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A common fixed point theorem connected to a result of B. Fisher (**)

1 - Introduction and statement of the main result

The study of common fixed points has started in the year 1936 by the well known result of Markov and Kakutani. Since this year, many works were devoted to Fixed point theory. The literature on the subject is now very rich. There are two major problems in metric fixed point theory. The first one consists of proving the existence of fixed or common fixed points for selfmappings in metric spaces, while the second one consists of finding and approximating them. In many situations, the proofs given for the existence of fixed or common fixed points give also effective methods of approximations and computations, but this is not the general case. The aim of this paper is a contribution to this area of investigations in metric fixed point theory and approximations.

Let (M, d) be a complete metric space. Let T be a fixed selfmapping and let $\alpha \in [0, 1[$. We define $B(T, \alpha)$ as the set of selfmappings S of M such that for all $x, y \in M$, the following condition is satisfied:

$$\text{(B)} \quad \begin{aligned} & d(Sx, TSy) \\ & \leqslant \alpha \max\left\{ d(x, Sy), \frac{d(x, Sx) + d(Sy, TSy)}{2}, \frac{d(x, TSy) + d(Sx, Sy)}{2} \right\}. \end{aligned}$$

For every selfmapping S of M, we denote F_S the mapping defined for all $x \in M$, by $F_S(x) := d(x, Sx)$. For all positive number c, we denote $L_{c,S} := \{x \in M : F_S(x) \leqslant c\}$.

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The purpose of this paper is to establish the following

Theorem 1.1. *Let* (M, d) *be a complete metric space. Let* $\alpha \in [0, 1[$ *and let* S, T *be two self-mappings of* M *such that* $S \in B(T, \alpha)$. *Then the following four assertions are true and are equivalent.*

(1) *There exists a unique point* $z \in M$ *such that* $\mathrm{Fix}(S) = \mathrm{Fix}(\{S, T\}) = \{z\}$.

(2) $\lim_{c \to 0^+} \mathrm{diam}(L_{c,S}) = 0$, *and the mapping* F_S *is an r.g.i. on* M.

(3) *There exists a (unique) point* $z \in M$, *such that, for each sequence* $\{x_n\} \subset M$; $\lim_n d(x_n, Sx_n) = 0$ *if and only if* $\{x_n\}$ *converges to* z.

(4) *There exists a (unique) point* $z \in \mathrm{Im}(S)$, *(the range of S) such that, for each sequence* $\{y_n\} \subset \mathrm{Im}(S)$, *we have* $\lim_{n\to\infty} y_n = z$, *if and only if,* $\lim_{n\to\infty} F_T(y_n) = 0$.

Moreover we have

(5) S *and* TS *are continuous at the point* z.

(6) *If* $\mathrm{Im}(T) \subset \mathrm{Im}(S)$ *then we have* $\mathrm{Fix}(S) = \mathrm{Fix}(T) = \mathrm{Fix}(\{S, T\}) = \{z\}$.

(7) *For every* $x_0 \in M$ *the Picard sequence* $\{S^n(x_0)\}$ *converges to the unique common fixed point* z *of* S *and* T.

We recall (see [**3**] and [**6**]) that a function $G : M \to \mathbb{R}$ is said to be a regular-global-inf (r.g.i.) at $x \in M$ if $G(x) > \inf_M(G)$ implies there exist $\varepsilon > 0$ such that $\varepsilon < G(x) - \inf_M(G)$ and a neighborhood N_x of x such that $G(y) > G(x) - \varepsilon$ for each $y \in N_x$. If this condition is satisfied for each $x \in M$, then G is said to be an r.g.i. on M. As we see, the r.g.i. condition may be considered as a weak type of regulatity. In the paper [**6**] this condition has been extensively used in many problems dealing with metric fixed points. Therefore, in Theorem 1.1 we see that not only all the conclusions of Theorem 4.3 (p. 149) of [**6**] are still valid for all selfmappings in the class $S \in B(T, \alpha)$ but that, in addition, they are equivalent.

In his paper [**4**], B. Fisher has proved the following result:

Theorem 1.2. [B. Fisher] *Let* (M, d) *be a complete metric space. Let* S, T *be two self-mappings of* M *such that*

(i) S *is continuous,*

(ii) S *and* T *satisfy the following contractive condition:*

$$\text{(F)}\qquad \begin{aligned} d(Sx, TSy) &\leqslant \alpha d(x, Sy) \\ &+ \beta[d(x, Sx) + d(Sy, TSy)] + \gamma[d(x, TSy) + d(Sx, Sy)], \end{aligned}$$

for all $x, y \in M$, *where* α, β, γ *are fixed nonnegative numbers such that* $\alpha + 2\beta + 2\gamma < 1$. *Then* S *and* T *have a unique common fixed point.*

It is clear that if S, T satisfy the condition (F) then $S \in B(T, q)$, where $q := \alpha + 2\beta + 2\gamma$. Since (in Theorem 1.1) we do not require any continuity condition on S, Theorem 1.1 improves actually Theorem 1.2. Also, we point out that L. Nova tried, in his paper [7], to improve Fisher's result but the assumptions used in [7] are still much stronger. So our paper solves the problem posed in [7].

