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Presymplectic Lagrangian systems subject to non-holonomic constraints (**)

1 - Introduction

In Classical Physics, a frame-independent description of the behaviour of a system $\mathcal{B}$, with n degrees of freedom, relies on the introduction of an $(n+1)$-dimensional manifold $\mathcal{V}_{n+1}$, called the *configuration space-time*, carrying a natural fibration $t : \mathcal{V}_{n+1} \to \Re$ over the real line, identified with the absolute time function [**5**], [**6**], [**7**], [**8**], [**9**], [**11**], [**12**], [**13**], [**14**], [**17**].

Every admissible evolution of the system is represented by a corresponding section $\gamma : \Re \to \mathcal{V}_{n+1}$. This leads to a natural identification of the velocity space of $\mathcal{B}$ with the first jet space $j_1(\mathcal{V}_{n+1})$, associated with the stated fibration.

However, when the system in study is subject to kinetic constraints, the totality of admissible kinetic states does no longer coincide with the whole manifold $j_1(\mathcal{V}_{n+1})$, but, in general, only with a subregion $\mathcal{A} \subset j_1(\mathcal{V}_{n+1})$.

The more important and significant cases dealt with in the literature are those in which $\mathcal{A}$ has the nature of an embedded submanifold of $j_1(\mathcal{V}_{n+1})$, fibered over

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$\mathcal{V}_{n+1}$, as expressed by the commutative diagram

(1.1)
$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & j_1(\mathcal{V}_{n+1}) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathcal{V}_{n+1} & = & \mathcal{V}_{n+1} . \end{array}$$

Diagrams of this kind include a wide variety of non-holonomic constraints, the only condition being that they are sufficiently smooth and two-sided (the submanifold $\mathcal{A}$ is without boundary).

In the last years, the study of non-holonomic systems has drawn a great and renewed interest: many Authors have devoted their papers to modern treatments of the subject, all — in spite of the variety of the different approaches proposed — having the diagram (1.1) as common geometrical framework. Among others, we refer to [**6**], [**8**], [**11**], [**13**], [**14**], [**15**], [**16**], [**17**], [**20**], [**21**], [**22**], [**23**], [**24**].

Again recently, Massa and co-workers have proposed a new mathematical setting for a gauge-invariant formulation of Lagrangian and Hamiltonian Dynamics [**9**], [**10**], [**12**].

The theory, establishing a natural link between Dynamics and connection theory, relies on the introduction of the bundle of *affine scalars*. The latter is a principal fiber bundle over the configuration space-time $\mathcal{V}_{n+1}$, with structural group $(\Re, +)$. The principal bundle structure of P gives rise to two distinguished actions of the group $\Re$ on the first jet-space $j_1(P, \Re)$ associated with the fibration $P \to \mathcal{V}_{n+1} \to \Re$. The quotient of $j_1(P, \Re)$ with respect these actions determines two fiber bundles $\mathcal{L}(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1})$ over the velocity space $j_1(\mathcal{V}_{n+1})$, called respectively the *Lagrangian* and the *co-Lagrangian* bundle. The situation is summarized into the commutative diagram

(1.2)
$$\begin{array}{ccc} j_1(P, \Re) & \longrightarrow & \mathcal{L}^c(\mathcal{V}_{n+1}) \\ \downarrow & & \downarrow \\ \mathcal{L}(\mathcal{V}_{n+1}) & \longrightarrow & j_1(\mathcal{V}_{n+1}) \end{array}$$

in which all arrows indicate principal fibrations, with structural groups isomorphic to $(\Re, +)$.

In the resulting geometrical set-up, the concept of Lagrangian function L is replaced by a *Lagrangian section* $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$, i.e. with a section of the Lagrangian bundle. Gauge-equivalent Lagrangians are then interpreted as different representations of one and the same section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$, under different choices of the trivialization, while equivalent sections are related to each other by the action of the gauge group. The important point is that, in this way, every Lagrangian section l induces a connection on the co-Lagrangian bundle $\mathcal{L}^c(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$, whose curvature 2-form, viewed as a (gauge invariant) field over $j_1(\mathcal{V}_{n+1})$, coincides — up to a sign — with the Poincaré-Cartan 2-form associated with the given class of Lagrangians.

In connection with this, in [**12**], a detailed analysis of the geometrical properties of the Lagrangian bundle $\mathcal{L}(\mathcal{V}_{n+1})$ has allowed the construction of a gauge-invariant presymplectic formalism for time-dependent Lagrangian Dynamics. Every Lagrangian section l is indeed seen to induce a connection in the principal fiber bundle $j_1(P, \mathfrak{R}) \to \mathcal{L}(\mathcal{V}_{n+1})$ too. The curvature 2-form of this connection, depending on whether l is regular or singular, endows $\mathcal{L}(\mathcal{V}_{n+1})$ either with a symplectic or with a presymplectic structure $\tilde{\Omega}_l$. By means of the latter one can then set up a pseudo problem of motion on $\mathcal{L}(\mathcal{V}_{n+1})$ consisting in the search for vector fields $\tilde{Z} \in D^1(\mathcal{L}(\mathcal{V}_{n+1}))$ solving the equation

$$\tilde{Z} \dashv \tilde{\Omega}_l = -d\varphi_l \tag{1.3}$$

φ_l denoting the trivialization of the bundle $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ associated with l, and playing the role of a «Hamiltonian». Also in [**12**], the mathematical equivalence between the problem (1.3) and the standard one formulated in the velocity space $j_1(\mathcal{V}_{n+1})$ has been proved both in the regular and in the singular case.

In this paper, the mathematical setting outlined so far is applied to the study of time-dependent non-holonomic Lagrangian systems.

The starting point is the construction, through a straightforward pull-back procedure, of a principal fiber bundle $\pi : \mathcal{L}(\mathcal{A}) \to \mathcal{A}$ over the *constraint manifold* $\mathcal{A}$, embedded into $\mathcal{L}(\mathcal{V}_{n+1})$, as expressed by the commutative diagram

$$\begin{tikzcd} \mathcal{L}(\mathcal{A}) \arrow[r, "i"] \arrow[d, "\pi"] & \mathcal{L}(\mathcal{V}_{n+1}) \arrow[d, "\pi"] \\ \mathcal{A} \arrow[r, "i"] & j_1(\mathcal{V}_{n+1}) . \end{tikzcd} \tag{1.4}$$

Next, we «lift» to the bundle $\mathcal{L}(\mathcal{A})$ the constrained problem of motion defined on the manifold $\mathcal{A}$, based on the usual Lagrange-Chetaev equations.

If the given Lagrangian section is regular, it is a straightforward matter to prove the mathematical equivalence between the standard problem on $\mathcal{A}$ and the one formulated on $\mathcal{L}(\mathcal{A})$. More in detail, borrowing from Ibort *et al* [**19**], we define an almost product structure $(\mathcal{P}, \mathcal{Q})$ on $\mathcal{L}(\mathcal{V}_{n+1})$ along $\mathcal{L}(\mathcal{A})$, in such a way that the constrained Dynamics is (π-related to) the projection $\mathcal{P}(\tilde{Z})$ of the solution of the unconstrained problem (1.3). Still following Ibort and co-workers, we construct

Dirac brackets for the present time-dependent systems which, as pointed out in [**19**], could be interesting for canonical quantization purposes.

After this, we examine singular Lagrangians. As it is well known, if the Lagrangian section is degenerate, the problem of motion may or may not admit solutions and, even when a solution exists, it is in general non-unique. Moreover, in the singular case, the equivalence of the problems on $\mathcal{A}$ and on $\mathcal{L}(\mathcal{A})$ is no longer automatically ensured.

To account for this fact, following the procedure implemented in [**12**], we construct a constraint algorithm extending to the present context the one developed by Gotay, Nester and Hinds [**1**], [**2**]. The aim is to obtain — in the singular case — necessary and sufficient conditions for the solvability of the constrained Dynamics in a differential sense.

To this end, we reinforce the problem of motion on $\mathcal{L}(\mathcal{A})$ by requiring that the solutions satisfy the further condition of annihilating the (pull-back to $\mathcal{L}(\mathcal{A})$ of the) so-called Chetaev bundle, i.e. the bundle where the virtual work of the reaction forces takes value [**8**], [**15**], [**22**], [**24**].

It will be seen that the request does not put any restriction on the problem of motion on $\mathcal{A}$, but it allows to discuss the second-order differential equation problem (the search for kinematically admissible solutions) under weaker hypotheses than those employed in other papers [**18**], [**19**]. Here, in fact, the SODE problem is dealt with under the assumption of *admissibility* for the Lagrangian section, while in [**18**], [**19**] the Authors work under the assumption of *almost regularity* which is slightly more restrictive. The non-holonomic constraints dealt with in this paper are affine on the velocities. However, all results stated for regular Lagrangians and the constraint algorithm developed for degenerate ones apply equally well to general (not necessarily affine) constraints. On the contrary, the affine assumption is strictly necessary in the discussion of the SODE problem proposed here.

2 - Geometrical preliminaries

2.1 - *Non-holonomic systems*

Let $\mathcal{V}_{n+1}$ be the configuration space-time of a (finite-dimensional) physical system $\mathcal{B}$. As it is well known, the first jet bundle $j_1(\mathcal{V}_{n+1})$ — identified with the velocity space of the system — is an affine space, modelled on the vertical bundle $V(\mathcal{V}_{n+1})$ associated with the fibration $t : \mathcal{V}_{n+1} \to \Re$, and canonically embedded into the tangent bundle $T(\mathcal{V}_{n+1})$, according to the identification

$$j_1(\mathcal{V}_{n+1}) \simeq \{X \in T(\mathcal{V}_{n+1}) \,|\, \langle X, dt \rangle = 1\}. \tag{2.1}$$

Any local fibered coordinate system $t, q^1, \dots, q^n$ on $\mathcal{V}_{n+1}$ gives rise to corresponding jet-coordinates $t, q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n$ on $j_1(\mathcal{V}_{n+1})$, subject to the transformation laws

$$\bar{t} = t + c, \qquad \bar{q}^i = \bar{q}^i(t, q^1, \dots, q^n), \qquad \dot{\bar{q}}^i = \frac{\partial \bar{q}^i}{\partial t} + \frac{\partial \bar{q}^i}{\partial q^k}\, \dot{q}^k.$$

In jet-coordinates, the identification (2.1) is made explicit by the relation

$$z = \left(\frac{\partial}{\partial t} + \dot{q}^i(z)\, \frac{\partial}{\partial q^i} \right)_{\pi(z)} \in j_1(\mathcal{V}_{n+1})$$

$\pi : j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$ denoting the natural projection.

A second relevant bundle is the second jet space $j_2(\mathcal{V}_{n+1})$ of the fibration $t : \mathcal{V}_{n+1} \to \Re$. The latter, referred to jet-coordinates $t, q^i, \dot{q}^i, \ddot{q}^i$, is an affine bundle over $j_1(\mathcal{V}_{n+1})$, modelled on the vertical bundle $V(j_1(\mathcal{V}_{n+1}))$ associated with the fibration $j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$.

Once again, we have a canonical identification between points $z \in j_2(\mathcal{V}_{n+1})$ and vectors on $j_1(\mathcal{V}_{n+1})$, expressed by the relation

$$z = \left(\frac{\partial}{\partial t} + \dot{q}^i(z)\, \frac{\partial}{\partial q^i} + \ddot{q}^i(z)\, \frac{\partial}{\partial \dot{q}^i} \right)_{\pi(z)} \in j_2(\mathcal{V}_{n+1}) \tag{2.2}$$

π denoting now the natural projection $\pi : j_2(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$.

