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Differential simplicity and dimension of a commutative ring (**)

1 - Preliminaries

All the rings considered in this paper are commutative with identity and all the fields are of characteristic zero (unless if it is otherwise stated).

A local ring R is understood to be a Noetherian ring with a unique maximal ideal; if R is not Noetherian, following the terminology of [**4**], we call it a quasi-local ring.

For special facts on commutative algebra we refer freely to [**1**], [**4**], [**7**] and [**16**].

Let d be a derivation of a ring R, then an ideal I of R is called a d-ideal if $d(I) \subseteq I$, and R is called a d-simple ring if it has no non zero, proper d-ideals (in order to simplify our notation we shall write dI instead of $d(I)$).

Non commutative d-simple rings exist in abundance, e.g. every simple ring is d-simple for every derivation d of R, and that is why our interest is turned to commutative d-simple rings only.

One should note that a d-simple ring R contains the field $F = \{x \in R : dx = 0\}$, and therefore is either of characteristic zero, or of a prime number p.

It is well known that if a ring R of characteristic zero is d-simple, then R is an integral domain, while if R contains the rationals and has no non zero prime d-ideals, then R is a d-simple ring ([**5**]; Corollary 1.5.)

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2 - Some remarks on the dimension of a d-simple ring

Given a ring R no criterion is known to decide whether or not there exists a derivation d of R (at least one) such that R is d-simple, but it seems that there is a connection between the d-simplicity and the dimension of R.

To verify this, let us consider first d-simple rings of characteristic $p \neq 0$. In this case it is easy to prove the following theorem:

Theorem 2.1. *Let R be a d-simple ring of prime characteristic p, then the nilradical of R is the unique prime ideal of R.*

Proof. It is sufficient to show that the nilradical N of R is a maximal ideal. For this, let M be a maximal ideal of R and let I be the ideal generated by the set $\{x^p : x \in M\}$.

Since ch $R = p$, I is a proper d-ideal of R. But R is d-simple ring, therefore $I = (0)$.

Thus $M \subseteq N$ and so $M = N$.

The previous theorem tells us that a d-simple ring of prime characteristic is a 0-dimensional, quasi-local ring. Further, if R is a domain, then R is a field.

The following case is a non trivial example of a d-simple ring of prime characteristic:

Example 2.2. Let $k[x]$ be a polynomial ring over a field k of prime characteristic, say p. Then $I = (x^p)$ is obviously a $\frac{d}{dx}$-ideal of $k[x]$, therefore $\frac{d}{dx}$ induces a derivation d of the ring $R = k[x]/I$ by $d(f+I) = \frac{df}{dx} + I$, for all f in $k[x]$.

Let A be a non zero d-ideal of R, then $A' = \{f \in k[x] : f + I \in A\}$ is obviously a $\frac{d}{dx}$-ideal of $k[x]$ containing properly I.

Given $f = \sum_{i=0}^{n} a_i x^i$ in A', with $n \geqslant p$, we can write $f = f_1 + f_2$, where $f_1 = \sum_{i=0}^{p-1} a_i x^i$ and $f_2 = \sum_{i=p}^{n} a_i x^i$. But f_2 is in I and therefore in A' and so f_1 is also in A'.

Thus $\frac{d^{p-1} f_1}{dx} \neq 0$ is in $A' \cap k$ and therefore $A = R$, i.e. R is a d-simple ring.

In the rest of this paper we shall be dealing with d-simple rings of characteristic zero, which are therefore integral domains containing the rational numbers (since they contain the field F: see section 1).

Seidenberg ([**10**] and [**11**]) proved that, if a d-simple algebra over a field k is either finitely generated, or a complete local ring, or a localization of such rings, then it is regular.

Hart ([**2**]; Corollary of Theorem 1) extended this result to the wider class of G-rings. Hart [**3**] further proved that, if R is a regular local ring of finitely generated type over a field k, then the converse is also true, i.e. there exists a derivation d of R admitting no non trivial d-ideals. He gave also examples [**3**]; examples (i), (iii) and (iv)) to show that the hypotheses «regular», «local» and «finitely generated type» are not superflous (i.e. they must hold all together).

The above results show that for a wide class of rings the d-simplicity is connected with the regularity, a property which requires from a ring R to have a «special» kind of dimension (namely in this case every maximal ideal M of R can be generated by $\dim R_M$ elements, where R_M denotes the localization of R at M).

But we shall proceed further on examining the connection between the d-simplicity and the dimension of a ring. For this, notice first that, since a d-simple ring R is a domain, if R is 0-dimensional, then it is a field. Further we give the following result, which is a straightforward consequence of Zariski's lemma for derivations:

Theorem 2.3. *Let R be a complete local ring and let d be a derivation of R admitting no non trivial d-ideals. Then R is a* 1-*dimensional ring.*

Proof. Assume that R is not 1-dimensional and let M be its unique maximal ideal. Then, since $dM \not\subset M$, there exists x in M, such that dx is in R-M, i.e dx is a unit of R. Thus, according to Zariski's lemma for derivations ([**15**]; Lemma 4), R contains a subring S such that $R = S[\![x]\!]$ (ring of formal power series), with $dS = 0$.