The proof of Theorem 1.1 will be given in the next section. At many places in this proof, the following simple elementary geometrical fact will be used.

Lemma 1.3. *If A and B are the end points of a real interval with midpoint $\frac{A+B}{2}$ and $A \leq \max\left\{B, \frac{A+B}{2}\right\}$, then A must be the left end point and B the right one.*

Before giving a proof to Theorem 1.1, let us give an example of two mappings satisfying condition (B).

Example. We take $X = \{1, 2, 3, 4\}$ equipped with the metric d given by $d(1, 2) = d(3, 4) = \frac{3}{5}$; $d(1, 4) = d(2, 3) = \frac{2}{5}$; $d(1, 3) = \frac{1}{5}$; and $d(2, 4) = 1$. Let S and T be two mappings defined on X by $S(2) = 1$, $S(1) = S(3) = S(4) = 3$; and $T(2) = 4$, $T(1) = T(3) = T(4) = 3$.

It is easy to show that $S \in B(T, \alpha)$ for every $\alpha \in \left[\frac{1}{2}, 1\right[$. Thus there exist mappings verifying the condition (B).

We have seen that the condition (F) implies the condition (B) but we do not know whether the converse is true or not.

2 - Proof of Thorem 1.1

2.1 - First, we begin by proving that (1) is true.

(a) Let x_0 be some point in M, and set

$$x_{2n} = Sx_{2n-1}, \qquad n = 1, 2, \ldots$$

$$x_{2n+1} = Tx_{2n}, \qquad n = 0, 1, 2, \ldots$$

We put $t_n := d(x_n, x_{n+1})$ for all integer n. Suppose that $n = 2m$ for some integer

m. Then

$$
\begin{aligned}
t_n &= d(x_{2m}, x_{2m+1}) = d(Sx_{2m-1}, Tx_{2m}) = d(Sx_{2m-1}, TSx_{2m-1}) \\
&\leqslant \alpha \max\left\{d(x_{2m-1}, x_{2m}), \frac{1}{2}[d(x_{2m-1}, x_{2m}) + d(x_{2m}, x_{2m+1})], \frac{1}{2} d(x_{2m-1}, x_{2m+1})\right\} \\
&\leqslant \alpha \max\left\{t_{n-1}, \frac{1}{2}[t_{n-1} + t_n], \frac{1}{2} d(x_{2m-1}, x_{2m+1})\right\} \\
&\leqslant \alpha \max\left\{t_{n-1}, \frac{1}{2}[t_{n-1} + t_n]\right\}.
\end{aligned} \tag{1}
$$

By using Lemma 1.3, and the inequality (1), we deduce that for every even integer greater than two, we have

$$0 \leqslant t_n \leqslant \alpha t_{n-1}. \tag{2}$$

By similar arguments, it is easy to see that the inequality (2) is still valid for odd integers. Now, from (2), we get $t_n \leqslant \alpha^n t_0$ for every integer greater than one. Since $0 \leqslant \alpha < 1$, the sequence $\{t_n\}$ is a strongly Cauchy sequence (i.e., Σt_n converges) and consequently $\{x_n\}$ is a Cauchy sequence. Since (M, d) is complete, this sequence must have a limit, say z, in M. Next, we shall prove that z is a common fixed point for S and T.

(b) For all positive integer n, we have

$$
\begin{aligned}
&d(Sz, x_{2n+1}) = d(Sz, Tx_{2n}) = d(Sz, TSx_{2n-1}) \\
&\leqslant \alpha \max\left\{d(z, x_{2n}), \frac{1}{2}[d(z, Sz) + d(x_{2n}, x_{2n+1})], \frac{1}{2}[d(z, x_{2n+1}) + d(Sz, x_{2n})]\right\}.
\end{aligned} \tag{3}
$$

By taking the limits in both sides of (3), we obtain

$$d(Sz, z) \leqslant \frac{\alpha}{2} d(Sz, z) < d(Sz, z).$$

Thus z must be fixed by S. Let us show that $Tz = z$. By use of the property (B), we have

$$
\begin{aligned}
d(z, Tz) &= d(Sz, TSz) \\
&\leqslant \alpha \max\left\{0, \frac{1}{2} d(z, Tz), \frac{1}{2} d(z, Tz)\right\}.
\end{aligned} \tag{4}
$$

(4) implies that $\left(1-\frac{\alpha}{2}\right) d(z, Tz) = 0$. Since $\alpha < 1$, we conclude that $d(z, Tz) = 0$ and then $z \in \mathrm{Fix}(\{S, T\})$. We deduce also that $\mathrm{Fix}(S) \subset \mathrm{Fix}(T)$.