Every section $Z : j_1(\mathcal{V}_{n+1}) \to j_2(\mathcal{V}_{n+1})$ is called a *dynamical flow* (or SODE) over $j_1(\mathcal{V}_{n+1})$. In local coordinates, consistently with eq. (2.2), we have the representation

$$Z = \frac{\partial}{\partial t} + \dot{q}^i\, \frac{\partial}{\partial q^i} + Z^i(t, q^k, \dot{q}^k)\, \frac{\partial}{\partial \dot{q}^i} \tag{2.3}$$

expressing every dynamical flow Z as a vector field on $j_1(\mathcal{V}_{n+1})$.

By eq. (2.3) it is easily seen that the integral curves of a dynamical flow Z are jet-extensions of sections of $\mathcal{V}_{n+1}$, and that the difference between two arbitrary dynamical flows is a vertical vector field over $j_1(\mathcal{V}_{n+1})$.

In what follows, we shall denote by $C(j_1(\mathcal{V}_{n+1}))$ the annihilator of the $(n+1)$-dimensional distribution over $j_1(\mathcal{V}_{n+1})$ generated by $j_2(\mathcal{V}_{n+1})$. It is a straightforward matter to verify that the bundle $C(j_1(\mathcal{V}_{n+1}))$ — usually referred to as the *contact bundle* — is spanned locally by the 1-forms $\omega^i = dq^i - \dot{q}^i\, dt$. Every section $\sigma : j_1(\mathcal{V}_{n+1}) \to C(j_1(\mathcal{V}_{n+1}))$ will be called *contact* 1-*form* over $j_1(\mathcal{V}_{n+1})$.

At each $z \in j_1(\mathcal{V}_{n+1})$, the affine character of the fibration $j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$ provides a canonical isomorphism between the vertical spaces $V_{\pi(z)}(\mathcal{V}_{n+1})$ and $V_z(j_1(\mathcal{V}_{n+1}))$. The algorithm relies on the fact that every vertical vector $V \in V_{\pi(z)}(\mathcal{V}_{n+1})$ identifies a corresponding vertical vector $\widehat{V} \in V_z(j_1(\mathcal{V}_{n+1}))$, namely the tangent vector to the straight line $\xi \to z + \xi V$ through z. In local coordinates, the correspondence $V \to \widehat{V}$ — known as the *vertical lift* of vectors — is expressed as

$$V = V^i \left(\frac{\partial}{\partial q^i} \right)_{\pi(z)} \to \widehat{V} = V^i \left(\frac{\partial}{\partial \dot{q}^i} \right)_z .$$

We may then set up a non singular pairing $\langle \, \| \, \rangle$ between vertical vectors and contact 1-forms on $j_1(\mathcal{V}_{n+1})$, based on the request [7]

$$\left\langle \frac{\partial}{\partial \dot{q}^i} \middle\| \omega^j \right\rangle_z := \left\langle \frac{\partial}{\partial q^i} \, , \, dq^j - \dot{q}^j(z)\, dt \right\rangle_{\pi(z)} = \delta^j_i \tag{2.4}$$

whence $\langle V \| \sigma \rangle = V^i \sigma_i$ for every $V = V^i \dfrac{\partial}{\partial \dot{q}^i}$, $\sigma = \sigma_i \omega^i$.

It is easily seen that this makes the vertical space $V_z(j_1(\mathcal{V}_{n+1})) \subset T_z(j_1(\mathcal{V}_{n+1}))$ isomorphic to the *dual* of $C_z(j_1(\mathcal{V}_{n+1}))$.

More in general, the pairing $\langle \, \| \, \rangle$ applies equally well to vertical vectors and semibasic 1-forms on $j_1(\mathcal{V}_{n+1})$, according to the relation

$$\langle V \| \eta \rangle = V^i \eta_i \tag{2.5}$$

with $\eta = \eta_i\, dq^i + \eta_0\, dt$.

Now, let us suppose that the given system $\mathcal{B}$ is subject to a set of affine nonholonomic constraints. These are expressed geometrically by an $(n-r)$-dimensional co-distribution D on $\mathcal{V}_{n+1}$, spanned locally by a set of linearly independent 1-forms having expression

$$g^\sigma = g^\sigma_i(t, q)\, dq^i + g^\sigma_0(t, q)\, dt \qquad \sigma = 1, \ldots, n-r . \tag{2.6}$$

In this case, the totality of admissible kinetic states of the system does no longer coincide with the whole manifold $j_1(\mathcal{V}_{n+1})$, but only with the region $\mathcal{A} := D^0 \cap j_1(\mathcal{V}_{n+1})$, D^0 denoting the annihilator of D in $T(\mathcal{V}_{n+1})$.

Under the usual assumption rank$\|g_i^\sigma\| = n - r$, $\mathcal{A}$ has the nature of an embedded submanifold of $j_1(\mathcal{V}_{n+1})$, fibered over $\mathcal{V}_{n+1}$, according to the diagram (1.1).

A local cartesian representation for the submanifold $\mathcal{A}$ is provided by the vanishing of the evaluation functions on $j_1(\mathcal{V}_{n+1})$ associated with the 1-forms g^σ, namely

$$\phi^\sigma(t, q^i, \dot{q}^i) := g_i^\sigma(t, q)\, \dot{q}^i + g_0^\sigma(t, q) = 0 \qquad \sigma = 1, \ldots, n - r\,. \tag{2.7a}$$

Referring $\mathcal{A}$ to local fibered coordinates $t, q^1, \ldots, q^n, z^1, \ldots, z^r$, we may describe equivalently the embedding $i : \mathcal{A} \to j_1(\mathcal{V}_{n+1})$ as

$$\dot{q}^i = \psi^i(t, q, z) := \psi^i_A(t, q)\, z^A + \psi^i_0(t, q) \qquad i = 1, \ldots, n \tag{2.7b}$$

with rank$\|\psi^i_A\| = r$.

The representations (2.7a, b) are related by the obvious identities

$$g_i^\sigma \psi^i_A = 0 \qquad \sigma = 1, \ldots, n - r, \qquad A = 1, \ldots, r\,. \tag{2.8}$$

Identifying $\mathcal{A}$ with its image $i(\mathcal{A}) \subset j_1(\mathcal{V}_{n+1})$, we shall call *admissible* every section $\gamma : \Re \to \mathcal{V}_{n+1}$ whose first jet extension $j_1(\gamma)$ is contained in $\mathcal{A}$. Any such section represents an evolution of the system allowed by the constraints.

The concepts of contact bundle, vertical vector and dynamical flow are easily adapted to the submanifold $\mathcal{A}$.

To start with, the contact bundle $C(\mathcal{A})$ over $\mathcal{A}$ is defined as the pull-back $C(\mathcal{A}) := i^*(C(j_1(\mathcal{V}_{n+1})))$, spanned locally by the 1-forms

$$\tilde{\omega}^k := i^*(\omega^k) = dq^k - \psi^k(t, q, z)\, dt \qquad k = 1, \ldots, n\,.$$

Every section $\sigma : \mathcal{A} \to C(\mathcal{A})$, expressed in coordinates as $\sigma = \sigma_k(t, q, z)\, \tilde{\omega}^k$, will be called contact 1-form over $\mathcal{A}$.

Also, the fibration $\pi : \mathcal{A} \to \mathcal{V}_{n+1}$ induces a corresponding vertical bundle $V(\mathcal{A})$, identified with the sub-bundle

$$V(\mathcal{A}) = \{V \in T(\mathcal{A}) \,|\, \langle V, dt\rangle = 0, \langle V, \tilde{\omega}^k\rangle = 0 \;\; k = 1, \ldots, n\}$$

yielding the characterization

$$V \in V(\mathcal{A}) \Leftrightarrow V = V^A \frac{\partial}{\partial z^A}\,.$$

Taking eq. (2.7b) into account, it is easily seen that the push-forward of the embedding i is an injection of $V(\mathcal{A})$ into $V(j_1(\mathcal{V}_{n+1}))$ summarized into the identities

$$i_*\left(\frac{\partial}{\partial z^A}\right) = \psi^i_A \frac{\partial}{\partial \dot{q}^i} .$$

This allows to look at every fibre of $V(\mathcal{A})$ as a vector sub-space of the corresponding fibre of $V(j_1(\mathcal{V}_{n+1}))$.

Moreover, let $\tau(\mathcal{A}) := T(\mathcal{A}) \cap j_2(\mathcal{V}_{n+1})$ be the intersection of the tangent bundle $T(\mathcal{A})$ over $\mathcal{A}$ with (the restriction to $\mathcal{A}$ of) the second jet bundle $j_2(\mathcal{V}_{n+1})$ viewed as an affine subspace of $T(j_1(\mathcal{V}_{n+1}))$. Recalling eq. (2.2), we have the identification

$$\tau(\mathcal{A}) = \{X \in T(\mathcal{A}) \mid \langle X, dt\rangle = 1, \langle X, \tilde{\omega}^i\rangle = 0 \quad i = 1, \dots, n\}. \tag{2.9}$$

From the latter, it is easily seen that $\tau(\mathcal{A})$ has the nature of affine bundle, modelled on the vertical bundle $V(\mathcal{A})$. Any section $Z : \mathcal{A} \to \tau(\mathcal{A})$ will be called *non-holonomic dynamical flow* and it will be expressed, in local coordinates, as

$$Z = \frac{\partial}{\partial t} + \psi^i(t, q, z) \frac{\partial}{\partial q^i} + Z^A(t, q, z) \frac{\partial}{\partial z^A} . \tag{2.10}$$

From eq. (2.10) it is a straightforward matter to verify that every integral curve of a dynamical flow on $\mathcal{A}$ is automatically the jet extension of an admissible section $\gamma : \Re \to \mathcal{V}_{n+1}$.

Now, let us consider the bilinear pairing (2.4) between vertical vectors and contact 1-forms, introduced above. This one operates on $\mathcal{A}$, through the obvious identification

$$\left\langle \tilde{\omega}^j \middle\| \frac{\partial}{\partial z^A} \right\rangle_z := \left\langle \omega^j \middle\| i_*\left(\frac{\partial}{\partial z^A}\right)\right\rangle_{i(z)} = (\psi^j_A)_z \tag{2.11}$$

$\left(\text{whence } \langle \nu \| V\rangle = \nu_j V^A \psi^j_A \text{ for every } \nu = \nu_j \tilde{\omega}^j,\ V = V^A \dfrac{\partial}{\partial z^A}\right)$.

Of course, the map $\nu, V \to \langle \nu \| V\rangle$ has now a *singular* character, it being clear that any 1-form $\nu = \nu_i \tilde{\omega}^i$ satisfying the conditions $\nu_i \psi^i_A = 0$, $A = 1, \dots, r$ annihilates all vertical vectors. The totality of such 1-forms generate a vector sub-bundle of the contact bundle, henceforth denoted by $\chi(\mathcal{A})$, and called the *Chetaev bundle*. Every section $\nu : \mathcal{A} \to \chi(\mathcal{A})$ is called a *Chetaev 1-form* on $\mathcal{A}$.

A local basis for the Chetaev bundle $\chi(\mathcal{A})$ is provided by any set of linearly independent contact 1-forms $\mu^\sigma = \mu_i^\sigma \tilde{\omega}^i$, $\sigma = 1, \ldots, n-r$ satisfying

$$\mu_i^\sigma \psi_A^i = 0, \qquad \sigma = 1, \ldots, n-r, \ A = 1, \ldots, r.$$

For example, if we adopt the cartesian representation (2.7a) for the submanifold $\mathcal{A}$, eq. (2.8) indicates that a possible choice is given by the ansatz

$$\mu^\sigma = g_i^\sigma \tilde{\omega}^i. \tag{2.12}$$

From this, denoting by D^v the vertical lift or, what is the same, the pull-back on $j_1(\mathcal{V}_{n+1})$ of the «constraint» co-distribution D over $\mathcal{V}_{n+1}$, it is easily seen that $\chi(\mathcal{A}) = D^v_{|\mathcal{A}}$ (1).