But $\dim S[\![x]\!] = 1 + \dim S$, therefore S is not a field. Thus, since $dS = 0$, every proper ideal of R generated by elements of S is a d-ideal, fact which contradicts our hypothesis that R is a d-simple ring.

As a staightforward consequence of the previous theorem a formal power series ring $k[\![x_1, x_2, \ldots, x_n]\!]$ over a field k has no derivation d admitting no non trivial d-ideals for any integer $n \geqslant 2$ (obviously $k[\![x_1]\!]$ is $\dfrac{d}{dx_1}$-simple).

Further, by Theorem 2.3, the completion of a d-simple local ring R is not d-simple if $\dim R > 1$ (it is well known that a derivation of a local ring can be extended to the completion and that completion preserves dimension).

One should note that there exist 1-dimensional rings R which are not d-simple for any derivation d of R; e.g. see example of [**6**].

The next theorem gives a necessary and sufficient condition for a 1-dimensional finitely generated k-algebra R to be d-simple, where d is a given k-derivation of R.

Theorem 2.4. *Let R be a 1-dimensional finitely generated algebra over a field k, say $R = k[y_1, y_2, \ldots, y_n]$ and let d be a k-derivation of R. Then R is a d-simple ring if, and only if, $R = (dy_1, dy_2, \ldots, dy_n)$.*

Proof. Assume first that R is d-simple and let M be a maximal ideal of R such that dy_i is in M for each $i = 1, 2, \ldots, n$. Then, given f in M, $df = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} dy_i$ is also in M, therefore $dM \subseteq M$, which is absurd. Thus $R = (dy_1, dy_2, \ldots, dy_n)$.

Conversely assume that $R = (dy_1, dy_2, \ldots, dy_n)$, and let M be a non-zero prime, and therefore (since $\dim R = 1$) maximal ideal of R, such that $dM \subseteq M$. Then dy_i is in M for each $i = 1, 2, \ldots, n$ ([**14**]; Lemma 2.2), which is a contradiction. Therefore R has no non zero prime d-ideals, fact which shows that it is a d-simple ring (see section 1).

Remark. By the first part of the proof of Theorem 2.4, if $R = k[y_1, y_2, \ldots, y_n]$ is a d-simple finitely generated algebra for some k-derivation d of R, then $R = (dy_1, dy_2, \ldots, dy_n)$.

The converse is not true in general. For example let d be the k-derivation of the polynomial ring $R = k[x_1, x_2]$ defined by $dx_1 = 1$ and $dx_2 = ax_2$, with $0 \neq a$ in k; then, although $R = (dx_1, dx_2)$, (x_2) is a non zero proper d-ideal of R.

One should note also that, if k is an algebraicaly closed field, then Theorem 2.4 is a straightforward consequence of Hilbert's Nullstellensatz ([4]; Theorem 32).

Finally, if k is the field Q of the rational numbers, it is easy to check that every derivation of k is a k-derivation, and therefore Theorem 2.4 is true for all derivations of k.

The following example illustrates the previous theorem:

Example 2.5. Consider the ideal $I = (x_1^2 + x_2^2 - 1)$ of the polynomial ring $R = k[x_1, x_2]$ and let d be the k-derivation of R defined by $dx_1 = x_2$ and $dx_2 = -x_1$. Then d induces a derivation of the ring R/I, denoted also by d. But R/I has dimension 1 and $x_2 dx_1 - x_1 dx_2 = 1$, therefore, by Theorem 2.4, R/I is a d-simple ring.

3 - Examples of d-simple rings of dimension greater than 1

Our first example will be a generalization of an unpublished result due to G. Bergman, which states that the polynomial ring $R = k[x_1, x_2]$ is d-simple for some derivation d of R; but first we need the following lemma:

Lemma 3.1. *Let R be a d-simple ring. Extend d to a derivation of the polynomial ring $R[x]$ by $dx = ax + b$, with a, b in R and assume that the equation $b + dy = ya$ has no solution y in R. Then $R[x]$ is also a d-simple ring.*

Proof. Assume that $R[x]$ is not a d-simple ring and let I be a non zero proper d-ideal of $R[x]$. Then it is easy to check that the set A of all leading coefficients of the elements of I of minimal degree, say n, together with zero is a non zero d-ideal of R and therefore $A = R$.

Thus there exists an element f of I of the form $f = x^n + \sum_{i=0}^{n-1} a_i x^i$, with a_i in R for each i.

Then df-naf is also in I and has degree less than n, therefore $df = naf$ (1).

