aligned}
& p'(R(\lambda_0, A_n)(T_n(t) - T(t))\, R(\lambda_0, A)\, y) \\
& \leqslant p'\left(\int_0^t T_n(t-s)[-A_n R(\lambda_0, A_n)\, T(s) + R(\lambda_0, A_n)\, T(s)\, A]\, R(\lambda_0, A)\, y ds \right) \\
& = p'\left(\int_0^t T_n(t-s)\{[Id + S(\lambda_0, A_n) - \lambda_0 R(\lambda_0, A_n)]\, R(\lambda_0, A) \right. \\
& \qquad \left. + R(\lambda_0, A_n)[\lambda_0 R(\lambda_0, A) - Id - S(\lambda_0, A)]\}\, T(s)\, y ds \right) \\
& \leqslant \int_0^t p'(T_n(t-s)[R(\lambda_0, A) - R(\lambda_0, A_n)]\, T(s)\, y)\, ds \\
& \qquad + \int_0^t p'(T_n(t-s)[S(\lambda_0, A_n) - S(\lambda_0, A)]\, R(\lambda_0, A)\, T(s)\, y)\, ds \\
& \qquad + \int_0^t p'(T_n(t-s)[R(\lambda_0, A) - R(\lambda_0, A_n)]\, S(\lambda_0, A)\, T(s)\, y)\, ds \\
& \leqslant \int_0^T q'([R(\lambda_0, A) - R(\lambda_0, A_n)]\, T(s)\, y)\, ds \\
& \qquad + \int_0^T q'([S(\lambda_0, A_n) - S(\lambda_0, A)]\, R(\lambda_0, A)\, T(s)\, y)\, ds \\
& \qquad + \int_0^T q'([R(\lambda_0, A) - R(\lambda_0, A_n)]\, S(\lambda_0, A)\, T(s)\, y)\, ds ,
\end{aligned}$$

which converges to zero uniformly on $[0, T]$ as n tends to infinity by assumption and the same compactness argument as above, where $q' \in P$ depends on p' and is taken as in Definition 11. Therefore, for each $y \in X$, we have

$$R(\lambda_0, A_n)(T_n(t) - T(t))\, R(\lambda_0, A)\, y \to 0$$

uniformly on $[0, T]$ as $n \to \infty$. Since $Rg\, R(\lambda_0, A)$ is dense in X, it follows that

$$R(\lambda_0, A_n)(T_n(t) - T(t))\, x \to 0$$

uniformly on $[0, T]$ as $n \to \infty$, thereby implying that $b_n(t) \to 0$ uniformly on $[0, T]$ as $n \to \infty$. Thus, with estimate (12), we obtain that

$$p(T_n(t)\, x - T(t)\, x) \to 0$$

uniformly on $[0, T]$ as $n \to \infty$.

$(b) \Rightarrow (a)$: For $x \in X$ we have, for every $p \in P$,

$$p(R(\lambda_0, A)\, x - R(\lambda_0, A_n)\, x) \leq \int_0^a e^{-\lambda_0 t} p(T(t)\, x - T_n(t)\, x)\, dt\,,$$

which converges to zero as $n \to \infty$. ■

Example 15. Let $C(\mathbb{R})$ be the space of continuous functions endowed with the compact-open topology τ_c. We consider the multiplication semigroup $(T_q(t))_{t \geq 0}$ defined as

$$T_q(t)\, f := e^{tq} f, \qquad t \geq 0,\ f \in C(\mathbb{R})\,,$$

for some function $q \in C(\mathbb{R})$. It can be easily verified that $(T_q(t))_{t \geq 0}$ is a locally equicontinuous semigroup on $(C(\mathbb{R}), \tau_c)$ and its generator is given by

$$Af = q \cdot f \quad \text{for all } f \in D(A) = C(\mathbb{R}).$$

Therefore, for fixed $a > 0$ and every $\lambda > 0$ the canonical asymptotic resolvent $R(\lambda, A)$ and the corresponding operator $S(\lambda, A)$ are given by

$$R(\lambda, A)\, f(s) = \frac{e^{(q(s) - \lambda)a} - 1}{q(s) - \lambda} f(s) \qquad \text{and} \qquad S(\lambda, A)\, f(s) = -e^{(q(s) - \lambda)a} f(s)$$

for all $f \in C(\mathbb{R})$ and $s \in \mathbb{R}$. Thus, the range of the canonical asymptotic resolvent is dense in $(C(\mathbb{R}), \tau_c)$.

Let us now consider a sequence $(q_n)_{n \in \mathbb{N}} \subset C(\mathbb{R})$ such that $(q_n)_{n \in \mathbb{N}}$ τ_c-converges to the function q. Then each A_n defined as $A_n f := q_n \cdot f$, $f \in C(\mathbb{R})$, generates a locally equicontinuous semigroup $(T_n(t))_{t\geq 0}$ given by

$$T_{q_n}(t)\, f = e^{tq_n} f, \qquad t \geq 0,\ f \in C(\mathbb{R}).$$

Furthermore, for every compact subset $K \subset \mathbb{R}$ we have

$$p_K(T_n(t)\, f) \leq \sup_{s\in K} |e^{tq_n(s)}|\, p_K(f) \leq e^{tM_K} p_K(f)$$

for all $f \in C(\mathbb{R})$, $t \geq 0$ and some constant $M_K := \sup_{n\in\mathbb{N}} \max_{s\in K} |q_n(s)|$. Consequently, $(T_n(t))_{t\geq 0}$, $n \in \mathbb{N}$, are uniformly locally equicontinuous semigroups on $(C(\mathbb{R}), \tau_c)$. Let $\lambda_0 > 0$. We have for each $a > 0$ that

$$S(\lambda_0, A_n)\, f = -e^{(q_n - \lambda_0)a} f \xrightarrow{\tau_c} S(\lambda_0, A)\, f = -e^{(q-\lambda_0)a} f$$

for all $f \in C(\mathbb{R})$. Finally, it is easy to see that $T_n(t)\, f \xrightarrow{\tau_c} T(t)\, f$ for all $f \in C(\mathbb{R})$ and $t \geq 0$.

The following result is very useful for applications because it permits us to conclude that an operator A is a generator of a locally equicontinuous semigroup only by assuming that a sequence $(A_n)_{n\in\mathbb{N}}$ of generators converges to it.

Theorem 16. *Let $\{(T_n(t))_{t\geq 0} : n \in \mathbb{N}\}$ be a sequence of uniformly locally equicontinuous semigroups on X with generators $(A_n, D(A_n))$. For each $n \in \mathbb{N}$ let $R(\lambda, A_n)$ be defined by* (1) *for the same $a > 0$ and $S(\lambda, A_n)$ defined by* (4). *Suppose that there exists $\lambda_0 > 0$ and $S_{\lambda_0} \in \mathcal{L}(X)$ such that $S(\lambda_0, A_n)\, x \to S_{\lambda_0} x$ for all $x \in X$ and $(Id + S_{\lambda_0})$ is bijective. Consider the following assertions:*

(a) *There exists a densely defined operator $(A, D(A))$ such that $A_n x \to Ax$ for all x in a core D of A and such that the range $Rg(\lambda_0 - A)$ is dense in X.*

(b) *There exists $R \in \mathcal{L}(X)$ such that $R(\lambda_0, A_n)\, x \to Rx$ for all $x \in X$, RgR is dense in X and $Ker R = \{0\}$.*

(c) The sequence $\{(T_n(t))_{t\geq 0} : n \in \mathbb{N}\}$ converges pointwise and uniformly in t on compact intervals of $\mathbb{R}_+$ to a locally equicontinuous semigroup $(T(t))_{t\geq 0}$ with generator $(B, D(B))$ such that $R = R(\lambda_0, B)$, where $R(\lambda_0, B)$ is an asymptotic resolvent of B.