Now, let $L \in \mathcal{F}(j_1(\mathcal{V}_{n+1}))$ be a lagrangian function expressing a given dynamics acting on the system $\mathcal{B}$.

Then, denoting by $\Omega := d\left(L\,dt + \dfrac{\partial L}{\partial \dot{q}^i}\,\omega^i\right)$ the associated Poincaré-Cartan 2-form, the corresponding problem of motion for the constrained system $\mathcal{B}$ may be formulated through the requirements

$$\begin{cases} Z \,\lrcorner\, \Omega_{\,|\mathcal{A}} \in \chi(\mathcal{A}) \\ \langle Z, dt \rangle = 1 \\ Z(\phi^\sigma)_{|\mathcal{A}} = 0 \qquad \forall \sigma = 1, \ldots, n-r \end{cases} \tag{2.13}$$

where $Z \in D^1(\mathcal{A})$ is the unknown.

In fact, under the standard regularity assumptions $\det \left\| \dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right\| \neq 0$ and $\operatorname{rank} \|g_i^\sigma\| = n - r$, eqs. (2.13) admit as unique solution the dynamical flow Z tangent to $\mathcal{A}$ whose integral curves satisfy the usual Lagrange-Chetaev equations [**6**], [8], [**13**], [**17**]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_\sigma g_i^\sigma, \qquad \dot{q}^i = \frac{dq^i}{dt}, \qquad g_i^\sigma \dot{q}^i + g_0^\sigma = 0 \tag{2.14}$$

for the unknowns $q^i(t)$, $\dot{q}^i(t)$, and $\lambda_\sigma(t)$.

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(1) This is a direct consequence of the affine nature of the given constraints. However, the construction of the Chetaev bundle proposed above is more general and concerns arbitrary (not necessarily affine) non-holonomic constraints.

2.2 - *The Lagrangian bundle*

For convenience of the reader, referring to [**9**], [**12**] for comments, notations and terminology, we recall here some geometrical properties of the Lagrangian bundle $\mathcal{L}(\mathcal{V}_{n+1})$.

By construction, the latter is a principal fiber bundle $\pi : \mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ over the velocity space, with structural group isomorphic to $(\Re, +)$. It is possible to refer $\mathcal{L}(\mathcal{V}_{n+1})$ to local fibered coordinates of the form $t, q^i, \dot{q}^i, \dot{u}$, subject to the transformation law

$$\bar{t} = t + c, \qquad \bar{q}^i = \bar{q}^i(t, q), \qquad \dot{\bar{q}}^i = \frac{\partial \bar{q}^i}{\partial q^k}\,\dot{q}^k + \frac{\partial \bar{q}^i}{\partial t}, \qquad \dot{\bar{q}} = \dot{u} + \frac{df}{dt} \tag{2.15}$$

with $\dfrac{df}{dt} := \dfrac{\partial f}{\partial q^k}\,\dot{q}^k + \dfrac{\partial f}{\partial t}$, $f = f(t, q) \in \mathcal{F}(\mathcal{V}_{n+1})$.

Of course, $\mathcal{L}(\mathcal{V}_{n+1})$ is also fibered, in a natural way, over the configuration space-time $\mathcal{V}_{n+1}$. The vertical bundle associated with this fibration is indicated by $V(\mathcal{L}(\mathcal{V}_{n+1}))$ and is spanned locally by the vector fields $\dfrac{\partial}{\partial \dot{u}}$, $\dfrac{\partial}{\partial \dot{q}^i}$ $i = 1, \ldots, n$.

Any section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ is called a *Lagrangian section*. In coordinates, every such l is described locally in the form

$$\dot{u} = L(t, q^i, \dot{q}^i) \tag{2.16}$$

involving a function $L \in \mathcal{F}(j_1(\mathcal{V}_{n+1}))$, henceforth called «the Lagrangian».

In [**12**], we have shown that the assignment of a Lagrangian section l induces on the manifold $\mathcal{L}(\mathcal{V}_{n+1})$ the following geometrical objects:

- a trivialization $\varphi_l := \dot{u} - L(t, q^i, \dot{q}^i)$ of the principal fiber bundle $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$;

- a smooth connection of $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$, whose connection 1-form is given by the differential $d\varphi_l = d\dot{u} - dL$. The related horizontal lift associates with every vector field $X = X^0 \dfrac{\partial}{\partial t} + X^i \dfrac{\partial}{\partial q^i} + \dot{X}^i \dfrac{\partial}{\partial \dot{q}^i} \in D^1(j_1(\mathcal{V}_{n+1}))$ a corresponding vector field X_l on $\mathcal{L}(\mathcal{V}_{n+1})$, invariant under the action of the structural group $\Big($i.e., under the 1-parameter group of diffeomorphisms generated by $\dfrac{\partial}{\partial \dot{u}}\Big)$ and expressed locally as

$$X_l = X^0 \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + \dot{X}^i \frac{\partial}{\partial \dot{q}^i} + X(L)\,\frac{\partial}{\partial \dot{u}} \tag{2.17}$$

- a $(1, 1)$ tensor field $\tilde{J}$ on $\mathcal{L}(\mathcal{V}_{n+1})$, having local expression

$$\tilde{J} = \omega^i \otimes \left(\frac{\partial L}{\partial \dot{q}^i} \frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{q}^i} \right) \tag{2.18}$$

where we have preserved the notation $\omega^i = dq^i - \dot{q}^i dt \;\; i = 1, \ldots, n$ for the pull-back to $\mathcal{L}(\mathcal{V}_{n+1})$ of the contact 1-forms on $j_1(\mathcal{V}_{n+1})$. It is immediate to see that $\tilde{J}$ is π-related to the fundamental tensor $J = \omega^i \otimes \dfrac{\partial}{\partial \dot{q}^i}$ of $j_1(\mathcal{V}_{n+1})$;

- an exact 2-form $\widetilde{\Omega}_l$ on $\mathcal{L}(\mathcal{V}_{n+1})$, expressed, in local fibered coordinates, as

$$\widetilde{\Omega}_l := d\dot{u} \wedge dt + d\left(\frac{\partial L}{\partial \dot{q}^i} \right) \wedge \omega^i - \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i \wedge dt\,. \tag{2.19}$$

Under the regularity assumption rank $\left\| \dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right\| = n$, it is a straightforward matter to verify that the 2-form (2.19) has maximal rank, thus endowing the bundle $\mathcal{L}(\mathcal{V}_{n+1})$ with a symplectic structure. When this is the case, following the standard terminology, the section l is said to be a *regular* Lagrangian section. On the contrary, when the regularity hypothesis is violated, but $\widetilde{\Omega}_l$ has constant rank everywhere, the 2-form (2.19) is *presymplectic*. In such a circumstance, we shall call l a *degenerate* (or *singular*) Lagrangian section. Furthermore, denoting by $\Omega_l := d\left(L\, dt + \dfrac{\partial L}{\partial \dot{q}^i} \omega^i \right)$ the Poincaré-Cartan 2-form on $j_1(\mathcal{V}_{n+1})$ associated with the Lagrangian function $L(t, q^i, \dot{q}^i)$, one has $\Omega_l = l^*(\widetilde{\Omega}_l)$.

By means of the 2-form $\widetilde{\Omega}_l$ and the trivialization φ_l mentioned above, we may construct equations of motion directly on the Lagrangian bundle $\mathcal{L}(\mathcal{V}_{n+1})$. The algorithm is based on the search for vector fields $\tilde{Z} \in D^1(\mathcal{L}(\mathcal{V}_{n+1}))$ satisfying the requirement

$$\tilde{Z} \rfloor \widetilde{\Omega}_l = -d\varphi_l. \tag{2.20}$$

In [**12**] we have indeed proved that — both in the regular and in the singular case — the problem (2.20) is mathematically equivalent to the standard one formulated on $j_1(\mathcal{V}_{n+1})$ through the cosymplectic (precosymplectic) structure (Ω_l, dt), namely through the equations

$$Z \rfloor \Omega_l = 0\,, \qquad \langle Z, dt \rangle = 1 \tag{2.21}$$

with unknown $Z \in D^1(j_1(\mathcal{V}_{n+1}))$. More precisely, we have shown that eq. (2.20) admits a solution if and only if eqs. (2.21) do, and that the solutions of both problems of motion (2.20), (2.21) are related, to each other, in a natural way.

3 - Presymplectic non-holonomic Lagrangian systems

3.1 - *Regular Lagrangian systems*

Let us denote by $\mathcal{L}(\mathcal{A})$ the pull-back on $\mathcal{A}$ of the Lagrangian bundle $\mathcal{L}(\mathcal{V}_{n+1})$, according to the commutative diagram

$$
\begin{array}{ccc}
\mathcal{L}(\mathcal{A}) & \xrightarrow{i} & \mathcal{L}(\mathcal{V}_{n+1}) \\
\downarrow \pi & & \downarrow \pi \\
\mathcal{A} & \xrightarrow{i} & j_1(\mathcal{V}_{n+1})\,.
\end{array}
\tag{3.1}
$$

In what follows, we shall systematically identify $\mathcal{L}(\mathcal{A})$ with its image $i(\mathcal{L}(\mathcal{A}))$ $\subset \mathcal{L}(\mathcal{V}_{n+1})$, and adopt for the latter a cartesian representation of the form (2.7a).

Also, we shall denote by $\widehat{\chi}(\mathcal{A})$ the pull-back on $\mathcal{L}(\mathcal{A})$ of the Chetaev bundle $\chi(\mathcal{A})$ over $\mathcal{A}$ ([2]).

Now, given a regular lagrangian section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$, expressed locally as $\dot{u} = L(t, q^i, \dot{q}^i)$, let $\widetilde{\Omega}_l$ be the symplectic 2-form on $\mathcal{L}(\mathcal{V}_{n+1})$ induced by l.

We set up a non-holonomic problem of motion on $\mathcal{L}(\mathcal{A})$, based on the requirements

$$
(\widehat{Z} \lrcorner\, \widetilde{\Omega}_l + d\varphi_l)_{|\mathcal{L}(\mathcal{A})} \in \widehat{\chi}(\mathcal{A})\,, \qquad \widehat{Z} \in D^1(\mathcal{L}(\mathcal{A}))
\tag{3.2}
$$

where, of course, $\widehat{Z}$ is the unknown, and $\varphi_l = \dot{u} - L$ denotes the trivialization of the principal fiber bundle $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ induced by l.

Taking the cartesian representation (2.7a), as well as the fact that the 1-forms $g_i^\sigma\, \omega^i{}_{|\mathcal{L}(\mathcal{A})}$ provide a local basis for $\widehat{\chi}(\mathcal{A})$ into account, it easily seen that the condi-

---

([2]) Once again, indicating by $\widehat{D^v}$ the pull-back on $\mathcal{L}(\mathcal{V}_{n+1})$ of the co-distribution D^v over $j_1(\mathcal{V}_{n+1})$, we have $\widehat{\chi}(\mathcal{A}) = \widehat{D^v}_{|\mathcal{L}(\mathcal{A})}$.

tions (3.2) are mathematically equivalent to the system of equations

$$
\begin{cases} (\widehat{Z} \dashv \widetilde{\Omega}_l + d\varphi_l)_{|\mathcal{L}(\mathcal{C})} = \lambda_\sigma g_i^\sigma \omega^i{}_{|\mathcal{L}(\mathcal{C})} \\ \widehat{Z}(\phi^\sigma)_{|\mathcal{L}(\mathcal{C})} = 0 \qquad \forall \sigma = 1, \ldots, n-r \end{cases} \tag{3.3}
$$

λ_σ indicating $(n-r)$ unknown Lagrange multipliers.

Under the regularity assumption for the lagrangian section l and the maximality condition on the rank of the matrix $\|g_i^\sigma\|$, it is a straightforward matter — left to the reader — to prove that the system (3.3) admits the unique solution

$$
\widehat{Z} = Z + Z(L) \frac{\partial}{\partial \dot{u}} \tag{3.4}
$$

Z denoting the dynamical flow on $\mathcal{C}$ (3) uniquely determined by eqs. (2.13).