(c) Suppose that there exists another point w fixed by S. Then by using the property (B), we have

$$\begin{aligned} d(w, z) &= d(Sw, TSz) \\ &\leqslant \alpha \max\{d(w, z), 0, d(w, z)\} \\ &\leqslant \alpha d(w, z) \end{aligned} \tag{5}$$

(5) implies that $w = z$. We conclude that $\mathrm{Fix}(S) = \mathrm{Fix}(\{S, T\}) = \{z\}$. This completes the proof of the first assertion.

2.2 - Suppose that (1) is satisfied, and let z be the unique common fixed point of S and T. Let x be some point in M. By using Property (B) and the triangular inequality, we have

$$\begin{aligned} d(Sx, z) &= d(Sx, TSz) \\ &\leqslant \alpha \max\left\{d(x, z), \frac{1}{2} d(x, Sx), \frac{1}{2}[d(x, z) + d(Sx, z)]\right\} \\ &\leqslant \max\left\{d(x, z), \frac{1}{2}[d(x, z) + d(Sx, z)]\right\}. \end{aligned} \tag{6}$$

By using (6) and lemma 1.3, we deduce that

$$d(Sx, z) \leqslant \alpha\, d(x, z) \qquad \forall x \in M. \tag{7}$$

By using (7) and the triangular inequality, we obtain

$$d(x, z) \leqslant \frac{1}{1-\alpha} d(x, Sx), \qquad \forall x \in M. \tag{8}$$

From (8) we deduce, for each positive number c, that $L_{c,S}$ is bounded. It is nonvoid since it contains z. Now, for all $x, y \in L_{c,S}$, we have

$$d(x, y) \leqslant d(x, z) + d(y, z) \leqslant \frac{2c}{1-\alpha}. \tag{9}$$

(9) shows that $\mathrm{diam}(L_{c,S})$ (the diameter of the set $L_{c,S}$) tends to zero when c tends to zero. In order to show that F_S is r.g.i., we use Proposition 1.2 of [**6**] and the inequality (8). Therefore (1) implies (2).

2.3 - Suppose that (2) is satisfied. Let x_0 be some point in M, and consider the associated sequence $\{x_n\}$ given by

$$x_{2n} = Sx_{2n-1}, \qquad n = 1, 2, \ldots$$
$$x_{2n+1} = Tx_{2n}, \qquad n = 0, 1, 2, \ldots$$

We observe that for every integer n, we have the following inequality:

$$F_S(x_n) \leq \frac{2+\alpha}{2-\alpha} t_n. \tag{10}$$

Indeed, if n is odd then $F_S(x_n) = t_n$, while if n is even (by using the property (B)) we have $F_S(x_n) \leq t_n + d(x_{n+1}, Sx_n) \leq t_n + \frac{\alpha}{2}(t_n + F_S(x_n))$. We deduce from (10) that $\lim_{n\to\infty} F_S(x_n) = 0$. Then every $L_{c,S}$ is nonempty and $\inf_M F_S = 0$. Consider $\{c_n\}$ a decreasing sequence of positive numbers converging to zero, and set $L_S := \cap_n \overline{L_{c_n,S}}$, (where $\overline{L_{c_n,S}}$ designates the closure of $L_{c_n,S}$). By applying Cantor's intersection theorem we ensure the existence of a unique element $z \in L_S$. For every nonzero integer n, we can find $y_n \in L_{c_n,S}$ such that $d(y_n, z) \leq \frac{1}{n}$. Therefore $\{y_n\}$ converges to z. For each integer n, we have $0 \leq F(y_n) \leq c_n$. Hence $\lim_n F_S(y_n) = 0$. Since F_S is supposed to be regular, then $F_S(z) = \inf_M F_S = 0$. Thus z is a fixed point of T. Since $S \in B(T, \alpha)$, z must be the unique common fixed point of S and T. Now, let $\{x_n\}$ be a sequence in M such that $\lim_n F_S(x_n) = 0$. Then by using the inequality (8), we deduce that $\lim_n x_n = z$. Conversely, according to (7), for every $x \in M$, we have

$$d(x, Sx) \leq d(x, z) + d(z, Sx) \leq (1+\alpha) d(x, z).$$

Thus, if $\lim_{n\to\infty} x_n = z$ then $\lim_{n\to\infty} F_S(x_n) = 0$. Hence, (2) implies (3).