Then the implications

$$(a) \Rightarrow (b) \Leftrightarrow (c)$$

hold. In particular, if (*a*) *holds, then B is an extension of $\overline{A}$.*

Proof. (*a*) $\Rightarrow$ (*b*): We observe that $\overline{(\lambda_0 - A)D} = X$. In fact, since $Rg(\lambda_0 - A)$ is dense in X, for fixed $x \in X$, $\varepsilon > 0$ and $p \in P$ there exists $y \in D(A)$ such that

$$p(x - (\lambda_0 - A)y) < \frac{\varepsilon}{2},$$

and there exists $z \in D$ such that

$$p((\lambda_0 - A)(y - z)) < \frac{\varepsilon}{2}.$$

Therefore,

$$p(x - (\lambda_0 - A)\, z) = p(x - (\lambda_0 - A)\, y + (\lambda_0 - A)(y - z)) < \varepsilon .$$

Take now $x \in D$ and put $y := (\lambda_0 - A)\, x$. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned} R(\lambda_0, A_n)\, y &= R(\lambda_0, A_n)[(\lambda_0 - A_n)\, x - (\lambda_0 - A_n)\, x + (\lambda_0 - A)\, x] \\ &= R(\lambda_0, A_n)(\lambda_0 - A_n)\, x + R(\lambda_0, A_n)(A_n x - Ax) \\ &= x + S(\lambda_0, A_n)\, x + R(\lambda_0, A_n)(A_n x - Ax). \end{aligned}$$

Hence $R(\lambda_0, A_n)\, y$ converges to $Ry := x + S_{\lambda_0} x$ since, by (10), for all $p \in P$ there exists $q \in P$ such that

$$p(R(\lambda_0, A_n)(A_n x - Ax)) \leqslant \frac{1}{\lambda_0}\, q(A_n x - Ax)$$

for all $n \in \mathbb{N}$. Moreover, by (10) again, for every $p \in P$ there exists $q \in P$ such that, for each $y \in (\lambda_0 - A)\, D$ and $n \in \mathbb{N}$,

$$p(R(\lambda_0, A_n)\, y) \leqslant \frac{1}{\lambda_0}\, q(y).$$

Therefore, letting $n \to +\infty$, we conclude that

$$p(Ry) \leqslant \frac{1}{\lambda_0}\, q(y) \tag{13}$$

for all $y \in (\lambda_0 - A)\, D$. This means that the linear operator R defined above is continuous on $(\lambda_0 - A)\, D$. Consequently, by the density of $(\lambda_0 - A)\, D$ in X, we can extend R continuously on X such that (13) still holds.

In the next step we show that

$$R(\lambda_0, A_n)\, x \to Rx \qquad \text{for all } x \in X.$$

Let $x \in X$, $\varepsilon > 0$ and $p \in P$. Corresponding to p there exists a continuous seminorm q on X such that (13) and

$$p(R(\lambda_0, A_n)\, x) \leq \frac{1}{\lambda_0} q(x) \qquad \text{for all } n \in \mathbb{N}$$

hold. Moreover, there exists $x_0 \in D$ such that

$$q(x - (\lambda_0 - A)\, x_0) < \frac{\lambda_0 \varepsilon}{3},$$

and there exists $n_0 \in \mathbb{N}$ such that

$$p(R(\lambda_0, A_n)(\lambda_0 - A)\, x_0 - R(\lambda_0 - A)\, x_0) < \frac{\varepsilon}{3}$$

for all $n \geq n_0$. By (13) it follows that, for each $n \geq n_0$,

$$\begin{aligned} p(R(\lambda_0, A_n)\, x - Rx) &\leq p(R(\lambda_0, A_n)\, x - R(\lambda_0, A_n)(\lambda_0 - A)\, x_0) \\ &\quad + p(R(\lambda_0, A_n)(\lambda_0 - A)\, x_0 - R(\lambda_0 - A)\, x_0) + p(R(\lambda_0 - A)\, x_0 - Rx) \\ &\leq \frac{2}{\lambda_0} q(x - (\lambda_0 - A)\, x_0) + \frac{\varepsilon}{3} \\ &\leq \varepsilon. \end{aligned}$$

We now show that RgR is dense in X. By definition we have $R(\lambda_0 - A) = Id + S_{\lambda_0}$, hence $R(\lambda_0 - A)\, D = (Id + S_{\lambda_0})\, D$. Since $Id + S_{\lambda_0}$ is surjective, we obtain

$$X = (Id + S_{\lambda_0})\, X = (Id + S_{\lambda_0})\, \overline{D} \subseteq \overline{(Id + S_{\lambda_0})\, D}.$$

Therefore, $X = \overline{R(\lambda_0 - A)\, D} = \overline{R\overline{(\lambda_0 - A) D}} = \overline{R(X)}$. Finally, $Ker R = \{0\}$ because $Id + S_{\lambda_0}$ is one-to-one and $(\lambda_0 - A)\, R = Id + S_{\lambda_0}$.

Since the implication $(c) \Rightarrow (b)$ holds, by Theorem 14, it remains to prove that $(b) \Rightarrow (c)$. To that purpose, we first show that the condition

(b′) For each $k \in \mathbb{N}$ there exists $R_k \in \mathcal{L}(X)$ such that $R^{(k)}(\lambda_0, A_n)\, x \to R_k x$ for all $x \in X$ and $Rg\, R_0$ is dense in X, implies (c). By Lemma 12 for each $\lambda > 0$ there exists $R_k(\lambda) \in \mathcal{L}(X)$ such that

$$R^{(k)}(\lambda, A_n)\, x \to R_k(\lambda)\, x$$

for all $x \in X$, $k \in \mathbb{N}$, $\lambda > 0$, and also, by Remark (13), that $R(\lambda)\, x$ is infinitely differentiable in $\lambda > 0$ with

$$\frac{d^k}{d\lambda^k} R(\lambda)\, x = R_k(\lambda)\, x$$

for all $x \in X$, $k \in \mathbb{N}$ and $\lambda > 0$. Moreover, by Remark 13, for every $p \in P$ there exists $q \in P$ such that

$$p\left(\frac{\lambda^{k+1}}{k!}\, \frac{d^k}{d\lambda^k} R(\lambda)\, x\right) \leq q(x) \tag{14}$$

for all $x \in X$, $k \in \mathbb{N}$ and $\lambda > 0$, where $R(\lambda_0) := R_0$. Furthermore, it is easy to verify that in this case $\{R(\lambda) : \lambda > 0\}$ is an asymptotic pseudo resolvent with $Rg\, R(\lambda_0)$ dense in X and $Ker\, R(\lambda_0) = \{0\}$. Thus, by Proposition 9, there exists a densely defined and closed operator $(B, D(B))$ such that $R(\lambda, B) = R(\lambda)$ for all $\lambda > \omega$, where $R(\lambda, B)$ is an asymptotic resolvent of B. Clearly, $R(\lambda, B)$ satisfies estimate (14), and therefore, by the generalized Hille-Yosida Theorem 5, $(B, D(B))$ generates a locally equicontinuous semigroup $(T(t))_{t \geq 0}$. We can now apply the implication $(a) \Rightarrow (b)$ from Theorem 14 in order to conclude that the semigroups $(T_n(t))_{t \geq 0}$ converge, in the desired way, to the semigroup $(T(t))_{t \geq 0}$.