Recalling the representation (2.17), the vector field $\widehat{Z}$ is recognized as the horizontal lift of the dynamical flow Z with respect to the smooth connection $d\varphi_l$ of $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ induced by l.

As in the unconstrained case [**12**], we thus come to the conclusion that the problem formulated through eqs. (3.3) and the standard one based on eqs. (2.13) are equivalent.

More specifically, if $\widehat{Z}$ satisfies eqs. (3.3) then it is π-projectable on $\mathcal{C}$, and its image $Z = \pi_*(\widehat{Z})$ is a solution of eqs. (2.13). Conversely, if Z is a solution of (2.13), its horizontal lift (3.4) satisfies eqs. (3.3).

Our aim is now to examine the geometrical meaning of eqs. (3.3), in connection with the symplectic structure $\widetilde{\Omega}_l$. To this end, we borrow from [**19**] adapting the argument to the present geometrical context.

To start with, let us indicate by $\sharp_l \colon T^*(\mathcal{L}(\mathcal{V}_{n+1})) \to T(\mathcal{L}(\mathcal{V}_{n+1}))$ the process of rising the indices induced by $\widetilde{\Omega}_l$, i.e. the linear map $\eta \in T^*(\mathcal{L}(\mathcal{V}_{n+1})) \to \sharp_l(\eta) \in T(\mathcal{L}(\mathcal{V}_{n+1}))$ defined by $\sharp_l(\eta) \dashv \widetilde{\Omega}_l = \eta$.

Let us then consider the symplectic complement $T(\mathcal{L}(\mathcal{C}))^\perp \subset T_{\mathcal{L}(\mathcal{C})}(\mathcal{L}(\mathcal{V}_{n+1}))$ of the tangent bundle $T(\mathcal{L}(\mathcal{C}))$.

Given a cartesian representation (2.7a) for the submanifold $\mathcal{L}(\mathcal{C})$, it is well known that a local basis for the bundle $T(\mathcal{L}(\mathcal{C}))^\perp$ is provided by (the restriction on $\mathcal{L}(\mathcal{C})$ of) the $(n-r)$ vector fields $X_{d\phi^\sigma} := \sharp_l(d\phi^\sigma)$. A straightforward evaluation

(3) More precisely, the flow Z is defined in a neighbourhood of the submanifold $\mathcal{C} \subset j_1(\mathcal{V}_{n+1})$ and is tangent to $\mathcal{C}$.

in local coordinates yields the expressions

$$X_{d\phi^\sigma} = -\Gamma^{ij} g_j^\sigma \frac{\partial}{\partial q^i} + V^\sigma \tag{3.5}$$

Γ^{ij} denoting the matrix inverse of $\Gamma_{ij} := \dfrac{\partial^2 L}{\partial \dot q^i \partial \dot q^j}$ and $V^\sigma \in V(\mathcal{L}(\mathcal{V}_{n+1}))$ indicating vertical vector fields whose explicit expression is not needed in the following discussion.

Also, we consider the symplectic complement of the annihilator $\hat\chi(\mathcal{A})^0 \subset T_{\mathcal{L}(\mathcal{A})}(\mathcal{L}(\mathcal{V}_{n+1}))$ of $\hat\chi(\mathcal{A})$, and denote it by $(\hat\chi(\mathcal{A})^0)^\perp$.

Setting, for simplicity, $\nu^\sigma := -g_i^\sigma \omega^i{}_{|\mathcal{L}(\mathcal{A})}$, as above we have that $(\hat\chi(\mathcal{A})^0)^\perp$ is generated locally by the vectors $X_{\nu^\sigma} := \sharp_l(\nu^\sigma)$. In local coordinates, it is an easy matter to prove the identities

$$X_{\nu^\sigma} = -\Gamma^{ij} g_j^\sigma \left(\frac{\partial}{\partial \dot q^i} + \frac{\partial L}{\partial \dot q^i} \frac{\partial}{\partial \dot u} \right). \tag{3.6}$$

Due to the linear independence of the 1-forms $d\phi^\sigma$ and ν^σ, the vectors $X_{d\phi^\sigma}$ and X_{ν^σ} are linearly independent too, so that one has $(\hat\chi(\mathcal{A})^0)^\perp \cap T(\mathcal{L}(\mathcal{A}))^\perp = \{0\}$.

In view of this, we state the following

Proposition 3.1. *The space $(\hat\chi(\mathcal{A})^0)^\perp \oplus T(\mathcal{L}(\mathcal{A}))^\perp$ is a symplectic subspace of $T_{\mathcal{L}(\mathcal{A})}(\mathcal{L}(\mathcal{V}_{n+1}))$.*

Proof. To start with, we observe that

$$\left((\hat\chi(\mathcal{A})^0)^\perp \oplus T(\mathcal{L}(\mathcal{A}))^\perp\right)^\perp = \hat\chi(\mathcal{A})^0 \cap T(\mathcal{L}(\mathcal{A}))\,.$$

Also, any $X \in (\hat\chi(\mathcal{A})^0)^\perp \oplus T(\mathcal{L}(\mathcal{A}))^\perp$ may be represented in the form $X = \lambda_\sigma X_{d\phi^\sigma} + \eta_\sigma X_{\nu^\sigma}$. Therefore, if, in addition, $X \in \hat\chi(\mathcal{A})^0 \cap T(\mathcal{L}(\mathcal{A}))$, from the representations (3.5), (3.6) we derive the relations

$$0 = \langle X, \nu^\sigma \rangle = \langle \lambda_\gamma X_{d\phi^\gamma}, \nu^\sigma \rangle = \lambda_\gamma C^{\gamma\sigma}$$

with $C^{\gamma\sigma} := \Gamma^{ij} g_i^\gamma g_j^\sigma$. Now, in view of the regularity assumptions on $\det \left\| \dfrac{\partial^2 L}{\partial \dot q^i \partial \dot q^j} \right\|$ and $\operatorname{rank} \| g_i^\sigma \|$, the matrix $C^{\gamma\sigma}$ is non-singular, so that $\lambda_\sigma = 0$, $\forall \sigma = 1, \ldots, n-r$.

In a similar way, we have

$$0 = \langle X, d\phi^\sigma \rangle = \langle \eta_\gamma X_{\nu^\gamma}, d\phi^\sigma \rangle = -\eta_\gamma C^{\gamma\sigma}$$

yielding $\eta_\sigma = 0$, $\forall\sigma = 1, \ldots, n-r$. We conclude that $X = 0$, so proving the Proposition. ■

From Proposition 3.1, the direct sum decomposition

$$T_{\mathcal{L}(\mathcal{A})}(\mathcal{L}(\mathcal{V}_{n+1})) = [(\hat{\chi}(\mathcal{A})^0)^\perp \oplus T(\mathcal{L}(\mathcal{A}))^\perp] \oplus [\hat{\chi}(\mathcal{A})^0 \cap T(\mathcal{L}(\mathcal{A}))] \tag{3.7}$$

follows. Associated with the latter, there are two projection operators

$$\mathcal{P}\colon T_{\mathcal{L}(\mathcal{A})}(\mathcal{L}(\mathcal{V}_{n+1})) \to \hat{\chi}(\mathcal{A})^0 \cap T(\mathcal{L}(\mathcal{A}))$$

and

$$\mathcal{Q}\colon T_{\mathcal{L}(\mathcal{A})}(\mathcal{L}(\mathcal{V}_{n+1})) \to (\hat{\chi}(\mathcal{A})^0)^\perp \oplus T(\mathcal{L}(\mathcal{A}))^\perp .$$

These (4) allow to clarify the geometrical meaning of eqs. (3.3). Indeed, we have

Theorem 3.1. *Let $\tilde{Z}$ be the solution of the unconstrained problem* (2.20). *Then the vector field $\hat{Z} := \mathcal{P}(\tilde{Z})$ is the solution of eqs.* (3.3).

Proof. First of all we recall that, in the regular case, the (unique) solution of (2.20) is nothing but the horizontal lift $\tilde{Z} := Z_l$ (see eq. (2.17)) of the unique dynamical flow Z over $j_1(\mathcal{V}_{n+1})$ solving eqs. (2.21) [**12**]. Furthermore, by definition, $\mathcal{P}(\tilde{Z})$ is tangent to $\mathcal{L}(\mathcal{A})$ and, according to eq. (3.7), may be expressed as

$$\mathcal{P}(\tilde{Z}) = \tilde{Z} - \mathcal{Q}(\tilde{Z}) = \tilde{Z} - \eta_\sigma X_{d\phi^\sigma} - \lambda_\sigma X_{\nu^\sigma} \tag{3.8}$$

for some functions λ_σ, η_σ. Moreover, we have necessarily $\langle \mathcal{P}(\tilde{Z}), \nu^\gamma \rangle = 0$, $\forall\gamma = 1, \ldots, n-r$. Therefore, from eqs. (3.5), (3.6), (3.8) we derive the relations

$$0 = -\eta_\sigma \langle X_{d\phi^\sigma}, \nu^\gamma \rangle = -\eta_\sigma C^{\sigma\gamma}$$

implying that $\eta_\sigma = 0$, $\forall\sigma = 1, \ldots, n-r$. We then get the expression

$$\mathcal{P}(\tilde{Z}) = \tilde{Z} - \lambda_\sigma X_{\nu^\sigma}$$

(4) More in particular, we may define the projectors $\mathcal{P}$ and $\mathcal{Q}$ on $T_U(\mathcal{L}(\mathcal{V}_{n+1}))$, U being a neighbourhood of $\mathcal{L}(\mathcal{A})$ in $\mathcal{L}(\mathcal{V}_{n+1})$.

mathematically equivalent to the equation

$$\left(\mathcal{P}(\widetilde{Z}) \dashv \widetilde{\Omega}_l + d\varphi_l\right)_{|\mathcal{L}(\mathcal{C})} = \lambda_\sigma g_i^\sigma \omega^i{}_{|\mathcal{L}(\mathcal{C})} \qquad \blacksquare$$

We notice that the multipliers λ_σ may be determined explicitly by imposing the tangency conditions $\widehat{Z}(\phi^\gamma)_{|\mathcal{L}(\mathcal{C})} = 0$. This yields $\lambda_\sigma = -C_{\sigma\gamma}\widetilde{Z}(\phi^\gamma)_{|\mathcal{L}(\mathcal{C})}$, $C_{\sigma\gamma}$ denoting the inverse matrix of $C^{\sigma\gamma}$. Indicating by $\{\, ,\}_l$ the Poisson brackets on $\mathcal{L}(\mathcal{V}_{n+1})$ induced by $\widetilde{\Omega}_l$, we may write $\lambda_\sigma = -C_{\sigma\gamma}\{\phi^\gamma, \varphi_l\}_{l|\mathcal{L}(\mathcal{C})}$.

Following Ibort and co-workers [**19**], making use of the projection $\mathcal{P}$, we may set up corresponding Dirac brackets $\{,\}_{\mathcal{L}(\mathcal{C})}$ on $\mathcal{L}(\mathcal{C})$, defined by

$$\{f, g\}_{\mathcal{L}(\mathcal{C})} := \widetilde{\Omega}_l(\mathcal{P}(X_{dg}), \mathcal{P}(X_{df}))$$

where $X_{df} := \sharp_l(df)$, $X_{dg} := \sharp_l(dg)$, f and g denoting arbitrary functions defined in a neighbourhood of $\mathcal{L}(\mathcal{C})$ in $\mathcal{L}(\mathcal{V}_{n+1})$.