2.4 - Suppose that (3) is satisfied. Then z must be fixed by both S and T. Let $x \in M$. We start by making the following estimation

$$\begin{aligned} d(z, TSx) &= d(Sz, TSx) \\ &\leq \alpha \max\left\{d(z, Sx),\ \frac{1}{2} d(Sx, TSx),\ \frac{1}{2}[d(z, TSx) + d(z, Sx)]\right\} \\ &\leq \alpha \max\left\{d(z, Sx),\ \frac{1}{2}[d(z, TSx) + d(z, Sx)]\right\}. \end{aligned} \tag{11}$$

From (11) and Lemma 1.3, we deduce that

$$d(z, TSx) \leq \alpha\, d(Sx, z), \qquad \forall x \in M. \tag{12}$$

Now, let $w = Sx$ be some element in the range $\operatorname{Im}(S)$ of S. Then according to the triangular inequality and (12), we have

$$
\begin{aligned}
F_T(w) = d(Sx,\, TSx) &\leqslant d(Sx,\, z) + d(z,\, TSx) \\
&\leqslant (1+\alpha)d(Sx,\, z) = (1+\alpha)d(w,\, z).
\end{aligned} \tag{13}
$$

From (13) we obtain the first implication in (4). To prove the converse, let again $w = Sx$ be an element of $\operatorname{Im}(S)$. According to (13), we have

$$
\begin{aligned}
d(w,\, z) = d(Sx,\, z) &\leqslant d(Sx,\, TSx) + d(TSx,\, z) \\
&\leqslant d(Sx,\, TSx) + \alpha\, d(Sx,\, z) = F_T(w) + \alpha\, d(w,\, z).
\end{aligned} \tag{14}
$$

From (14), we get

$$d(w,\, z) \leqslant \frac{1}{1-\alpha} F_T(w).$$

Thus, for every sequence $\{w_n\}$ of points in $\operatorname{Im}(S)$, if $\lim_{n\to\infty} F_T(w_n) = 0$, then the sequence $\{w_n\}$ converges to z in M. Hence, (3) implies (4).

2.5 - Suppose that (4) is satisfied. Then the point z involved in (4) must be fixed by T. It remains to show that z is fixed by S. Let $y \in M$ such that $z = Sy$. According to Property (B), we have

$$
\begin{aligned}
d(Sz,\, z) &= d(Sz,\, TSy) \\
&\leqslant \alpha \max\left\{0,\, \frac{1}{2} d(z,\, Sz),\, \frac{1}{2} d(Sz,\, z)\right\} \\
&= \frac{\alpha}{2} d(Sz,\, z)
\end{aligned} \tag{15}
$$

(15) shows that necessarily $Sz = z$. Thus, (4) implies (1), and this proves the equivalence of the four properties quoted in Theorem 1.1.

2.6 - Let z be the unique common fixed point of S and T. The continuity of S at z is a consequence from the inequality (7), and the continuity of TS at z is an immediate consequence from the inequalities (12) and (7).

2.7 - Suppose that $\operatorname{Im}(T) \subset \operatorname{Im}(S)$. Then, from Subsection (b) of 2.1, we already know that $\operatorname{Fix}(S) \subset \operatorname{Fix}(T)$. Now, let $w \in \operatorname{Fix}(T)$, then $w \in \operatorname{Im}(S)$. Let $u \in M$, such that $w = Tw = Su$. Then, by using the property (B), we obtain

$$\begin{aligned} & d(Sw, w) = d(Sw, Tw) = d(Sw, TSu) \\ (16) \qquad & \leqslant \alpha \max\left\{0, \frac{1}{2} d(w, Sw), \frac{1}{2} d(Sw, w)\right\}. \end{aligned}$$

(16) implies that $[2 - \alpha]\, d(Sw, w) = 0$. Therefore $Sw = w$.

2.8 - Let z be the unique common fixed point of S and T. Let x_0 be any arbitrary point in M. We observe that (7) implies $d(S^n x_0, z) \leqslant \alpha^n d(x_0, z)$, therefore $\lim_{n \to \infty} S^n x_0 = z$. This completes the proof of Theorem 1.1. ■

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Abstract

Let (M, d) be a complete metric space, let $0 \leqslant \alpha < 1$, and let S, T be two selfmappings of M. We suppose that S belongs to the class $B(T, \alpha)$ (i.e. the condition (B) below is satis-

fied). Our main result is Theorem 1.1, *in which we prove that* S, T *have a unique common fixed point. Although we do not suppose any continuity assumption neither for* T *nor for* S, *we conclude some regularity properties. Indeed, we show that* S *and* TS *must be continuous at the unique common fixed point and that the mapping* $F_S\colon x \mapsto d(x, Sx)$ *is an r.g.i. mapping. We establish four equivalent properties characterizing the existence and uniqueness of the common fixed point for* S, T, *and give sequences of points approximating this fixed point. In particular, we show that all the Picard sequences defined by* S *converge to this common fixed point. This paper provides improvements to a well known result of B. Fisher* (*see* [**4**]).

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