Now, we observe that conditions (b) and (b′) are equivalent. In fact, (b) follows from (b′) by taking $R = R_0$. In order to obtain (b′) from (b) it remains to show that for each $k \in \mathbb{N}$, $k > 0$, there exists $R_k \in \mathcal{L}(X)$ such that

$$R^{(k)}(\lambda_0, A_n)\, x \to R_k x \qquad \text{for all } x \in X$$

as $n \to \infty$. We proceed by induction. Put $R_0 := R$. By assumption, the assertion for $k = 0$ holds. We assume that for some $k \geq 1$ there exists $R_k \in \mathcal{L}(X)$ such that

$$R^{(k)}(\lambda_0, A_n)\, x \to R_k x$$

for all $x \in X$. To obtain the result for $k + 1$, it is sufficient to show that there exists $R_{k+1} \in \mathcal{L}((\lambda_0 - A)\, D, X)$ such that, for all $x \in D$,

$$R^{(k+1)}(\lambda_0, A_n)(\lambda_0 - A)\, x \to R_{k+1}(\lambda_0 - A)\, x.$$

Let $x \in D$ and put $y := (\lambda_0 - A)x$. Then, by using (3), we have

$$R^{(k+1)}(\lambda_0, A_n)\, y = R^{(k+1)}(\lambda_0, A_n)(\lambda_0 - A_n)\, x + R^{(k+1)}(\lambda_0, A_n)(A_n x - Ax)$$

$$= S^{(k+1)}(\lambda_0, A_n)\, x - (k+1) R^{(k)}(\lambda_0, A_n)\, x + R^{(k+1)}(\lambda_0, A_n)(A_n x - Ax),$$

which converges to $S_{\lambda_0}^{(k+1)} x - (k+1) R_k x := R_{k+1} y$ by assumption and (5).

In the final step, we show that (a) implies B is an extension of $\overline{A}$. By assumption we have that

$$T_n(t)\, A_n x - T(t)\, Ax = T_n(t)(A_n x - Ax) + (T_n(t) - T(t))\, Ax$$

converges to zero as $n \to \infty$ for all $x \in D$ and uniformly for t in compact intervals in $\mathbb{R}_+$. By [7], Prop. 1.2, (2), we have

$$T_n(t)\, x - x = \int_0^t T_n(s)\, A_n x ds$$

and therefore, by letting $n \to \infty$,

$$T(t)\, x - x = \int_0^t T(s)\, Axds$$

for all $t \geqslant 0$. Thus

$$Bx = \lim_{t \searrow 0} \frac{T(t)\, x - x}{t} = \lim_{t \searrow 0} \int_0^t T(s)\, Axds = Ax\,,$$

and we obtain $D \subseteq D(B)$. We now fix $x \in D(A)$. Since D is a core for A, there exists a net $(x_\alpha)_{\alpha \in I} \subseteq D \subseteq D(B)$ such that $x_\alpha \to x$ and $Ax_\alpha \to Ax$. But $Bx_\alpha = Ax_\alpha$ for every $\alpha \in I$ and hence $Bx_\alpha \to Ax$. Since B is a closed operator, we obtain $x \in D(B)$ and $Bx = Ax$. Therefore, we have shown that $D(A) \subseteq D(B)$ and $Bx = Ax$ for all $x \in D(A)$.

Therefore, the proof is complete. ■

Example 17. Let $F \in C(\mathbb{R}^n; \mathbb{R}^n)$ be a globally Lipschitz function whose Lipschitz constant is $L > 0$. It follows from standard results that there exists a continuous function $\Phi : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi(t+s, x) = \Phi(t, \Phi(s, x))$ and

$\Phi(0, x) = x$ for every $t, s \geqslant 0$ and $x \in \mathbb{R}^n$, which solves the differential equation

$$\frac{\partial \Phi}{\partial t}(t, x) = F(\Phi(t, x)) \tag{15}$$

for all $t \geqslant 0$ and $x \in \mathbb{R}^n$. Moreover, by Gronwall's lemma, Φ also satisfies

$$\|\Phi(t, x)\| \leqslant (\|x\| + t\|F(0)\|)\, e^{Lt} \tag{16}$$

for all $t \geqslant 0$ and $x \in \mathbb{R}^n$.

For each $t \geqslant 0$, $f \in C(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$T(t)\, f(x) = f(\Phi(t, x)). \tag{17}$$

Then $(T(t))_{t \geqslant 0}$ is a locally equicontinuous semigroup on the space $C(\mathbb{R}^n)$ of the continuous functions on $\mathbb{R}^n$ endowed with the compact-open topology τ_c. Moreover, if $(A, D(A))$ is its generator and $f \in C^1(\mathbb{R}^n)$, $f \in D(A)$ and $Af = \langle \nabla f, F \rangle$.

Indeed, fix $f \in C(\mathbb{R}^n)$ and $R > 0$. If $\|x\| \leqslant R$ and $C_R = (R + \|F(0)\|)\, e^L$, then, by (16), for each $0 \leqslant t \leqslant 1$, by Lagrange's theorem,

$$\|\Phi(t, x) - \Phi(0, x)\| = \|F(\overline{\Phi}(t, x)\| t \leqslant t \sup_{\|y\| \leqslant C_R} \|F(y)\|, \tag{18}$$

where $\overline{\Phi}(t, x) = (\Phi_1(t_1, x), \Phi_2(t_2, x), \dots, \Phi_n(t_n, x))$ for suitable $t_1, t_2, \dots, t_n \in]0, 1[$ which depend only on t and x.

Since f is continuous on $\mathbb{R}^n$ and hence uniformly continuous on the compact set $B = \{x \in \mathbb{R}^n ;\ \|x\| \leqslant C_R\}$, for each $\varepsilon > 0$ there is $\delta > 0$ such that $|f(y) - f(y')| < \varepsilon$ whenever $y, y' \in B$ and $\|y - y'\| < \delta$. Therefore, if $0 < t < \delta(\sup_{\|y\| \leqslant C_R} \|F(y)\|)^{-1}$, by (18), we get that

$$|T(t) f(x) - f(x)| = |f(\Phi(t, x)) - f(\Phi(0, x))| < \varepsilon.$$

We have thus shown that $\tau_c - \lim_{t \to 0^+} T(t)\, f = f$.

Next, let $s > 0$, $R > 0$ and $f \in C(\mathbb{R}^n)$. Then, put $C_{s, R} = (R + s\|F(0)\|)\, e^{Ls}$, by (16), for each $0 \leqslant t \leqslant s$,

$$\sup_{\|x\| \leqslant R} |T(t)\, f(x)| = \sup_{\|y\| \leqslant (R + t\|F(0)\|) e^{Lt}} |f(y)| \leqslant \sup_{\|y\| \leqslant C_{s, R}} |f(y)|. \tag{19}$$

Since $(C(\mathbb{R}^n), \tau_c))$ is a Fréchet space and hence it is barrelled, by Remark 2, we can conclude that $(T(t))_{t \geqslant 0}$ is a locally equicontinuous semigroup on $(C(\mathbb{R}^n), \tau_c)$.

Let $f \in C^1(\mathbb{R}^n)$. Then, for each $x \in \mathbb{R}^n$ and $0 < t \leq 1$, by Lagrange's theorem again, we have that

$$\frac{T(t)\, f(x) - f(x)}{t} = \langle \nabla f(\overline{\Phi}(t, x)), F(\overline{\Phi}(t, x)) \rangle,$$

where $\overline{\Phi}(t, x)$ is as above. Consequently, for each $t \in]0, 1]$ and $\|x\| \leq R$,

$$\left| \frac{T(t)\, f(x) - f(x)}{t} - \langle \nabla f(x), F(x) \rangle \right| = |\langle \nabla f(\overline{\Phi}(t, x)), F(\overline{\Phi}(t, x)) \rangle - \langle \nabla f(x), F(x) \rangle|$$

$$\leq \|F(\overline{\Phi}(t, x))\| \, \|\nabla f(\overline{\Phi}(t, x)) - \nabla f(\Phi(0, x))\| + \|\nabla f(x)\| \, \|F(\overline{\Phi}(t, x)) - F(\Phi(0, x))\|$$

$$\leq \sup_{0 \leq s \leq t} \sup_{\|x\| \leq R} \|F(\Phi(s, x))\| \, \|\nabla f(\overline{\Phi}(t, x)) - \nabla f(\Phi(0, x))\| +$$

$$+ \sup_{\|x\| \leq R} \|\nabla f(x)\| L \|\overline{\Phi}(t, x) - \Phi(0, x)\|.$$

Since Φ and ∇f are continuous functions and hence uniformly continuous on every compact sets, it follows exactly as before that $\tau_c - \lim_{t \to 0^+} \dfrac{T(t)\, f - f}{t} = \langle \nabla f, F \rangle$.