In this connection, we state

Proposition 3.2. *The Dirac brackets $\{\, ,\}_{\mathcal{L}(\mathcal{C})}$ satisfy the following properties:*

i) *the constraint functions ϕ^σ are Casimir functions, i.e.*

$$\{\phi^\sigma, f\}_{\mathcal{L}(\mathcal{C})} = 0 \quad \forall f \in \mathcal{F}(\mathcal{L}(\mathcal{V}_{n+1}))\,;$$

ii) *if f is an observable (i.e. f is (the pull-back of) a function on $j_1(\mathcal{V}_{n+1})$), its evolution law is expressed as*

$$\frac{df}{dt} = \{f, \varphi_l\}_{\mathcal{L}(\mathcal{C})}$$

Proof. i) By definition, $\{\phi^\sigma, f\}_{\mathcal{L}(\mathcal{C})} = \widetilde{\Omega}_l(\mathcal{P}(X_{df}), \mathcal{P}(X_{d\phi^\sigma})) = 0$ since $\mathcal{P}(X_{d\phi^\sigma}) = 0$. ii) Analogously, one has $\{f, \varphi_l\}_{\mathcal{L}(\mathcal{C})} = \{-\varphi_l, f\}_{\mathcal{L}(\mathcal{C})} = \widetilde{\Omega}_l(\mathcal{P}(X_{df}), \mathcal{P}(\widetilde{Z})) = \widetilde{\Omega}_l(X_{df} - \lambda_\sigma X_{d\phi^\sigma} - \eta_\sigma X_{\nu^\sigma}, \widehat{Z})$, for some functions λ_σ, η_σ. Then, since $\widehat{Z} \in \widehat{\chi}(\mathcal{C})^0 \cap T(\mathcal{L}(\mathcal{C})) = ((\widehat{\chi}(\mathcal{C})^0)^\perp \oplus T(\mathcal{L}(\mathcal{C}))^\perp)^\perp$, we deduce $\{f, \varphi_l\}_{\mathcal{L}(\mathcal{C})} = \widetilde{\Omega}_l(X_{df}, \widehat{Z}) = \langle \widehat{Z}, df\rangle = \dfrac{df}{dt}$. $\blacksquare$

Proposition 3.3. *The Dirac brackets $\{\, ,\}_{\mathcal{L}(\mathcal{C})}$ satisfy the Jacobi identity if and only if the constraints are integrable*

Proof. By a general result on Poisson manifolds [**25**] (see also [**19**]), the Dirac brackets $\{\, ,\}_{\mathcal{L}(\mathcal{C})}$ satisfy the Jacobi identity if and only if $\widehat{\chi}(\mathcal{C})^0 \cap T(\mathcal{L}(\mathcal{C}))$ is an involutive distribution. Moreover, given arbitrary vector fields $X, Y \in \hat{\chi}(\mathcal{C})^0 \cap T(\mathcal{L}(\mathcal{C}))$ and any Chetaev 1-form $\nu \in \hat{\chi}(\mathcal{C})$, one has $\langle [X, Y], \nu \rangle = d\nu(Y, X)$. The mathematical equivalence between the involutiveness of $\hat{\chi}(\mathcal{C})^0 \cap T(\mathcal{L}(\mathcal{C}))$ and the integrability of the constraints follows then by the fact that a set of kinetic constraints is integrable if and only if the ideal generated by the corresponding module of Chetaev 1-forms is a differential ideal [8]. ■

As a concluding remark, we notice that the whole machinery developed so far applies equally well to the case of general (not necessarily affine) non-holonomic constraints.

3.2 - *Singular Lagrangian systems*

Let us now suppose that the given lagrangian section l is degenerate and that the 2-form $\tilde{\Omega}_l$ is presymplectic.

In this case, in general, eqs. (2.13) (respectively (3.2)) may admit no solution at all, or, when a solution exists, it may be non-unique. Moreover, the equivalence between the problems of motion formulated respectively on $\mathcal{L}(\mathcal{V}_{n+1})$ and $j_1(\mathcal{V}_{n+1})$ is no longer so immediate as in the regular case.

To account for this situation, we shall restrict our attention on the solutions Z $(\hat{Z})$ of (2.13) ((3.2)) satisfying the additional requirement

$$\langle Z, \nu \rangle = 0 \ \ (\langle \hat{Z}, \hat{\nu} \rangle = 0) \qquad \forall \nu \in \chi(\mathcal{C}) \ \ (\hat{\nu} \in \hat{\chi}(\mathcal{C}))\,. \tag{3.9}$$

It is easily seen that the request (3.9) is not restrictive in order to find dynamical flows $Z \in D^1(\mathcal{C})$ solving eqs. (2.13) (i.e, in order to find kinematically admissible solutions). Indeed, since $\chi(\mathcal{C}) \subset C(\mathcal{C})$ — taking the characterization (2.9) into account — it is clear that if the problem (2.13), (3.9) has no solution, then the (singular) constrained problem of motion for $\mathcal{B}$ is unsolvable too.

Therefore, in the subsequent discussion, the condition (3.9) will be systematically embodied into the scheme.

We shall now prove that the two problems (2.13) and (3.2), both completed with the additional requirement (3.9), are mathematically equivalent, at least in an algebraic sense.

To this end, following [**12**], let us suppose that there exists a maximal submanifold $M \subset \mathcal{L}(\mathcal{C})$ on which eqs. (3.3), (3.9) admit a solution $\hat{X}$, namely $\forall z \in M \ \exists \hat{X} \in T_z(\mathcal{L}(\mathcal{C}))$ such that $(\hat{X} \lrcorner \tilde{\Omega}_l + d\varphi_l)_{|\mathcal{L}(\mathcal{C})} \in \hat{\chi}(\mathcal{C})$ and $\langle \hat{X}, \hat{\nu} \rangle = 0 \ \forall \hat{\nu} \in \hat{\chi}(\mathcal{C})$.

Then, making use of the straightforward fact that the forms $\tilde{\Omega}_l$ and $d\varphi_l$, as well as the bundle $\hat{\chi}(\mathcal{C})$, are invariant under the action of the structural group, i.e. under translations along the fibres of $\mathcal{L}(\mathcal{C}) \to \mathcal{C}$, we have the following

Proposition 3.4. *Let $\pi : \mathcal{L}(\mathcal{A}) \to \mathcal{A}$ denote the canonical projection. Then, for each $z \in M$, the whole fiber $\pi^{-1}(\pi(z))$ over $\pi(z)$ is contained in M.*

Proof. Let $\widehat{X} \in T_z(\mathcal{L}(\mathcal{A}))$ be a solution of the problem (3.2), (3.9) at $z \in M$. For each $\bar{z} \in \pi^{-1}(\pi(z))$, denote by $\psi_{\xi}: \mathcal{L}(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ the translation satisfying $\psi_{\xi}(\bar{z}) = z$, and consider the vector $\widehat{Y} \in T_{\bar{z}}(\mathcal{L}(\mathcal{A}))$ such that $\widehat{X} = (\psi_{\xi})_* \widehat{Y}$. It is then a straightforward matter — left to the reader — to verify that $\widehat{Y}$ satisfies the requirements (3.2), (3.9) at the point $\bar{z}$. ■

Definition 3.1. *A pair $(M, \widehat{X})$, where $M \subset \mathcal{L}(\mathcal{A})$ is a submanifold and $\widehat{X}: M \to T_M(\mathcal{L}(\mathcal{A}))$ is a vector field satisfying the requirements* (3.2) *and* (3.9), *will be called an algebraic solution of the problem* (3.2), (3.9). *In a similar way, an algebraic solution of eqs.* (2.13), (3.9) *will be understood as a vector field X, defined on a submanifold N of $\mathcal{A}$ and satisfying eqs.* (2.13), (3.9) *everywhere on N.*

Proposition 3.5. *Problem* (3.2), (3.9) *admits an algebraic solution if and only if eqs.* (2.13), (3.9) *do.*

Proof. Still following [**12**], let

$$M_0 := \{z \in \mathcal{L}(\mathcal{A}) \mid \dot{u}(z) = L(\pi(z))\} \tag{3.10}$$

denote the image of $\mathcal{A}$ under the lagrangian section l. Given any algebraic solution $(M, \widehat{X})$ of (3.2), (3.9), we have necessarily

$$\langle \widehat{X}, d\varphi_l \rangle = 0 \tag{3.11}$$

everywhere on M, and, therefore, also on $\Sigma := M \cap M_0$. It follows that $\widehat{X}_{|\Sigma}$ is tangent to M_0. Accordingly, there exists a unique vector field $X : \pi(\Sigma) \to T(\mathcal{A})$ π-related to $\widehat{X}$, i.e. satisfying $X = \pi_*(\widehat{X})$, or, what is the same, $l_*(X) = \widehat{X}_{|\Sigma}$. A straightforward argument, left to the reader, shows that X is then a solution of eqs. (2.13), (3.9) on $\pi(\Sigma)$ $(= \pi(M))$.

Conversely, if (N, X) is an algebraic solution of eqs. (2.13), (3.9), then it is easily seen that the horizontal lift X_l (2.17) satisfies eqs. (3.2), (3.9) on the submanifold $\pi^{-1}(N) \subset \mathcal{L}(\mathcal{A})$. Indeed, the push forward $l_*(X)$ satisfies (3.2), (3.9) on the image space $l(N) \subset \mathcal{L}(\mathcal{A})$. The conclusion then follows from the fact that eqs. (3.2), (3.9) are invariant under the action of the structural group, while the lift X_l is obtained by pushing $l_*(X)$ along the fibres via the action of the group itself. ■

3.3 - *The constraint algorithm*

The aim of this subsection is to set up a constructive method for finding differential solutions for eqs. (3.3), (3.9) — or, equivalently, for eqs. (2.13), (3.9) — in the singular case.

To this end, using the tools of presymplectic geometry, we shall adapt the constraint algorithm proposed in [**12**] for «free» degenerate Lagrangians ([5]) to the present constrained case.

To start with, we shall look for solutions of (3.2), (3.9) only at points of M_0, i.e. on the image of $\mathcal{A}$ under the lagrangian section l. There is no loss in generality in doing this. Indeed, solving the problem on a submanifold $M \subseteq M_0$ and «pushing forward» the result along the fibres (as we did in Proposition 3.4), we obtain a solution on the totality of fibres $\pi^{-1}(\pi(z)) \subset \mathcal{L}(\mathcal{A})$, $z \in M$. Moreover, as seen in Proposition 3.5, the knowledge of a solution on $M \subseteq M_0$ is sufficient to identify a corresponding vector field on $\pi(M) \subseteq \mathcal{A}$ solving eqs. (2.13), (3.9).

With this in mind, let us begin by assuming that there exists a submanifold $M_1 \subseteq M_0$ on which the problem (3.2), (3.9) admits at least an algebraic solution X.

Denoting by $\flat : T(\mathcal{L}(\mathcal{V}_{n+1})) \to T^*(\mathcal{L}(\mathcal{V}_{n+1}))$ the linear map $\flat(X) = X^\flat := X \rfloor \tilde{\Omega}_l$, the submanifold M_1 may be easily characterized as the subset

$$M_1 = \{z \in M_0 \mid d\varphi_l(z) \in (TM_0 \cap \hat{\chi}(\mathcal{A})^0)^\flat + \hat{\chi}(\mathcal{A})\}. \tag{3.12}$$

Of course, in order for an algebraic solution X to be dynamically significant, it is necessary that it be tangent to the submanifold M_1 itself.

In general, this requirement will be satisfy only on a subregion M_2 of M_1, identified with the totality of points of M_1 at which the problem (3.2), (3.9) admits at least a solution X belonging to TM_1.

Once again, by assuming that M_2 has a submanifold structure, we may express it as

$$M_2 = \{z \in M_1 \mid d\varphi_l(z) \in (TM_1 \cap \hat{\chi}(\mathcal{A})^0)^\flat + \hat{\chi}(\mathcal{A})\}. \tag{3.13}$$

As above, we must now require that at least one solution X on M_2 be tangent to

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([5]) This algorithm, in its turn, extended to the time-dependent Lagrangians the procedure developed by Gotay and co-workers in [**2**], [**3**].

