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The Penrose transform for currents (**)

1 - Introduction

Let the real manifold X parametrise a family of compact holomorphic submanifolds of a CR manifold Z such that the set of points in X incident with each $z \in Z$ is contractible and of dimension d. Then under certain conditions (see [**1**] and [**2**]) there are spectral sequences (called smooth Penrose transforms)

$$E_1^{p,q} = \Gamma(X, V_{p,q}) \Rightarrow H^{p+q}(Z, \mathcal{Q}(V))$$

and

$$E_1^{p,q} = \Gamma_c(X, V_{p,q}) \Rightarrow H_c^{p+q-d}(Z, \mathcal{Q}(V))$$

where $\Gamma(X, V_{p,q})$ denotes the space of smooth sections of certain vector bundles $V_{p,q}$ over X, $H^{p+q}(Z, \mathcal{Q}(V))$ is the smooth involutive cohomology with respect to the CR structure on Z and the subscript «c» refers to the compactly supported sections and cohomology. In this paper we shall prove the following result.

Theorem 1. *In the situation of the smooth Penrose transforms* (*see* [**1**] *and* [**2**]) *there are spectral sequences*

$$E_1^{p,q} = \mathcal{J}^0(X, V_{p,q}) \Rightarrow H^{p+q}(Z, \mathcal{Q}'(V))$$

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and

$$E_1^{p,q} = \mathcal{J}_c^0(X, V_{p,q}) \Rightarrow H_c^{p+q-d}(Z, \mathcal{Q}'(V))$$

where $\mathcal{J}^0(X, V_{p,q})$ *denotes the space of* 0*-currents on* X *with values in* $V_{p,q}$ *and the superscript «′» refers to the distribution cohomology. The vector bundles and the operators which arise on* X *are exactly the same as those which arise at the smooth Penrose transform, except that the operators act on currents and not on smooth sections.*

Our definition of CR structures includes the complex case. If Z is a complex manifold the involutive cohomology on Z is the Dolbeault cohomology which is the same if calculated with smooth sections or currents. In this case the operators which arise on X are elliptic and the Penrose transform for currents proves regularity results.

If Z is a real hyper-surface in a complex manifold, it inherits a CR structure from the complex structure of the manifold in which it is embedded. In this case the cohomology on Z is the $\bar{\partial}_b$ cohomology, which is different if calculated with smooth sections or currents. The operators which arise from the Penrose transforms admit distribution solutions which are not necessarily smooth.

Penrose transforms are of interest since they provide a method of studying partial differential equations using geometrical tools.

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2 - Currents and involutive structures

The manifolds we will consider will always be real, smooth, oriented and without boundary. If M is an m-dimensional manifold and $V \to M$ is a complex vector bundle over M, a k-current on M with values in V is a continuous linear functional defined on the space $\Gamma_c(M, \mathcal{E}^{m-k} \otimes V^*)$ (where $\mathcal{E}^{m-k}$ denotes the bundle of complexified $m-k$ forms on M) on which we consider the uniform topology. The set of all k currents on M with values in V will be denoted $\mathcal{J}^k(M, V)$ (or $\mathcal{J}^k(M)$ when V is trivial of rank one) and its subset of compactly supported currents will be denoted $\mathcal{J}_c^k(M, V)$. By duality most of the operations on forms extend to operations on currents compatible with the natural inclusion of forms in currents. In particular if $\eta : F \to Z$ is an (oriented) fiber-bundle there is a pull-back $\eta^*: \mathcal{J}^0(Z) \to \mathcal{J}^0(F)$ defined as the dual of the fiber-integral on forms. It extends to currents with values in vector bundles in an obvious way.

Recall now that an involutive structure on M is a complex sub-bundle $\mathcal{A}^{1,0}$ of $\mathcal{E}^1$ which satisfies $d(\mathcal{A}^{1,0}) \subset \mathcal{A}^{1,0} \wedge \mathcal{E}^1$ (with «d» meaning the exterior derivative). We define $\mathcal{A}^{0,1}$ by the exactness of

$$0 \to \mathcal{A}^{1,0} \to \mathcal{E}^1 \to \mathcal{A}^{0,1} \to 0$$

and $\mathcal{A}^{0,q}$ by $\mathcal{A}^{0,q} := \wedge^q \mathcal{A}^{0,1}$. A complex vector bundle $V \to M$ is said to be compatible with the involutive structure $\mathcal{A}$ (or $\mathcal{A}$-compatible) if there is given a linear operator (called the partial connection)

$$\bar{\partial}: \Gamma(M, V) \to \Gamma(M, V \otimes \mathcal{A}^{0,1})$$

such that

$$\bar{\partial}(fs) = f\bar{\partial}(s) + (\bar{\partial}f)\, s\,, \qquad \forall f \in \mathcal{E}(M),\ s \in \Gamma(M, V)$$