Also, for a fixed $a > 0$ and for every $\lambda > 0$ the canonical asymptotic resolvent $R(\lambda, A)$ and the corresponding operator $S(\lambda, A)$ are given by

$$R(\lambda, A)\, f(x) = \int_0^a e^{-\lambda t} f(\Phi(t, x))\, dt \quad \text{and} \quad S(\lambda, A)\, f(x) = -e^{-\lambda a} f(\Phi(a, x)) \tag{20}$$

for all $f \in C(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Now, taking a sequence $(F_m)_m \subset C(\mathbb{R}^n; \mathbb{R}^n)$ of globally Lipschitz functions with Lipschitz's constant L_m respectively and the corresponding sequence $(\Phi_m)_m$ of continuous functions which solve the differential equation (15) with F_m instead of F, the family $\{(T_m(t))_{t \geq 0} : m \in \mathbb{N}\}$ of operators, defined as in (17), is a sequence of locally equicontinuous semigroups on $(C(\mathbb{R}^n), \tau_c)$ with generators $(A_m, D(A_m))$ respectively such that

$$A_m f(x) = \langle \nabla f(x), F_m(x) \rangle$$

for all $f \in C^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Moreover, by (19), the family $\{(T_m(t))_{t \geq 0} : m \in \mathbb{N}\}$ is uniformly locally equicontinuous if and only if the sequence $(L_m)_m$ is bounded.

Next, suppose that $L := \sup_{m \in \mathbb{N}} L_m < +\infty$ and $(F_m)_m$ τ_c-converges to some function $F \in C(\mathbb{R}^n; \mathbb{R}^n)$. Clearly, F is also a globally Lipschitz function with Lipschitz constant $\leq L$. In particular, denoting by Φ the continuous function which solves (15) with respect to F, the family $(T(t))_{t \geq 0}$ of operators, defined as in (17), is a locally equicontinuous semigroup on $(C(\mathbb{R}^n), \tau_c)$ with generator $(A, D(A))$ such that, for each $f \in C^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$Af(x) = \langle \nabla f(x), F(x) \rangle$$

and, by [3], Chapt. II, Section 3.28-Proposition, the space $D = C_c^1(\mathbb{R}^n)$ is a core of A and $Rg(1-A) \supset D$; hence $Rg(1-A) \supset D$ is dense in $(C(\mathbb{R}^n), \tau_c)$. On the other hand, we have that $(A_m f)_m$ τ_c-converges to Af for all $f \in D$.

Moreover, since $\|F_m(x) - F_m(y)\| \leq L\|x-y\|$ for all $m \in \mathbb{N}$ and $x, y \in \mathbb{R}^n$, it follows, by Gronwall's lemma, that, for each $m \in \mathbb{N}$, $t \geq 0$ and $x \in \mathbb{R}^n$,

$$\|\Phi_m(t, x) - \Phi(t, x)\| \leq e^{Lt} \int_0^t \|F_m(\Phi(s, x)) - F(\Phi(s, x))\| \, ds \,,$$

which implies that $(S(\lambda, A_m) f)_m$ τ_c-converges to $S(\lambda, A) f$ for all $f \in C(\mathbb{R}^n)$ and $\lambda > 0$. Consequently, by the above theorem, we can conclude that the sequence $\{(T_m(t))_{t \geq 0} : m \in \mathbb{N}\}$ τ_c-converges pointwise and uniformly in t on compact intervals of $\mathbb{R}_+$ to the locally equicontinuous semigroup $(T(t))_{t \geq 0}$.

Remark 18. In many applications, e.g., in Section 5.2 below, we have the situation that the semigroup considered is locally equicontinuous and the resolvent defined as $\int_0^\infty e^{-\lambda t} T(t) x dt$ exists for suitable $\lambda \in \mathbb{C}$. Consequently, the corresponding operator $S(\lambda, A)$ is zero. Moreover, we obtain in Theorem 16 that if (a) holds, then $B = \overline{A}$. In fact, by Theorem 16, we have that B is an extension of $\overline{A}$. Furthermore, $(\lambda_0 - A)^{-1}$ exists and its closure $(\lambda_0 - \overline{A})^{-1}$ is contained in $R(\lambda_0, B)$. Since $R(\lambda_0, B)$ is continuous, we obtain that $(\lambda_0 - \overline{A})^{-1}$ is continuous. Further, the domain $D((\lambda_0 - \overline{A})^{-1})$ contains the range $rg(\lambda_0 - A)$ which is dense in X by assumption. This implies that $D((\lambda_0 - \overline{A})^{-1}) = X$. Consequently, we obtain $R(\lambda_0, B) = (\lambda_0 - \overline{A})^{-1}$, and therefore $B = \overline{A}$.

5 - Applications

To apply our results we concentrate on the Lie-Trotter product formula which goes back to [15]. We will obtain an explicit product formula for locally equicontinuous semigroups whose generator is the sum of two generators. We then apply this formula to the Ornstein-Uhlenbeck semigroup on $C_b(\mathbb{R}^n)$.

5.1 - *The Lie-Trotter product formula*

A first application of Theorem 16 is a version of the Lie-Trotter product formula for locally equicontinuous semigroups. To that purpose, we restate Lemma III, 5.1 from [3] replacing the norm $\|\cdot\|$ by seminorms $p, q \in P$.

Lemma 19. *Let $S \in \mathcal{L}(X)$. Assume that for each $p \in P$ there exists $q \in P$ such that*

$$p(S^m x) \leq q(x) \quad \text{for all } x \in X \text{ and } m \in \mathbb{N}.$$

We then have that for each $p \in P$ there exists $q \in P$ such that

$$p(e^{n(S - Id)} x - S^n) \leq \sqrt{n}\, q(Sx - x)$$

for all $x \in X$ and $n \in \mathbb{N}$.

Now, we are able to state the Lie-Trotter product formula for locally equicontinuous semigroups.

Theorem 20. *Let $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ be locally equicontinuous semigroups on X with generators $(A, D(A))$ and $(B, D(B))$, respectively. Assume that there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that for all $s > 0$ and $p \in P$ there exists $q \in P$ such that*

$$p([T(t)S(t)]^m x) \leq M e^{m\omega t} q(x)$$

for all $x \in X$, $0 \leq t \leq s$ and $m \in \mathbb{N}$. Consider the sum $A + B$ on a subspace $D \subseteq D(A) \cap D(B)$ and assume that D and $(\lambda_0 - A - B) D$ are dense in X for some $\lambda_0 > \omega$. Then the closure of $A + B$ exists and generates a locally equicontinuous semigroup $(U(t))_{t \geq 0}$ given by the Lie-Trotter product formula

$$U(t)\, x = \lim_{n \to \infty} \left[T\left(\frac{t}{n} \right) S\left(\frac{t}{n} \right) \right]^n x\,, \tag{21}$$

where the limit exists for all $x \in X$ and uniformly for t in compact intervals in $\mathbb{R}_+$.

Proof. Let $V(t) := T(t)\,S(t)$ for all $t \geqslant 0$. By a rescaling argument we can assume $\omega = 0$ without loss of restriction. For $\tau > 0$ we define

$$(A+B)_n := \frac{V\left(\frac{\tau}{n}\right) - Id}{\frac{\tau}{n}} \in \mathcal{L}(X), \qquad n \in \mathbb{N},$$

and observe that $(A+B)_n x \to Ax + Bx$ for $x \in D$ as $n \to \infty$. Since for all $p \in P$ there exists $q \in P$ such that

$$p(e^{t(A+B)_n} x) \leqslant e^{-\frac{tn}{\tau}}\, p\left(\sum_{m=0}^{\infty} \frac{\left(\frac{tn}{\tau}\right)^m}{m!} \left[V\left(\frac{\tau}{n}\right)\right]^m x\right)$$

$$\leqslant Mq(x)$$

for all $x \in X$, $t \geqslant 0$ and $n \in \mathbb{N}$, the semigroups $(e^{t(A+B)_n})_{t \geqslant 0}$ are equicontinuous semigroups. Therefore, by a result of K. Yosida [**16**, p. 241], we obtain that the resolvent

$$R(\lambda, (A+B)_n)\, x = \int_0^{\infty} e^{-\lambda t} e^{t(A+B)_n} x\, dt$$

exists for all $x \in X$ and $\lambda \in \mathbb{C}$ with $Re\,\lambda > 0$. This shows that the assumptions of Theorem 16 are fulfilled with, in particular, $S(\lambda, (A+B)_n) = 0$ for all $\lambda > 0$. Hence, by Theorem 16 and Remark 18, the closure of $A+B$ generates a locally equicontinuous semigroup $(T(t))_{t \geqslant 0}$ satisfying

$$e^{t(A+B)_n} x \to T(t)\, x \tag{22}$$

for all $x \in X$ and uniformly for $t \in [0, T]$.