M_2. So, by iterating the process, we obtain a sequence of «constraint manifolds» ([6])

$$\mathcal{L}(\mathcal{A}) \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$$

where, for $\forall k \geqslant 0$ we have

$$M_{k+1} = \{z \in M_k \,|\, d\varphi_l(z) \in (TM_k \cap \widehat{\chi}(\mathcal{A})^0)^\flat + \widehat{\chi}(\mathcal{A})\}\,. \tag{3.14}$$

The algorithm is said to stabilize if there exists an integer $\bar{k} > 0$ such that $M_{\bar{k}+1} = M_{\bar{k}}$ and $\dim M_{\overline{K}} > 0$. In this circumstance (and only in this) the problem (3.2), (3.9) admits at least one differential solution, namely a submanifold $M := M_{\bar{k}}$ carrying a solution X tangent to M itself. Following the usual terminology [**1**], [**2**], [**3**], [**4**], [**5**], [**12**], [**18**], we shall call M the *final constraint manifold*. Also, it is easily seen that M (when existing) is *maximal*, i.e. if N is any other submanifold of M_0 along which eqs. (3.3), (3.9) possess a differential solution, then N is contained in M.

As a concluding remark, we notice that a useful characterization of the constraint submanifolds M_k may be obtained by observing that $(TM_k \cap \widehat{\chi}(\mathcal{A})^0)^\flat + \widehat{\chi}(\mathcal{A}) = [(TM_k \cap \widehat{\chi}(\mathcal{A})^0)^\perp \cap \widehat{\chi}(\mathcal{A})^0]^0$. This allows to express each constraint submanifold as ([7])

$$M_{k+1} = \{z \in M_k \,|\, \langle (TM_k \cap \widehat{\chi}(\mathcal{A})^0)^\perp \cap \widehat{\chi}(\mathcal{A})^0,\, d\varphi_l\rangle(z) = 0\}, \qquad k \geqslant 0\,. \tag{3.15}$$

It follows that, whenever the algorithm (3.15) terminates with a final constraint manifold M, the relation

$$\langle (TM \cap \widehat{\chi}(\mathcal{A})^0)^\perp \cap \widehat{\chi}(\mathcal{A})^0,\, d\varphi_l\rangle(z) = 0 \tag{3.16}$$

holds identically $\forall z \in M$.

3.4 - *The second-order differential equation problem*

As it is well known, finding a «final constraint manifold» M along which eqs. (3.3), (3.9) admit solutions is not yet enough. Indeed, in principle, such sol-

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([6]) Here it is systematically assumed that every subset $M_k \subset M_0$ arising in the course of the constraint procedure is a submanifold.

([7]) At each $k \geqslant 0$, we may suppose that the subspaces $(TM_k \cap \widehat{\chi}(\mathcal{A})^0)^\perp \cap \widehat{\chi}(\mathcal{A})^0$ are not trivial. Indeed, if $(TM_0 \cap \widehat{\chi}(\mathcal{A})^0)^\perp \cap \widehat{\chi}(\mathcal{A})^0 = \{0\}$ then $(TM_0 \cap \widehat{\chi}(\mathcal{A})^0)^\flat + \widehat{\chi}(\mathcal{A}) = T^*(\mathcal{L}(\mathcal{V}_{n+1}))_{|M_0}$, so that the constraint algorithm does not need to start. Moreover, we have obviously $(TM_0 \cap \widehat{\chi}(\mathcal{A})^0)^\perp \cap \widehat{\chi}(\mathcal{A})^0 \subset (TM_k \cap \widehat{\chi}(\mathcal{A})^0)^\perp \cap \widehat{\chi}(\mathcal{A})^0$ $\forall k \geqslant 1$.

utions may have no dynamical meaning at all, i.e. may fail to be dynamical flows (or SODEs).

Accordingly, we shall now deal with the problem of finding out *kinematically admissible* solutions of eqs. (3.3), (3.9). The topic is known in the literature as the second-order differential equation problem.

To start with, let Ω_l be the Poincaré-Cartan 2-form on $j_1(\mathcal{V}_{n+1})$ induced by the (degenerate) lagrangian section l, and let D^v be the vertical lift to $j_1(\mathcal{V}_{n+1})$ of the «constraint» co-distribution D over $\mathcal{V}_{n+1}$.

At each $z \in j_1(\mathcal{V}_{n+1})$, we consider the intersection $\mathcal{D}_{lz} := \ker \Omega_{lz} \cap V_z(D^v)$ with $V_z(D^v) \subset V_z(j_1(\mathcal{V}_{n+1}))$ denoting the annihilator of D^v_z with respect to the pairing (2.5).

By supposing that the subspaces $\mathcal{D}_{lz}$ are not trivial and have constant rank at each $z \in j_1(\mathcal{V}_{n+1})$ (8), it is a straightforward matter — left to the reader — to verify that $\mathcal{D}_l := \cup_{z \in j_1(\mathcal{V}_{n+1})} \mathcal{D}_{lz}$ is an involutive distribution.

In local coordinates, expressing Ω_l as

$$\Omega_l = \frac{\partial L}{\partial q^i}\, dq^i \wedge dt + \frac{\partial^2 L}{\partial t \partial \dot{q}^i}\, dt \wedge \omega^i + \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i}\, dq^j \wedge \omega^i + \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i}\, d\dot{q}^j \wedge \omega^i$$

and adopting a local basis $\{g^\sigma = g^\sigma_i(t, q)\, dq^i + g^\sigma_0(t, q)\, dt,\ \sigma = 1, \dots, n-r\}$ for D^v, it is easily seen that every vector V belonging to $\mathcal{D}_l$ is necessarily of the form

$$V = V^i \frac{\partial}{\partial \dot{q}^i} \tag{3.17a}$$

with the components V^i subject to the restrictions

$$V^i \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = 0, \quad V^i g^\sigma_i = 0 \quad j = 1, \dots, n, \quad \sigma = 1, \dots, n-r. \tag{3.17b}$$

We have now enough geometrical tools to introduce the following

Definition 3.2. *A lagrangian section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ is called admissible with respect to the constraints if and only if the leaf space $\mathfrak{I}$:*

(8) In what follows, we restrict our analysis to the only systems obeying such requirements.

$= j_1(\mathcal{V}_{n+1})/\mathcal{D}_l$ *of the foliation generated by* $\mathcal{D}_l$ *admits a manifold structure such that the canonical projection* $\varrho : j_1(\mathcal{V}_{n+1}) \to \mathfrak{J}$ *is a submersion.*

It is worth noticing that, by construction, the distribution $\mathcal{D}_l$ is automatically tangent to the constraint submanifold $\mathcal{A}$, thus foliating it. It is then an easy matter to verify that, whenever the lagrangian section l is admissible, the quotient space $\mathfrak{A} := \mathcal{A}/\mathcal{D}_{l|\mathcal{A}}$ — identified with the image of $\mathcal{A}$ under the map $\varrho_{\mathcal{A}} := \varrho \circ i$ — inherits an embedded submanifold structure from $\mathfrak{J}$, such that $\varrho_{\mathcal{A}} := \varrho \circ i$ is a submersion.

Moreover, since $\mathcal{D}_l$ is vertical, both manifolds $\mathfrak{J}$ and $\mathfrak{A}$ are fibered over $\mathcal{V}_{n+1}$, thus giving rise to the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{i} & \mathfrak{J} \\
\Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi} \\
\mathcal{V}_{n+1} & = & \mathcal{V}_{n+1}\,.
\end{array}
\tag{3.18}
$$

i denoting the stated embedding.

Next, we focus attention on the fact that the assignment of the lagrangian section l allows to foliate $\mathcal{L}(\mathcal{V}_{n+1})$ in terms of the 1-parameter family of leaves

$$\Sigma_\xi := \{z \in \mathcal{L}(\mathcal{V}_{n+1}) \mid \dot{u}(z) = L(\pi(z)) + \xi\},\ \ \xi \in \mathfrak{R}\,. \tag{3.19}$$

Every such leaf is clearly the image of $j_1(\mathcal{V}_{n+1})$ under the section $l_\xi : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ described locally by $\dot{u} = L(t, q^i, \dot{q}^i) + \xi$. In a similar way, the images

$$M_\xi := \{z \in \mathcal{L}(\mathcal{A}) \mid \dot{u}(z) = L(\pi(z)) + \xi\}, \qquad \xi \in \mathfrak{R} \tag{3.20}$$

of $\mathcal{A}$ under the sections l_ξ give rise to a foliation of the submanifold $\mathcal{L}(\mathcal{A})$.

Now, making use of the horizontal lift (2.17) induced by the section l (or, what is the same, by any other section l_ξ), we lift the distribution $\mathcal{D}_l$ over $j_1(\mathcal{V}_{n+1})$ to an involutive distribution $\widetilde{\mathcal{D}}_l$ over $\mathcal{L}(\mathcal{V}_{n+1})$. Equivalently, we may define $\widetilde{\mathcal{D}}_{lz} := l_{\xi *}(\mathcal{D}_{l|\pi(z)})$ at each $z \in \Sigma_\xi$ and then put $\widetilde{\mathcal{D}}_l := \cup_{z \in \mathcal{L}(\mathcal{V}_{n+1})} \widetilde{\mathcal{D}}_{lz}$.

Directly from eqs. (2.17), (3.17), we derive that every vector $\widetilde{V} \in \widetilde{\mathcal{D}}_l$ is expressed locally as

$$\widetilde{V} = \widetilde{V}^i \left(\frac{\partial L}{\partial \dot{q}^i} \frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{q}^i} \right) \tag{3.21}$$

with the components $\widetilde{V}^i$ obeying the requirements (3.17b).

Moreover, by construction, the distribution $\widetilde{\mathcal{D}}_l$ is tangent to every surface (3.19) and (3.20), so that the leaves of the foliation generated by $\widetilde{\mathcal{D}}_l$ lie on the surfaces Σ_ξ and M_ξ itselves.

We then come to the conclusion that, whenever the lagrangian section l is admissible, both quotient spaces $\mathfrak{L} := \mathcal{L}(\mathcal{V}_{n+1})/\widetilde{\mathcal{D}}_l$ and $\mathfrak{L}(\mathfrak{A}) := \mathcal{L}(\mathcal{C})/\widetilde{\mathcal{D}}_{l|\mathcal{L}(\mathcal{C})}$ have a manifold structure and that the canonical projections $\xi : \mathcal{L}(\mathcal{V}_{n+1}) \to \mathfrak{L}$ and $\xi_{\mathfrak{L}(\mathfrak{A})} : \mathcal{L}(\mathcal{C}) \to \mathfrak{L}(\mathfrak{A})$ are submersions.

Also, we have that $\mathfrak{L}(\mathfrak{A})$ is naturally embedded into $\mathfrak{L}$ and that both manifolds $\mathfrak{L}$ and $\mathfrak{L}(\mathfrak{A})$ are principal fiber bundles respectively over $\mathfrak{I}$ and $\mathfrak{A}$, with structural group isomorphic to $(\mathfrak{R}, +)$.

The situation is summarized into the commutative diagram

(3.22)
$$
\begin{tikzcd}
 & \mathcal{L}(\mathcal{V}_{n+1}) \arrow[rr, "\pi"] \arrow[dd, "\xi"] & & j_1(\mathcal{V}_{n+1}) \arrow[dd, "\varrho"] \\
\mathcal{L}(\mathcal{C}) \arrow[ur, "i"] \arrow[rr, "\pi"] \arrow[dd, "\xi_{\mathfrak{L}(\mathfrak{A})}"'] & & \mathcal{C} \arrow[ur, "i"] \arrow[dd, "\varrho_{\mathcal{C}}"] & \\
 & \mathfrak{L} \arrow[rr, "\pi"] & & \mathfrak{I} \\
\mathfrak{L}(\mathfrak{A}) \arrow[ur, "i"] \arrow[rr, "\pi"] & & \mathfrak{A} \arrow[ur, "i"] &
\end{tikzcd}
$$

where, for simplicity, we have denoted by i the various embeddings, and by π the principal fibrations arising from the previous discussion.