and such that the natural extension to

$$\bar{\partial}: \Gamma(M, V \otimes \mathcal{A}^{0,k}) \to \Gamma(M, V \otimes \mathcal{A}^{0,k+1})$$

satisfies $\bar{\partial}^2 = 0$. Since every differential operator which acts on the space of smooth sections of a vector bundle extends naturally to the space of 0-currents with values in that vector bundle there is an induced complex

$$\mathcal{J}^0(M, V) \xrightarrow{\bar{\partial}} \mathcal{J}^0(M, V \otimes \mathcal{A}^{0,1}) \xrightarrow{\bar{\partial}} \mathcal{J}^0(M, V \otimes \mathcal{A}^{0,2}) \xrightarrow{\bar{\partial}} \dots$$

which calculates the distribution involutive cohomology $H^*(M, \mathcal{A}'(V))$. The subcomplex of the above complex consisting of compactly supported currents defines the distribution compactly supported involutive cohomology $H^*(M, \mathcal{A}'_c(V))$. Via the isomorphism $\mathcal{E}^m \otimes (\mathcal{A}^{0,k})^* \cong \mathcal{A}^{n, m-n-k}$ (where n is the rank of $\mathcal{A}^{1,0}$) the partial connection $\bar{\partial}$ on $\mathcal{J}^0(M, \mathcal{A}^{0,k})$ is the dual of the partial connection (on compactly supported sections) which makes the bundle $\mathcal{A}^{n,0} \otimes V^*$ (with $\mathcal{A}^{n,0} := \wedge^n \mathcal{A}^{1,0}$) compatible with the involutive structure $\mathcal{A}$ (see [**4**]).

3 - The transform for currents

Our initial data is a double fibration of smooth oriented manifolds

$$\begin{array}{ccc} & F & \\ {}^{\eta}\swarrow & & \searrow^{\tau} \\ Z & & X \end{array}$$

which satisfies the following conditions (see also [**1**] or [**2**]):

1. On Z there is a CR-structure $\mathcal{Q}$. (This means that $\mathcal{Q}$ is an involutive structure for which $T^{0,1} \cap \overline{T^{0,1}} = \{0\}$, where $T^{0,1}$ is the complex sub-bundle of the complexified tangent bundle of Z which is annihilated by $\mathcal{Q}^{1,0}$).

2. The maps η and τ are fibre-bundle projections. The fibers of η have finite dimensional de Rham cohomology (with compact and non-compact supports) and the fibers of τ are compact complex manifolds.

3. The map η embeds the fibres of τ as holomorphic submanifolds of Z. (A submanifold of a CR-manifold is holomorphic if the involutive structure of the CR-manifold restricts to give a complex structure on the submanifold).

The transform for currents has three steps.

1. **Step 1**. For the first step we need the following lemma.

Lemma 2. *Consider the involutive structure* $\mathcal{C}^{1,0} := \eta^*(\mathcal{E}_Z^1)$ *on* F. *The map*

$$\eta^* : \mathcal{J}^0(Z, V) \to \mathcal{J}^0(F, \eta^*(V))$$

is injective and its image is $H^0(F, \mathcal{C}'(\eta^* V))$. *If* η *has contractible fibers then for every* $k \geqslant 1$

$$H^k(F, \mathcal{C}'(\eta^* V)) = 0 .$$

Proof. The first statement is an easy calculation. The second statement follows by considering the homotopy formula for compactly supported forms along the fibers of η and dualising it to get an homotopy formula for currents. ■

Using this lemma the rest of step one follows like in [**1**] or [**2**]. We define an involutive structure $\mathcal{A}$ on F by $\mathcal{A}^{1,0} := \eta^* \mathcal{Q}^{1,0}$ and we obtain the spectral sequence

$$E_2^{p,q} = H^p(Z, \mathcal{Q}(V \otimes \mathcal{H}^q)) \Rightarrow H^{p+q}(F, \mathcal{A}'(\eta^* V)).$$

(Here $\mathcal{H}^q \to Z$ is the bundle whose fiber over $z \in Z$ is the de Rham cohomology group $H^q(\eta^{-1}(z), \mathbb{C})$).