On the other hand, by Lemma 19, for $p \in P$ there exists $q \in P$ such that

$$\begin{aligned} p\left(e^{\tau(A+B)_n} x - \left[V\left(\frac{\tau}{n}\right)\right]^n x\right) &\leqslant \sqrt{n} M q\left(V\left(\frac{\tau}{n}\right) x - x\right) \\ &= \frac{\tau M}{\sqrt{n}}\, q((A+B)_n x), \qquad x \in X, \end{aligned} \tag{23}$$

which converges to zero as n tends to infinity for all $x \in D$ and uniformly for

$\tau \in [0, T]$. Finally, since for all $p \in P$ there exists $q \in P$ such that

$$p\left(e^{t(A+B)_n} x - \left[V\left(\frac{\tau}{n}\right)\right]^n x\right) \leq 2Mq(x)$$

for all $x \in X$, the combination of (22), (23) yields the convergence on a dense subset which then holds by [**12**], Chapt. III, Thm. 4.5 on all of X. ■

Example 21. On $C(\mathbb{R})$ endowed with the compact-open topology τ_c we take the locally equicontinuous (right) translation semigroup $(T(t))_{t \geq 0}$ with generator A and the multiplication semigroup $(T_q(t))_{t \geq 0}$ defined as in Example 15 with generator B. For $f \in C(\mathbb{R})$ we can calculate the Lie-Trotter products

$$\left[T\left(\frac{t}{n}\right) T_q\left(\frac{t}{n}\right)\right]^n f(s) = exp\left(\sum_{k=1}^{n} q\left(s - \frac{kt}{n}\right)\frac{t}{n}\right) f(s-t)$$

for $t \geq 0$ and $s \in \mathbb{R}$ which converge to $U(t)\, f$ with respect to τ_c with

$$U(t)\, f := exp\left(\int_{s-t}^{s} q(r)\, dr\right) f(s-t)\,.$$

The operators $(U(t))_{t \geq 0}$ form a locally equicontinuous semigroup on $(C(\mathbb{R}), \tau_c)$.

5.2 - *Application to the Ornstein-Uhlenbeck semigroup on $C_b(\mathbb{R}^n)$*

In this section we are concerned with the Ornstein-Uhlenbeck operator which has been studied, e.g., in [**2**]. For any symmetric, positive definite matrix $A := (a_{ij})$ and a matrix $B := (b_{ij}) \in \mathcal{L}(\mathbb{R}^n)$, *the Ornstein-Uhlenbeck operator* is defined as

$$\begin{aligned}[\mathcal{O}f](x) &:= \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} D_{ij} f(x) + \sum_{i,j=1}^{n} b_{ij} x_j D_i f(x) \\ &=: \langle \nabla, A\nabla f(x)\rangle + \langle Bx, \nabla f(x)\rangle \\ &=: \mathcal{A}f(x) + \mathcal{B}f(x)\end{aligned} \tag{24}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing functions, $x \in \mathbb{R}^n$, $\nabla := \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$. The related semigroup $(\mathcal{P}(t))_{t \geq 0}$ has the following representation due to Kolmogorov (see [2]):

$$(25)\qquad (\mathcal{P}(t)\, f)(x) = \begin{cases} \dfrac{1}{(2\pi)^{n/2}(det A_t)^{1/2}} \displaystyle\int_{\mathbb{R}^n} e^{-\frac{\langle A_t^{-1} y,\, y\rangle}{2}} f(e^{tB} x - y)\, dy, & \text{if } t > 0, \\ f(x), & \text{if } t = 0, \end{cases}$$

for all $f \in C_b(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, where $A_t := \int_0^t e^{sB} A e^{sB'} ds$. In [2] it has been shown that $(\mathcal{P}(t))_{t \geq 0}$ is not strongly continuous on $UC_b(\mathbb{R}^n)$, the space of bounded uniformly continuous functions on $\mathbb{R}^n$ with respect to the supremum norm $\|\cdot\|_\infty$ (and hence on $(C_b(\mathbb{R}^n), \|\cdot\|_\infty)$). Therefore, Lie-Trotter's formula in its classical formulation does not apply.

In the following, we show that, if $C_b(\mathbb{R}^n)$ is endowed with a suitable locally convex topology τ finer than the compact-open topology, then $(C_b(\mathbb{R}^n), \tau)$ is sequentially complete and $(\mathcal{P}(t))_{t \geq 0}$ is a locally equicontinuous semigroup on it. Moreover, the Lie-Trotter product formula from Section 5.1 holds for the Ornstein-Uhlenbeck semigroup $(\mathcal{P}(t))_{t \geq 0}$.

We begin by constructing the topology τ. To that purpose, we define a family P of seminorms on $C_b(\mathbb{R}^n)$ generating a locally convex topology τ on $C_b(\mathbb{R}^n)$ such that the inclusion maps

$$(C_b(\mathbb{R}^n), \|\cdot\|_\infty) \hookrightarrow (C_b(\mathbb{R}^n), \tau) \hookrightarrow (C_b(\mathbb{R}^n), \tau_c)$$

are continuous, where τ_c denotes the compact-open topology on $C_b(\mathbb{R}^n)$. The construction of P is similar to the one given in [4], Section 2. Let

$$\Gamma := \{\gamma \in C_0(\mathbb{R}^n) : \gamma > 0,\ \lim_{\|x\| \to \infty} \|x\|^2 \gamma(x) =: l \text{ exists in } \mathbb{R}\}.$$

Clearly, Γ is not empty. Indeed, each function defined as

$$(26)\qquad \gamma(x) := \begin{cases} l & \text{if } \|x\| \leq r, \\ \dfrac{lr^2}{\|x\|^2} & \text{if } \|x\| > r, \end{cases}$$

with $l, r > 0$ arbitrary, belongs to Γ. Moreover, if $(D_m)_{m \in \mathbb{N}}$ is an exhaustion of $\mathbb{R}^n$ (i.e. $\overline{D}_m$ is compact, $\overline{D}_m \subset D_{m+1}$ for all $m \in \mathbb{N}$, and $\bigcup_{m=0}^{\infty} D_m = \mathbb{R}^n$), and $(\gamma_m)_{m \in \mathbb{N}} \subseteq C_0(\mathbb{R}^n)$ such that for all $m \in \mathbb{N}$

$$0 \leq \gamma_m \leq 1 \quad \text{on } \mathbb{R}^n, \quad \gamma_m = 1 \text{ on } \overline{D}_{m-1} \text{ and } \gamma_m = 0 \quad \text{on } \mathbb{R}^n \setminus D_m,$$

then each function defined as

$$(27) \qquad \gamma(x) := \sum_{m=1}^{\infty} \frac{1}{2^{l_m}} \gamma_m(x)$$

for $x \in \mathbb{R}^n$ belongs to Γ too, where $(l_m)_{m \in \mathbb{N}}$ is an increasing sequence of integers such that $l_m \geqslant \max\{m, d(0, D_m)\}$ and $l_m \in \mathbb{N}$ for all $m \in \mathbb{N}$. Furthermore, we have the following property.