Another important fact is that $\widetilde{\mathcal{D}}_l \subset \ker \widetilde{\Omega}_l$, $\widetilde{\Omega}_l$ denoting the presymplectic 2-form induced by the lagrangian section l through the algorithm explained in § 2.2. This point is easily seen by a direct check, taking the representation

$$\widetilde{\Omega}_l = d\dot{u} \wedge dt + \frac{\partial^2 L}{\partial t \partial \dot{q}^i} dt \wedge \omega^i + \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} dq^j \wedge \omega^i + \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i} d\dot{q}^j \wedge \omega^i - \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i \wedge dt$$

as well as the characterization (3.21) explicitly into account. We have then $\widetilde{\mathcal{D}}_l \rfloor \widetilde{\Omega}_l = 0$. Again by eq. (3.21), it is also immediately verified that $\langle \widetilde{\mathcal{D}}_l, d\varphi_l \rangle = 0$, $\varphi_l = \dot{u} - L$ indicating the trivialization of $\mathcal{L}(\mathcal{V}_{n+1})$ associated with l. Accordingly, we may conclude that there exist a presymplectic 2-form $\overline{\Omega}_l$ over $\mathfrak{L}$ and a trivialization $\overline{\varphi}_l$ of the principal fiber bundle $\pi : \mathfrak{L} \to \mathfrak{I}$, such that $\widetilde{\Omega}_l = \xi^*(\overline{\Omega}_l)$ and $\varphi_l = \xi^*(\overline{\varphi}_l)$.

In addition to this, since the manifold $\mathfrak{L}$ is fibered over $\mathcal{V}_{n+1}$ — with projection $\tau : \mathfrak{L} \to \mathcal{V}_{n+1}$ — we may pull-back on $\mathfrak{L}$ the co-distribution D over $\mathcal{V}_{n+1}$. We denote by $\overline{D}^v := \tau^*(D)$ the resulting co-distribution over $\mathfrak{L}$.

At this point, collecting all stated results, we may set up a reduced non-holonomic problem of motion on the quotient space $\mathfrak{L}$, consisting in the search for vector fields $\overline{Z}$ satisfying the requirements

$$\tag{3.23} \begin{cases} (\overline{Z} \dashv \overline{\Omega}_l + d\overline{\varphi}_l)_{|\mathfrak{L}(\mathfrak{A})} \in \overline{D}^v_{|\mathfrak{L}(\mathfrak{A})} \\ \langle \overline{Z}, \overline{D}^v_{|\mathfrak{L}(\mathfrak{A})} \rangle = 0 \\ \overline{Z} \in D^1(\mathfrak{L}(\mathfrak{A})). \end{cases}$$

The solvability of the problem (3.23) may be analysed through a presymplectic constraint algorithm identical to the one outlined in § 3.3.

In this connection, it is possible to show that the constraint procedures or, more generally, the constrained dynamics respectively on $\mathcal{L}(\mathcal{V}_{n+1})$ and $\mathfrak{L}$ are intimately related to each other.

To see this point, let us return to the surface $M_0 \subset \mathcal{L}(\mathcal{A})$, and consider the final constraint submanifold $M \subset M_0$ generated by the constraint algorithm. We have then

Proposition 3.6. *For every constraint submanifold M_k, the restriction $\widetilde{\mathcal{O}}_{l|M_k}$ is an involutive distribution in TM_k, foliating M_k. The corresponding leaf spaces $\mathfrak{M}_k := M_k / \widetilde{\mathcal{O}}_{l|M_k}$ are submanifolds embedded in $\mathfrak{L}(\mathfrak{A})$ and the induced projections $\xi_{M_k}: M_k \to \mathfrak{M}_k$ are submersions.*

Proof. The proof is given by induction on the constraint submanifolds M_k and is essentially identical to that of Proposition 3.5 stated in [**12**] in the case of free (unconstrained) Lagrangians (see also Proposition 1 in [**3**] for free time-independent systems). Referring the reader to [**12**] for a detailed account, here we notice that the only differences with respect to the unconstrained case are:

- the different characterization of the constraint submanifold M_{k+1} expressed, in view of eq. (3.15), by the vanishing of functions of the form $\phi = \langle Z, d\varphi_l \rangle$, with $Z \in (TM_k \cap \widehat{\chi}(\mathcal{A})^0)^\perp \cap \widehat{\chi}(\mathcal{A})^0$;
- the straightforward fact that, given two arbitrary vector fields $Y \in \widetilde{\mathcal{O}}_l$ and $W \in \widehat{\chi}(\mathcal{A})^0$, one has $[Y, W] \in \widehat{\chi}(\mathcal{A})^0$. ■

As a direct consequence of the previous Proposition, we may state the following Equivalence Theorem

Theorem 3.2.

i) *The presymplectic algorithm terminates with a final constraint submanifold M in $\mathcal{L}(\mathcal{A})$ if and only if the corresponding «reduced» algorithm terminates* *with a final constraint submanifold* $\overline{M}$ *in* $\mathfrak{L}(\mathfrak{A})$. *Moreover, the submanifold* $\overline{M}$ *is diffeomorphic to the leaf space* $\mathfrak{M} := M/\widetilde{\mathcal{O}}_{l|M}$.

ii) *The problem of motion* (3.2), (3.9) *is equivalent to the reduced one* (3.23), *in the sense that:*

1) *for every solution* X *of eqs.* (3.3) *and* (3.9), *if* $\xi_*(X)$ *exists, then it satisfies the reduced problem* (3.23);

2) *if* $\overline{X}$ *satisfies the relations* (3.23), *then every* X ξ*-related to* $\overline{X}$ *solves eqs.* (3.3), (3.9).

Proof. Taking Proposition 3.6 into account and using that $\widetilde{\Omega}_l = \xi^*(\overline{\Omega}_l)$, $\varphi_l = \xi^*(\overline{\varphi}_l)$ and ξ is a submersion, one may easily prove (9) that the every restriction ξ_{M_k} of the submersion ξ to the constraint submanifold M_k is a submersion from M_k onto $\overline{M}_k$ ($\overline{M}_k$ denoting the corresponding constraint submanifold for the reduced problem (3.23)) and that the fibers of ξ by the points of $\overline{M}_k$ are entirely contained in M_k. Every quotient space $\mathfrak{M}_k$ is then diffeomorphic to the corresponding submanifold $\overline{M}_k$. From this all assertions follow (10). ■

It is worth observing that Theorem 3.2 ensures the existence of solutions of (3.2), (3.9) projecting to $\mathfrak{L}(\mathfrak{A})$. Indeed, given a solution $\overline{X}$ of (3.23), any vector field X ξ-related to $\overline{X}$ solves the problem (3.2), (3.9) and projects to $\mathfrak{L}(\mathfrak{A})$.

By analogy with the terminology adopted in [**3**], [**12**], every such solution will be called *prolongable*, while a solution prolongable modulo $V(\mathcal{L}(\mathcal{A}))$ will be called *semi-prolongable*.

As a further step in the discussion of the SODE problem, we have

Proposition 3.7. *Let* $X \in D^1(M)$ *be a vector field solution of the problem* (3.2), (3.9). *Then* $\widetilde{J}(X) \in \widetilde{\mathcal{O}}_{l|M}$

Proof. On one hand, by a direct calculation, it is easily seen that every solution Z of (3.2), (3.9) automatically satisfies the conditions $\langle X, \omega^i \rangle \dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = 0$ and $g_i^\sigma \langle X, \omega^i \rangle = 0$ $\forall j = 1, \ldots, n$, $\sigma = 1, \ldots, n-r$. On the other hand, recalling

(9) The reader may adapt the arguments stated in [**5**] to the present geometrical context.

(10) In 1) of ii), one has to verify that the Chetaev 1-form in the right-hand-side of the first equation (3.3) is necessarily the pull-back of a corresponding 1-form on $\mathfrak{L}(\mathfrak{A})$. However, also this point is nothing but a straightforward check.

eq. (2.18), one has $\tilde{J}(X) = \langle X, \omega^i \rangle \left(\dfrac{\partial L}{\partial \dot{q}^i} \dfrac{\partial}{\partial \dot{u}} + \dfrac{\partial}{\partial \dot{q}^i} \right)$. The result then follows from eqs. (3.17b), (3.21). ■

Proposition 3.8. *Let l be an admissible lagrangian section and $X \in D^1(M)$ a semi-prolongable solution of eqs.* (3.3), (3.9). *Then there exists a unique point in each leaf of the foliation of M generated by $\widetilde{\mathcal{D}}_{l|M}$ at which X is* (π*-related to*) *a SODE.*

Proof. Once again, the proof is essentially identical to the one concerning the unconstrained case [3], [**12**]. The basic idea is to consider the integral curves of the vertical vector field $-\tilde{J}(X) = -\langle X, \omega^i \rangle \left(\dfrac{\partial L}{\partial \dot{q}^i} \dfrac{\partial}{\partial \dot{u}} + \dfrac{\partial}{\partial \dot{q}^i} \right)$. Due to Proposition 3.7, these are vertical trajectories $\gamma(\sigma) : (t, q^i, \dot{q}^i(\sigma), \dot{u}(\sigma))$ contained in the leaves of the foliation of M generated by the distribution $\widetilde{\mathcal{D}}_{l|M}$. Given then a point $m \in M$, let $\mathfrak{L}(\mathfrak{A})_m \subset M$ denote the leaf through m and $\gamma_m(\sigma)$ the integral curve of $-\tilde{J}(X)$ starting at m for $\sigma = 0$. Since $\mathfrak{L}(\mathfrak{A})_m$ is closed and $\gamma_m(\sigma) \in \mathfrak{L}(\mathfrak{A})_m$ $\forall \sigma$, as $\sigma \to -\infty$ the limit point n_X of $\gamma_m(\sigma)$ lies on $\mathfrak{L}(\mathfrak{A})_m$. Finally, it is a straightforward matter to see that n_X is independent of the choice of $m \in \mathfrak{L}(\mathfrak{A})_m$ and that it is the unique point in the leaf $\mathfrak{L}(\mathfrak{A})_m$ at which X is (π-related to) a SODE, namely at which the relation $\tilde{J}(X)(n_X) = 0$ holds. For further comments and details the reader is referred to [**12**]. ■

On the basis of the Proposition 3.8, we may define an injection $\alpha_X \colon \mathfrak{M} \to M$ associating with every leaf $\mathfrak{m} \in \mathfrak{M}$ the corresponding point $n_X(\mathfrak{m}) \in M$. The image of $\mathfrak{M}$ under α_X is clearly the union S_X of all the points $n_X(\mathfrak{m})$, one for each leaf. The injection α_X is a global section of the fibration $\xi_M \colon M \to \mathfrak{M}$, so that S_X is an embedded submanifold of M, diffeomorphic to $\mathfrak{M}$. We have thus found a submanifold $S_X \subset M$ on which the vector field X is a SODE. Note that, by construction, if X and Y are two semi-prolongable solutions such that $\tilde{J}(X) = \tilde{J}(Y)$, then $S_X = S_Y$. In this case, X and Y will be said to be $\tilde{J}$-equivalent.

A final difficulty now arises from the fact that, in general, $X_{|S_X}$ may fail to be tangent to S_X. To account for this problem, we observe that, by construction, one has the direct sum decomposition $T_{S_X} M = TS_X \oplus \widetilde{\mathcal{D}}_{l|S_X}$. Accordingly, we may express uniquely $X_{|S_X}$ in the form $X_{|S_X} = Z + V$ with $Z \in TS_X$ and $V \in \widetilde{\mathcal{D}}_{l|S_X}$. Since $V \in \widetilde{\mathcal{D}}_l$, it is then clear that Z too is a SODE (tangent to S_X), satisfying both conditions (3.2), (3.9).