2. **Step 2**. For the second step we define the involutive structure $\mathcal{E}$ on F by $\mathcal{E}^{1,0}$ being the annihilator of the τ vertical vectors which are $(0, 1)$ vectors in the

complex structure of the fibers of τ. We notice that $\mathcal{A}^{1,0} \subset \mathcal{E}^{1,0}$ and we define the vector bundle $B^1 \to F$ by $B^1 := \mathcal{E}^{1,0}/\mathcal{A}^{1,0}$. We obtain (see [**1**] or [**2**]) the spectral sequence

$$E_1^{p,q} = H^q(F, \mathcal{E}'(\eta^* V \otimes B^p)) \Rightarrow H^{p+q}(F, \mathcal{A}'(\eta^* V)).$$

3. **Step 3**. For the third step we need the identification

$$H^k(F, \mathcal{E}'(W)) \cong \mathfrak{J}^0(X, \tau_*^k(W))$$

where $W \to F$ is an $\mathcal{E}$-compatible vector bundle and $\tau_*^k(W)$ is the vector bundle whose fiber over $x \in X$ is the k-Dolbeault cohomology group of the restriction of W to $\tau^{-1}(x)$. (We suppose that the dimension of this cohomology group is constant with respect to $x \in X$). This identification follows by considering a hermitian metric on the fibers of τ varying smoothly with respect to the base and a hermitian metric on the vector bundle W. Then the Green operators and the codifferential operators on each fiber vary smoothly with respect to the base (see [**5**]). Applying fiber by fiber a Hodge de Rham decomposition on forms and dualising it at the level of currents one can show that the natural map

$$: \mathfrak{J}^0(\triangle, \tau_*^k(W)) \to H^k(F, \mathcal{E}'(W))$$

is an isomorphism.

Let $V_{p,q} := \tau_*^q(\eta^* V \otimes B^p)$. Using the above mentioned identification the spectral sequence of step two becomes the spectral sequence

$$E_1^{p,q} = \mathfrak{J}^0(X, V_{p,q}) \Rightarrow H^{p+q}(F, \mathcal{A}'(\eta^* V)).$$

This completes the third step of the transform.

Combining the above steps in the case when the fibres of η are contractible we arrive at the Penrose transform for currents as stated in Theorem 1 of the introduction.

4 - The compactly supported transform

We retain the set-up of Section 3.

Lemma 3. *If the typical fiber of $\eta : F \to Z$ is R^d then for every vector bundle $V \to Z$ there is a canonical isomorphism*

$$H^d(F, \mathcal{C}'_{cv}(\eta^* V)) \cong \mathfrak{J}^0(Z, V)$$

where the subscript «cv» *refers to compact vertical support. The other cohomology groups are zero.*

Proof. Let m be the dimension of Z. Currents in $\mathcal{J}^0_{\mathrm{cv}}(F, \mathcal{C}^{0,d} \otimes \eta^* V)$ are applied to sections of $\eta^*(\mathcal{E}^m \otimes V^*)$ whose support project to compact subsets of Z. The isomorphism is induced by the map

$$T : \mathcal{J}^0(F, \mathcal{C}^{0,d} \otimes \eta^* V) \to \mathcal{J}^0(Z)$$

defined by

$$T(\omega)(\gamma) = (\omega, \eta^*(\gamma))$$

where $\omega \in \mathcal{J}^0(F, \mathcal{C}^{0,d} \otimes \eta^* V)$ and $\gamma \in \Gamma_c(Z, \mathcal{E}^m \otimes V^*)$. Using the homotopy formula on forms along the fibers of η dualised at the level of currents we obtain that the other cohomology groups are zero. ■

The spectral sequence of Theorem 1 for compactly supported currents follows now as in the smooth case (see [**1**]) using the same techniques as in the previous section. We notice that the above lemma induces a shift in the transform by the fiber-dimension of η.

5 - Examples

In our examples we shall consider only the non-compactly supported transforms. The compactly supported transforms work similarily.

A complex twistor space

Consider $X = R^3$ and Z the total space of the holomorphic tangent bundle of CP^1, thought of as the space of all oriented lines in R^3. There is a natural double fibration with F defined by incidence. The Penrose transforms (see [**6**] and Theorem 1) identify the cohomology group $H^1(Z, \mathcal{O}(-2))$ (where $\mathcal{O}(-2)$ on Z is the pull-back of $\mathcal{O}(-2)$ on CP^1 via the tangent bundle projection) with the kernel of the Laplace operator on R^3, acting on smooth functions or currents on R^3.

A CR example

Consider the original Penrose transform of Penrose, restricted to the real Minkowski space R^4 (see [**1**]). In this case Z is a CR-manifold (being a real hypersurface of CP^3). The smooth Penrose transform identifies the $\bar{\partial}_b$ cohomology of $\mathcal{L}(-2)$ (the restriction of $\mathcal{O}(-2)$ to Z) in degree one with the kernel of the wave operator acting on smooth functions on R^4, and the $\overline{\partial}_b$ cohomology in degree two with the cokernel of this operator. From Theorem 1 we obtain similar isomorphisms, but for distribution cohomology and currents.

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Abstract

We extend the smooth Penrose transform to the distribution cohomology.

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