(i) Let A be a non-zero, real matrix and $\gamma \in \Gamma$. For each $s > 0$ the function γ'_s defined as

$$\gamma'_s(x) := \sup_{0 \leqslant t \leqslant s} \gamma(e^{-t\|A\|} x), \qquad x \in \mathbb{R}^n,$$

belongs to Γ and $\gamma \leqslant \gamma'_s$.

We now consider the family of seminorms $P := \{p_\gamma : \gamma \in \Gamma\}$ on $C_b(\mathbb{R}^n)$ defined as

$$p_\gamma(f) := \sup_{x \in \mathbb{R}^n} \gamma(x)\,|f(x)| \quad \text{for all } f \in C_b(\mathbb{R}^n).$$

Clearly, P generates a locally convex topology τ coarser than the topology of uniform convergence on $\mathbb{R}^n$. Since for each $\gamma \in \Gamma$ there exists $M := \sup_{x \in \mathbb{R}^n} \gamma(x) > 0$ such that

$$p_\gamma(f) = \sup_{x \in \mathbb{R}^n} \gamma(x)\,|f(x)| \leqslant M\|f\|_\infty$$

for all $f \in C_b(\mathbb{R}^n)$, the inclusion map $(C_b(\mathbb{R}^n), \|\cdot\|_\infty) \hookrightarrow (C_b(\mathbb{R}^n), \tau)$ is continuous. Also the inclusion map $(C_b(\mathbb{R}^n), \tau) \hookrightarrow (C_b(\mathbb{R}^n), \tau_c)$ is continuous. Indeed, for each $m \in \mathbb{N}$ there exists $\gamma \in \Gamma$, where γ is given as in (26) by taking $l = 1$ and $r = m$ such that

$$p_m(f) = \sup_{\|x\| \leqslant m} |f(x)| \leqslant \sup_{x \in \mathbb{R}^n} \gamma(x)\,|f(x)| = p_\gamma(f)$$

for all $f \in C_b(\mathbb{R}^n)$. Moreover, by repeating the proof in [4, Prop. 2.4] with minor changes and using functions γ defined as in (27), we obtain the following results:

(ii) The space $C_0(\mathbb{R}^n)$ is dense in $(C_b(\mathbb{R}^n), \tau)$;

(iii) A sequence $(f_n)_n$ converges in $(C_b(\mathbb{R}^n), \tau)$ to f if and only if $(f_n)_n$ is uniformly bounded and converges uniformly to f on each compact set of $\mathbb{R}^n$;

(iv) The space $(C_b(\mathbb{R}^n), \tau)$ is sequentially complete.

In the sequel, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ from (24) are generators of locally equicontinuous semigroups on $(C_b(\mathbb{R}^n), \tau)$.

Proposition 22. *The semigroup* $(\mathcal{S}(t))_{t \geqslant 0}$ *given by*

$$(\mathcal{S}(t)\, f)(x) = f(e^{tB} x) \quad \text{for } t \geqslant 0,\ f \in C_b(\mathbb{R}^n),\ x \in \mathbb{R}^n \tag{28}$$

is locally equicontinuous on $(C_b(\mathbb{R}^n), \tau)$ *and its generator coincides with the closure of the operator*

$$\mathcal{B}f(x) := \sum_{i,j=1}^{n} b_{ij} x_j D_i f(x) = \langle Bx, \nabla f(x) \rangle$$

defined for every $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Clearly, for each compact subset $K \subset \mathbb{R}^n$ we have

$$\lim_{t \to 0^+} \sup_{x \in K} \|e^{tB} x - x\| = 0\,. \tag{29}$$

Let $f \in C_b(\mathbb{R}^n)$, $M := \sup_{x \in \mathbb{R}^n} |f(x)|$ and $\gamma \in \Gamma$. Then $\gamma > 0$ is continuous on $\mathbb{R}^n$ and $\lim_{\|x\| \to \infty} \|x\|^2 \gamma(x) = 0$. Let $\varepsilon > 0$. There exists $r > 0$ such that $0 < \gamma(x) < \dfrac{\varepsilon}{4M}$ for all $x \in \mathbb{R}^n$ with $\|x\| > r$. Thus,

$$\sup_{\|x\| > r} \gamma(x)\, |f(e^{tB} x) - f(x)| \leqslant \frac{\varepsilon}{4M} \sup_{\|x\| > r} |f(e^{tB} x) - f(x)| \leqslant \frac{\varepsilon}{4M} 2M = \frac{\varepsilon}{2} \tag{30}$$

for all $t \geqslant 0$. Now, let $K := \{x \in \mathbb{R}^n : \|x\| \leqslant r\}$ and $0 < d := \max_{x \in K} \gamma(x) < \infty$. Then, by (29), there exists $\delta > 0$ such that

$$\sup_{\|x\| \leqslant r} |f(e^{tB} x) - f(x)| < \frac{\varepsilon}{2d}$$

for all $t \in]0, \delta[$, and hence

$$\sup_{\|x\| \leqslant r} \gamma(x)\, |f(e^{tB} x) - f(x)| \leqslant d \sup_{\|x\| \leqslant r} |f(e^{tB} x) - f(x)| < d \frac{\varepsilon}{2d} = \frac{\varepsilon}{2}\,. \tag{31}$$

Combining (30) and (31), we obtain for all $0 < t < \delta$ that

$$\sup_{x \in \mathbb{R}^n} \gamma(x) |f(e^{tB} x) - f(x)| \leqslant \sup_{\|x\| \leqslant r} \gamma(x) |f(e^{tB} x) - f(x)| + \sup_{\|x\| > r} \gamma(x) |f(e^{tB} x) - f(x)| < \varepsilon.$$

Therefore,

$$\tau - \lim_{t \to 0+} \mathcal{S}(t)\, f = f$$

for all $f \in C_b(\mathbb{R}^n)$. Next, let $s > 0$ and $\gamma \in \Gamma$. Then, taking $\gamma'(y) := \sup_{0 \leq t \leq s} \gamma(e^{-tB} y)$, $y \in \mathbb{R}^n$, so that $\gamma' \in \Gamma$ by (i), we have

$$\begin{aligned} p_\gamma(\mathcal{S}(t)\, f) &= \sup_{x \in \mathbb{R}^n} \gamma(x)\, |f(e^{tB} x)| \\ &\leq \sup_{y \in \mathbb{R}^n} \gamma'(y)\, |f(y)| \\ &= p_{\gamma'}(f) \end{aligned}$$

for all $f \in C_b(\mathbb{R}^n)$ and $0 \leq t \leq s$. Thus, by Remark 2, $(\mathcal{S}(t))_{t \geq 0}$ is a locally equicontinuous semigroup on $(C_b(\mathbb{R}^n), \tau)$.

Let $(\widetilde{\mathcal{B}}, D(\widetilde{\mathcal{B}}))$ be the generator of $(\mathcal{S}(t))_{t \geq 0}$. It is not difficult to verify directly that $\mathcal{S}(\mathbb{R}^n) \subset D(\widetilde{\mathcal{B}})$. On the other hand, $\mathcal{S}(\mathbb{R}^n)$ is invariant under $(\mathcal{S}(t))_{t \geq 0}$ and dense in $(C_b(\mathbb{R}^n), \tau)$. So, it is a core by Proposition 7. This completes the proof. ■