Collecting all previous results, we conclude by stating the following

Theorem 3.3. *Let $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ be an admissible* (*degenerate*) *lagrangian section with final constraint submanifold M embedded in $M_0 \subset \mathcal{L}(\mathcal{A})$.*

Then there exists at least one submanifold S of M along which the problem (3.2), (3.9) *admits at least one kinematically admissible solution Z.*

As a concluding remark, it is worth underlining an important difference between free and constrained Lagrangian systems. Indeed, in the previous discussion, nothing ensures the uniqueness of the solution on the submanifold S, which, on the contrary, is automatically guaranteed in the unconstrained case [**3**], [**12**]. This is due to the fact that the vectors belonging to the distribution $\widetilde{\mathcal{D}}_l$ do not represent all the gauge freedom of the theory. The latter is in fact related to the totality of vectors $V \in V(\mathcal{L}(\mathcal{A}))$ satisfying $V \dashv \widetilde{\Omega}_l \in \hat{\chi}(\mathcal{A})$, $V(\mathcal{L}(\mathcal{A}))$ denoting the vertical bundle associated with the fibration $\mathcal{L}(\mathcal{A}) \to \mathcal{V}_{n+1}$.

3.5 - *Example*

Consider a 5-dimensional configuration space-time $\mathcal{V}_{4+1}$, referred to (global) coordinates t, x, y, z, w. These, together with any trivialization $u : P \to \Re$ of the associated bundle of affine scalars $P \to \mathcal{V}_{4+1}$ (see [**9**], [**12**]), induce respectively on $j_1(\mathcal{V}_{4+1})$ and $\mathcal{L}(\mathcal{V}_{4+1})$ fibered coordinates of the form $t, x, y, z, w, \dot{x}, \dot{y}, \dot{z}, \dot{w}$ and $t, x, y, z, w, \dot{x}, \dot{y}, \dot{z}, \dot{w}, \dot{u}$.

In this context, let $l : j_1(\mathcal{V}_{4+1}) \to \mathcal{L}(\mathcal{V}_{4+1})$ denote the degenerate Lagrangian section expressed locally as

$$\dot{u} = L(t, x, y, z, w, \dot{x}, \dot{y}, \dot{z}, \dot{w}) := \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + w(\dot{x} + t) + z \, .$$

Recalling eq. (2.19), the presymplectic 2-form on $\mathcal{L}(\mathcal{V}_{4+1})$ associated with l has then local expression

$$\widetilde{\Omega}_l = d\dot{u} \wedge dt + d\dot{x} \wedge (dx - \dot{x}\,dt) + dw \wedge (dx - \dot{x}\,dt) + d\dot{y} \wedge (dy - \dot{y}\,dt) - (\dot{x} + w)\,d\dot{x} \wedge dt - \dot{y}\,d\dot{y} \wedge dt$$

while the differential of the trivialization $\varphi_l = \dot{u} - L$ of $\mathcal{L}(\mathcal{V}_{4+1})$ induced by l is described by

$$d\varphi_l = d\dot{u} - \dot{x}\,d\dot{x} - \dot{y}\,d\dot{y} - (\dot{x} + t)\,dw - w\,d\dot{x} - w\,dt - dz \, .$$

Let now subject the system to the (ideal) kinetic constraint

$$\dot{z} - w = 0 \, . \tag{3.24}$$

The latter gives rise to two corresponding submanifolds $\mathcal{C}$ and $\mathcal{L}(\mathcal{C})$, embedded respectively into $j_1(\mathcal{V}_{4+1})$ and $\mathcal{L}(\mathcal{V}_{4+1})$, and represented in cartesian form by the same eq. (3.24). Furthermore, it is easily seen that (the pull-back to $\mathcal{L}(\mathcal{C})$ of) the associated (1-dimensional) Chetaev bundle $\widehat{\chi}(\mathcal{C})$ is spanned locally by the 1-form $(dz - \dot{z}\,dt)_{|\mathcal{L}(\mathcal{C})}$.

For such a system, taking explicitly the requirement (3.9) into account, the problem of motion formulated on the Lagrangian bundle $\mathcal{L}(\mathcal{V}_{4+1})$ relies on the system

$$\begin{cases} (\widehat{Z} \rfloor \widetilde{\Omega}_l + d\varphi_l)_{|\mathcal{L}(\mathcal{C})} = +\lambda(dz - \dot{z}\,dt)_{|\mathcal{L}(\mathcal{C})} \\ \langle \widehat{Z}, (dz - \dot{z}\,dt)\rangle_{|\mathcal{L}(\mathcal{C})} = 0 \\ \widehat{Z}(\dot{z} - w)_{|\mathcal{L}(\mathcal{C})} = 0 \end{cases} \tag{3.25}$$

with unknown

$$\widehat{Z} = Z^0 \frac{\partial}{\partial t} + Z^x \frac{\partial}{\partial x} + Z^y \frac{\partial}{\partial y} + Z^z \frac{\partial}{\partial z} + Z^w \frac{\partial}{\partial w} + \dot{Z}^x \frac{\partial}{\partial \dot{x}} + \dot{Z}^y \frac{\partial}{\partial \dot{y}} + \dot{Z}^z \frac{\partial}{\partial \dot{z}} + \dot{Z}^w \frac{\partial}{\partial \dot{w}} + \dot{Z}^u \frac{\partial}{\partial \dot{u}}\,.$$

As pointed out in § 3.3, in order to solve the system (3.25), it is sufficient to focus attention on the submanifold

$$M_0 := \{z \in \mathcal{L}(\mathcal{C}) \mid \dot{u}(z) = L(\pi(z))\}$$

image of $\mathcal{C}$ under the section l. Accordingly, it is a straightforward matter to see that the problem (3.25) admits solutions only on the submanifold $M_1 \subset M_0$ expressed in coordinates as

$$M_1 := \{z \in M_0 \mid \dot{x}(z) + t(z) = 0\}\,.$$

More precisely, eqs. (3.25) possess on M_1 a family of (algebraic) solutions consisting of the totality of vector fields of the form

$$\widehat{Z} = Z_{|M_1} + Z(L)\,\frac{\partial}{\partial \dot{u}_{|M_1}} \tag{3.26a}$$

with

$$Z = \frac{\partial}{\partial t} + \dot{x}\,\frac{\partial}{\partial x} + \dot{y}\,\frac{\partial}{\partial y} + \dot{z}\,\frac{\partial}{\partial z} + Z^w\,\frac{\partial}{\partial w} - Z^w\,\frac{\partial}{\partial \dot{x}} + Z^w\,\frac{\partial}{\partial \dot{z}} + \dot{Z}^w\,\frac{\partial}{\partial \dot{w}} \tag{3.26b}$$

Z^w and $\dot{Z}^w$ being arbitrary differentiable functions.

The question is now to check whether among the vector fields (3.26) there is at least one tangent to M_1. To this end, by imposing the tangency condition $\widehat{Z}(\dot{x} + t)_{|M_1} = 0$, it is easily seen that indeed the family (3.26) includes infinitely many differential solutions along M_1, all having expression (3.26a) with

$$Z = \frac{\partial}{\partial t} + \dot{x}\,\frac{\partial}{\partial x} + \dot{y}\,\frac{\partial}{\partial y} + \dot{z}\,\frac{\partial}{\partial z} + \frac{\partial}{\partial w} - \frac{\partial}{\partial \dot{x}} + \frac{\partial}{\partial \dot{z}} + \dot{Z}^w\,\frac{\partial}{\partial \dot{w}}\,. \tag{3.27}$$

This proves that the constraint algorithm stabilizes at the first step, with final constraint manifold $M := M_1$.

However, it is self-evident that none of the solutions (3.26a), (3.27) is kinematically admissible on the whole submanifold M_1.

Accordingly, following step by step the procedure explained in § 3.4, we shall now discuss the second-order differential equation problem associated with the system in study.

To start with, we let the reader verify that the given Lagrangian section l is effectively admissible, and that the related involutive distribution $\widetilde{\mathcal{D}}_l$ is locally spanned by the single vector field $\dfrac{\partial}{\partial \dot{w}}$, namely $\widetilde{\mathcal{D}}_l = \operatorname{Span}\left(\dfrac{\partial}{\partial \dot{w}}\right)$.

By Proposition 3.6, one has then that the restriction $\widetilde{\mathcal{D}}_{l|M}$ foliates the submanifold M itself. We may refer the corresponding leaf space $\mathfrak{M} := M/\widetilde{\mathcal{D}}_{l|M}$ to natural local coordinates t, x, y, z, w, $\dot{y}$. It follows that every solution $\widehat{Z}$ of the kind (3.27) is semi-prolongable.

In addition to this, recalling eq. (2.18), the tensor field $\tilde{J}$ on $\mathcal{L}(\mathcal{V}_{4+1})$ induced by l is expressed locally as

$$\tilde{J} = \left((\dot{x} + w)\,\frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{x}}\right) \otimes (dx - \dot{x}\,dt) + \left(\dot{y}\,\frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{y}}\right) \otimes (dy - \dot{y}\,dt)$$
$$+ \frac{\partial}{\partial \dot{z}} \otimes (dz - \dot{z}\,dt) + \frac{\partial}{\partial \dot{w}} \otimes (dw - \dot{w}\,dt)\,.$$

For every $\widehat{Z}$ of the form (3.26a), (3.27) we have then the relation

$$\tilde{J}(\widehat{Z}) = (1 - \dot{w}) \frac{\partial}{\partial \dot{w}}$$

showing that all these vector fields are $\tilde{J}$-equivalent.

Given any such $\widehat{Z}$ — by proceeding as in Proposition 3.8 and subsequent discussion — we may therefore define an injection $\alpha : \mathfrak{M} \to M$ described in local coordinates by $\mathfrak{m} = (t, x, y, z, w, \dot{y}) \in \mathfrak{M} \to \alpha(\mathfrak{m}) := n_{\widehat{Z}}(\mathfrak{m}) = (t, x, y, z, w,$ $\dot{x} = -t, \dot{y}, \dot{z} = w, \dot{w} = 1, \dot{u} = L(t, x, y, z, w, -t, \dot{y}, w, 1)) \in M$ (for more details, see [**12**]). The image $S := \alpha(\mathfrak{M}) \subset M$ of the injection α is then the (unique) submanifold of M on wich all solutions (3.26a), (3.27) are (π-related to) dynamical flows.

To sum up, due to the direct sum decomposition $T_S M = TS \oplus \widetilde{\mathcal{D}}_{l|S}$, every such solution $\widehat{Z}$ admits unique representation of the form $\widehat{Z}_{|S} = \overline{Z}_{|S} + V_{|S}$ with

$$\overline{Z}_{|S} = \left(\frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} + \frac{\partial}{\partial w} - \frac{\partial}{\partial \dot{x}} + \frac{\partial}{\partial \dot{z}} + (\dot{z} + t) \frac{\partial}{\partial \dot{u}} \right)_{|S} \in TS \tag{3.28}$$

and

$$V_{|S} = \left(\dot{Z}^w \frac{\partial}{\partial \dot{w}} \right)_{|S} \in \widetilde{\mathcal{D}}_{l|S}.$$

The vector field (3.28) is thus the unique (11) kinematically admissible solution of the problem (3.25) along the submanifold S.

(11) Note that the uniqueness is here due to the peculiarity of the system in study. In general, as already pointed out, the results stated in § 3.4 do not ensure the uniqueness of the solution.

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Abstract

A presymplectic (symplectic) setting for degenerate (regular) time-dependent Lagrangians subject to non-holonomic constraints is proposed. In the resulting geometrical framework a constraint algorithm for the singular case is developed and the associated second-order differential equation problem is solved for a wide class of systems. An explicit example is given.

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