Proposition 23. *The heat semigroup* $(\mathcal{T}(t))_{t \geq 0}$ *given by*

$$(\mathcal{T}(t)\, f)(x) = \begin{cases} ((2\pi t)^{n/2} (\det A)^{1/2})^{-1} & \\ \cdot \int_{\mathbb{R}^n} \exp\left(-\dfrac{1}{2t} \langle A^{-1}(x-y), (x-y) \rangle \right) f(y)\, dy & \text{if } t > 0, \\ f(x) & \text{if } t = 0 \end{cases}$$

for all $t \geq 0$, $f \in C_b(\mathbb{R}^n)$ *and* $x \in \mathbb{R}^n$ *is locally equicontinuous on* $(C_b(\mathbb{R}^n), \tau)$ *and its generator coincides with the closure of the operator*

$$\mathcal{A} f(x) := \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} D_{ij} f(x) = \langle \nabla, A \nabla f(x) \rangle$$

defined for every $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. We first prove the local equicontinuity of $(\mathcal{C}(t))_{t\geq 0}$. Let $\gamma \in \Gamma$, $f \in C_b(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Then

$$\begin{aligned}
&\gamma(x)\,|\mathcal{C}(t)\,f(x)\,| \\
&\leq \frac{\gamma(x)}{(2\pi t)^{n/2}(det A)^{1/2}} \int_{\mathbb{R}^n} exp\left(-\frac{1}{2t}\langle A^{-1}(x-y),(x-y)\rangle\right) \\
&\quad \cdot (1+\|y\|^2)\,\frac{|f(y)\,|}{1+\|y\|^2}\,dy \\
&\leq \frac{\gamma(x)}{(2\pi t)^{n/2}(det A)^{1/2}} \int_{\mathbb{R}^n} exp\left(-\frac{1}{2t}\langle A^{-1}(x-y),(x-y)\rangle\right)(1+\|y\|^2)\,dy \\
&\quad \cdot \sup_{z\in\mathbb{R}^n} \frac{|f(z)\,|}{1+\|z\|^2} \\
&\leq \gamma(x)(1+\|x\|^2+nt\|A\|^2)\sup_{z\in\mathbb{R}^n}\frac{|f(z)\,|}{1+\|z\|^2} \\
&\leq M(1+t)\sup_{z\in\mathbb{R}^n}\frac{|f(z)\,|}{1+\|z\|^2}\,,
\end{aligned}$$

where $M := 2\max\{M_\gamma,\, n\|A\|^2\}$ with $M_\gamma := \sup_{x\in\mathbb{R}^n}(1+\|x\|^2)\,\gamma(x) < \infty$.

Put $\gamma'(z) := \dfrac{M}{1+\|z\|^2}$, $z \in \mathbb{R}^n$, so that $\gamma' \in \Gamma$. It follows that for each $f \in C_b(\mathbb{R}^n)$

$$p_\gamma(\mathcal{C}(t)\,f) \leq (1+t)\,p_{\gamma'}(f). \tag{32}$$

Therefore, $(\mathcal{C}(t))_{t\geq 0}$ is locally equicontinuous in $(C_b(\mathbb{R}^n),\,\tau)$.

To prove the strong τ-continuity of $(\mathcal{C}(t))_{t\geq 0}$, it is well known that $(\mathcal{C}(t))_{t\geq 0}$ is a strongly continuous semigroup on $(C_0(\mathbb{R}^n),\,\|\cdot\|_\infty)$. Consequently, we have

$$\tau - \lim_{t\to 0^+} \mathcal{C}(t)\,f = f$$

for all $f \in C_0(\mathbb{R}^n)$, the space of continuous functions vanishing at infinity, because the topology τ is coarser than the topology of uniform convergence on $\mathbb{R}^n$. Using the density of $C_0(\mathbb{R}^n)$ in $(C_b(\mathbb{R}^n), \tau)$ and formula (32) we easily get

$$\tau - \lim_{t \to 0^+} \mathcal{C}(t) f = f$$

for all $f \in C_b(\mathbb{R}^n)$ and, by Remark 2, $(\mathcal{C}(t))_{t \geq 0}$ is a locally equicontinuous semigroup on $(C_b(\mathbb{R}^n), \tau)$.

Let $(\widetilde{\mathcal{A}}, D(\widetilde{\mathcal{A}}))$ be the generator of $(\mathcal{C}(t))_{t \geq 0}$. Note that $(\mathcal{C}(t))_{t \geq 0}$ is strongly continuous on $C_0(\mathbb{R}^n)$ with the $\|\cdot\|$-closure of $(\mathcal{A}, \mathcal{S}(\mathbb{R}^n))$ as its generator. Further, $\mathcal{S}(\mathbb{R}^n)$ is invariant under $(\mathcal{C}(t))_{t \geq 0}$. On the other hand, $\mathcal{S}(\mathbb{R}^n)$ is dense in $(C_b(\mathbb{R}^n), \tau)$. So, it is a core by Proposition 7. This completes the proof. ■

With the previous propositions we are able to approximate $(\mathcal{P}(t))_{t \geq 0}$ by the Lie-Trotter products of $(\mathcal{C}(t))_{t \geq 0}$ and $(\mathcal{S}(t))_{t \geq 0}$.

Theorem 24. *Let $(\mathcal{C}(t))_{t \geq 0}$ and $(\mathcal{S}(t))_{t \geq 0}$ be the locally equicontinuous semigroups on $(C_b(\mathbb{R}^n), \tau)$ given in Propositions 22 and 23 and generated by $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{B}, D(\mathcal{B}))$, respectively. Then the Ornstein-Uhlenbeck semigroup on $(C_b(\mathbb{R}^n), \tau)$ given by (25) is a τ-locally equicontinuous semigroup generated by the closure of $\mathcal{A} + \mathcal{B}$ and represented by the Lie-Trotter product formula, i.e.*

$$\mathcal{P}(t) f = \tau - \lim_{n \to \infty} \left[\mathcal{C}\left(\frac{t}{n} \right) \mathcal{S}\left(\frac{t}{n} \right) \right]^n f$$

for all $f \in C_b(\mathbb{R}^n)$ and uniformly for t in compact intervals of $\mathbb{R}_+$.

Proof. Put $l_t := (2\pi t)^{n/2} (\det A)^{1/2}$ for $t > 0$.

It is not difficult to verify that, for each $m \in \mathbb{N}$, $t \geq 0$, $f \in C_b(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$,

$$|[\mathcal{C}(t)\, \mathcal{S}(t)]^m f(x)| \leq \frac{1}{l_t^m} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t} \langle A^{-1}(x - y_1), (x - y_1) \rangle \right) dy_1 \cdot \ldots$$

$$\cdot dy_{m-1} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t} \langle A^{-1}(e^{tB} y_{m-1} - y_m), (e^{tB} y_{m-1} - y_m) \rangle \right) |f(e^{tB} y_m)| \, dy_m$$

$$\leq \sup_{z \in \mathbb{R}^n} \frac{|f(e^{tB} z)|}{1 + \|z\|^2} \cdot \frac{1}{l_t^m} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t} \langle A^{-1}(x - y_1), (x - y_1) \rangle \right) dy_1 \cdot \ldots$$

$$\cdot\, dy_{m-1} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t} \langle A^{-1}(e^{tB} y_{m-1} - y_m), (e^{tB} y_{m-1} - y_m) \rangle \right) (1 + \|y_m\|^2) \, dy_m.$$

Now, fix $s > 0$, $\gamma \in \Gamma$ and put $\tilde{\gamma}_0(z) := \sup_{0 \leq t \leq s} \dfrac{1}{1 + \|e^{-tB} z\|^2}$, $z \in \mathbb{R}^n$, so that $\tilde{\gamma}_0 \in \Gamma$. By standard computation of integrals with respect to Gaussian measures, it follows that, for each $0 \leq t \leq s$,

$$\gamma(x)\,|[\mathcal{T}(t)\,\mathcal{S}(t)]^m f(x)\,| \leq \sup_{z \in \mathbb{R}^n} \tilde{\gamma}_0(z)\,|f(z)\,|$$

$$\cdot\left(\gamma(x) + \frac{\gamma(x)}{l_t^m} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(x - y_1), (x - y_1)\rangle\right) dy_1 \cdot \ldots\right.$$

$$\cdot\, dy_{m-2} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB} y_{m-2} - y_{m-1}), (e^{tB} y_{m-2} - y_{m-1})\rangle\right)$$

$$\left.\cdot\,(\|e^{tB} y_{m-1}\|^2 + nt\|A\|^2)\, dy_{m-1}\right) \leq \sup_{z \in \mathbb{R}^n} \tilde{\gamma}_0(z)\,|f(z)\,|$$

$$\cdot\left(\gamma(x)(1 + nt\|A\|^2) + \frac{\gamma(x)\,e^{2t\|B\|}}{l_t^{m-1}} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(x - y_1), (x - y_1)\rangle\right) dy_1 \cdot \ldots\right.$$

$$\left. dy_{m-2} \cdot \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB} y_{m-2} - y_{m-1}), (e^{tB} y_{m-2} - y_{m-1})\rangle\right) \|y_{m-1}\|^2\, dy_{m-1}\right)$$

$$\leq \sup_{z \in \mathbb{R}^n} \tilde{\gamma}_0(z)\,|f(z)\,|\,\gamma(x)(1 + mnt\|A\|^2 + e^{2mt\|B\|}\,\|x\|^2),$$

and hence

$$\gamma(x)\,|[\mathcal{T}(t)\ \mathcal{S}(t)]^m f(x)\,| \leq \sup_{z \in \mathbb{R}^n} \tilde{\gamma}_0(z)\,|f(z)\,|\,\gamma(x)(1 + mnt\|A\|^2 + e^{2mt\|B\|}\,\|x\|^2). \tag{33}$$

Take $w := \max\{2\|B\|, 1\}$ which is independent of γ, s and f, and $M := 2\max\{M_\gamma, n\|A\|^2\}$ with $M_\gamma := \sup_{x \in \mathbb{R}^n}(1 + \|x\|^2)\,\gamma(x) < \infty$. It follows by (33) that there exists $\tilde{\gamma} := M\tilde{\gamma}_0 \in \Gamma$ such that

$$p_\gamma([\mathcal{T}(t)\ \mathcal{S}(t)]^m f) \leq e^{mwt} p_{\tilde{\gamma}}(f)$$

for all $f \in C_b(\mathbb{R}^n)$, $0 \leq t \leq s$ and $m \in \mathbb{N}$. Since γ and s were arbitrary, we conclude that there exists $w \in \mathbb{R}_+$ such that for $\gamma \in \Gamma$ and $s > 0$ there exists $\tilde{\gamma} \in \Gamma$ such that

$$p_\gamma\left(\left[\mathcal{T}\left(\frac{t}{m}\right)\ \mathcal{S}\left(\frac{t}{m}\right)\right]^m f\right) \leq e^{wt} p_{\tilde{\gamma}}(f)$$

for all $f \in C_b(\mathbb{R}^n)$, $0 \leq t \leq s$, and $m \in \mathbb{N}$.

As stated in (ii) at the beginning of this subsection, the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $(C_b(\mathbb{R}^n), \tau)$. Moreover, it is a subset of $D(\mathcal{A}) \cap D(\mathcal{B})$. On $C_0(\mathbb{R}^n)$ the Ornstein-Uhlenbeck semigroup $(\mathcal{P}(t))_{t \geqslant 0}$ is strongly continuous and is represented by the Lie-Trotter Product Formula (see [8], Prop. 12). In particular, its generator coincides with $\mathcal{A} + \mathcal{B}$ restricted to $\mathcal{S}(\mathbb{R}^n)$. Hence, by the invariance of the Schwartz space under $(\mathcal{P}(t))_{t \geqslant 0}$, we obtain that $(\lambda - \mathcal{A} - \mathcal{B})\,\mathcal{S}(\mathbb{R}^n)$ is dense in $(C_b(\mathbb{R}^n), \tau)$ for $\lambda > 0$. Applying Theorem 20 we obtain that the closure of $\mathcal{A} + \mathcal{B}$ generates the locally equicontinuous semigroup $(\mathcal{P}(t))_{t \geqslant 0}$ on $C_b(\mathbb{R}^n)$ given by the Lie-Trotter product formula

$$\mathcal{P}(t)\, f = \tau - \lim_{m \to \infty} \left[\mathcal{T}\left(\frac{t}{m} \right) \mathcal{S}\left(\frac{t}{m} \right) \right]^m f$$

for all $f \in C_b(\mathbb{R}^n)$ and uniformly for t in compact intervals of $\mathbb{R}^n$. ■

Remark 25. We consider on $C_b(\mathbb{R}^n)$ the topology σ given by the norm

$$|||f||| := \sup_{x \in \mathbb{R}^n} \frac{|f(x)|}{1 + \|x\|^2}, \qquad f \in C_b(\mathbb{R}^n).$$

Since τ is finer than σ, and by setting $\gamma(x) = \dfrac{1}{1 + \|x\|^2}$ for $x \in \mathbb{R}^n$, we obtain in the same way as before that the Ornstein-Uhlenbeck semigroup $(\mathcal{P}(t))_{t \geqslant 0}$ given by (25) is a strongly continuous semigroup on $(C_b(\mathbb{R}^n), |||\cdot|||)$ and

$$\mathcal{P}(t) f = |||\cdot||| - \lim_{n \to \infty} \left[\mathcal{T}\left(\frac{t}{n} \right) \mathcal{S}\left(\frac{t}{n} \right) \right]^n f$$

for all $f \in C_b(\mathbb{R}^n)$ and uniformly for t in compact intervals of $\mathbb{R}_+$. We point out that the space $(C_b(\mathbb{R}^n), \sigma)$ is not complete.

References

[1] P. Cannarsa and G. Da Prato, *On a functional analysis approach to parabolic equations in infinite dimensions*, J. Funct. Anal. **118** (1993), 22-42.

[2] G. Da Prato and A. Lunardi, *On the Ornstein-Uhlenbeck operator in spaces of continuous functions*, J. Funct. Anal. **131** (1995), 94-114.

[3] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York 2000.

[4] M. Giga, *Abstract elliptic operator and its associated semigroup in a locally convex space*, Hokkaido Math. J. **22** (1993), 225-244.

[5] C. GROSU, *The Trotter product formula in locally convex spaces, I*, Rev. Roumaine Math. Pures Appl. **31** (1986), 29-42.

[6] H. KOMATSU, *Semi-groups of operators in locally convex spaces*, J. Math. Soc. Japan **16** (1964), 230-262.

[7] T. KŌMURA, *Semigroups of operators in locally convex spaces*, J. Funct. Anal. **2** (1968), 258-296.

[8] F. KÜHNEMUND and M. WACKER, *Commutator conditions implying the convergence of the Lie-Trotter products*, Proc. Amer. Math. Soc. **129** (2001), 3569-3582.

[9] I. MIYADERA, *Semigroups of operators in Fréchet spaces and applications to partial differential equations*, Tôhoku Math. J. **11** (1959), 162-183.

[10] S. ŌUCHI, *Semi-groups of operators in locally convex spaces*, J. Math. Soc. Japan **25** (1973), 265-276.

[11] A. PAZY, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, N.Y. 1992.

[12] H. H. SCHAEFER, *Topological vector spaces*, Graduate Texts in Mathematics, vol. 3, Springer-Verlag, N.Y. 1980.

[13] G. TESSITORE and J. ZABCZYK, *Comments on transition semigroups and stochastic invariance*, preprint Scuola Normale Superiore di Pisa, 1998.

[14] G. TESSITORE and J. ZABCZYK, *Trotter's formula for transitions semigroups*, Semigroup Forum **63** (2001), 114-126.

[15] H. F. TROTTER, *On the product of semi-groups of operators*, Proc. Amer. Math. Soc. **10** (1959), 545-551.

[16] K. YOSIDA, *Functional Analysis*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol.123, Springer-Verlag, N.Y.-Heidelberg 1974.

Abstract

In this paper we prove Trotter-Kato approximation results and the Lie-Trotter product formula for locally equicontinuous semigroups on sequentially complete locally convex spaces. These results are then applied to the Ornstein-Uhlenbeck semigroup on the space of bounded continuous functions on $\mathbb{R}^n$ *endowed with a locally convex topology agreeing with the compact-open topology on norm-bounded sets.*

* * *