But $df = nx^{n-1}(ax + b) + da_{n-1}x^{n-1} + a_{n-1}(n-1)\,x^{n-2}(ax + b) + \ldots + da_0 = nax^n + [nb + da_{n-1} + (n-1)\ a\ a_{n-1}]\,x^{n-1} + \ldots + da_0$ and therefore on comparing the coefficients of x^{n-1} in relation (1) we get that $nb + da_{n-1} + (n-1)\ a\ a_{n-1} = n\ a\ a_{n-1}$ or $b + d\left(\frac{1}{n} a_{n-1}\right) = a\left(\frac{1}{n} a_{n-1}\right)$, with $y = \frac{1}{n} a_{n-1}$ in R, which is a contradiction.

We are ready now to prove:

Theorem 3.2. *Let $R_n = k[x_1, x_2, \ldots, x_n]$ be a polynomial ring in n indeterminates over a field k, and let d be the k-derivation of R_n defined by $dx_1 = 1$ and $dx_i = x_{i-1}x_i + 1$, for each $i = 1, 2, \ldots, n$. Then R_n is a d-simple ring.*

Proof. We shall apply induction on n. The polynomial ring R_1 is d-simple because it is a PID and d lowers the degree. Assume that R_{n-1} is d-simple, then we must show that $R_n = R_{n-1}[x_n]$ is also a d-simple ring.

To show this, according to the previous lemma, it is enough to show that the equation

$$1 + dy = yx_{n-1} \tag{1}$$

has no solution y in R_{n-1}.

In fact, if y is in k, then (1) gives that $1 = yx_{n-1}$, which is absurd. Also by (1) becomes evident that we can not have $y = y(x_1, x_2, \ldots, x_{n-2})$.

Finally if there exists $y = \sum_{i=0}^{k} g_i(x_1, x_2, \ldots, x_{n-2})\, x_{n-1}^i$ in R_{n-1} satisfying (1), then obviously the degree of dy with respect to x_{n-1} will be at most k, a fact which contradicts (1) again.

Given a derivation d and a pime ideal P of a ring R, d extends to a derivation of the local ring R_p (denoted also by d) by the usual rule of differentials for quotients. Further an algebra R over a field k is d-simple if, and only if, R is a domain and R_M is d-simple, for all maximal ideals M of R ([**9**]; Corollary 2.2). This result combined with Theorem 3.2 provides us examples of local d-simple rings of dimension greater than 1.

Using also results from [8], where conditions are obtained, which guarantee that certain rings are polynomial rings in one indeterminate over a field, we get some additional cases where one can recognize that a given ring has a derivation d admitting no non trivial d-ideals.

The following corollary gives an example of a d-simple ring with infinite dimension:

Corollary 3.3. *Let $R = k[x_1, x_2, \dots, x_n, \dots]$ be a polynomial ring in infinitely many indeterminates over k. Define a k-derivation d of R recursively by $dx_1 = 1$ and by $dx_i = 1 + x_i x_{i-1}$, for all $i \geqslant 2$. Then R is a d-simple ring.*

Proof. Set $R_n = k[x_1, x_2, \dots, x_n]$, $i = 1, 2, \dots,$ then $dR_n \subseteq R_n$ and, by the previous theorem, R_n is a d-simple ring.

Let I be a non zero proper d-ideal of R, then $I \cap R_n \neq 0$ for some $n < \infty$ and $d(I \cap R_n) \subseteq I \cap R_n$. But $I \cap R_n$ is obviously a proper ideal of R_n, therefore we have a contradiction to the d-simplicity of R_n.

Corollary 3.4. *Let R_n and d be as in Theorem 3.2 and let $K = k(x_1, x_2, \dots, x_{n-1})$ be the quotient field of R_{n-1}. Then d extends to a derivation of the polynomial ring $A = K[x_n]$ and A is a d-simple ring.*

Proof. It is easy to check that A is isomorphic to the ring of fractions $(R_n)_S$, where S is the set of all non zero polynomials of R_{n-1} and therefore d extends to a derivation of A by the usual rule of differentials for quotients. Further, if I is a non zero proper d-ideal of A, then $R_n \cap I$ is obviously a proper non zero d-ideal of R_n, which is a contradiction, since, by Theorem 3.2, R_n is a d-simple ring.

A similar argument to that applied in Lemma 3.1 gives the following result:

Theorem 3.5. Let $R = k[x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}]$ be a Laurent polynomial ring over a field k, and let d be the k-derivation of R defined by $dx_i = a_i x_i$, $a_i \in k$, $i = 1, 2, \dots, n$.

Then R is a d-simple ring if, and only if, the a_i's are linearly independent over the ring Z of integers.

Proof. Assume first that the a a_i's are linearly independent over Z, then we want to show that R is a d-simple ring. We shall apply induction on n.

The case n=0 is true, since k is a field.

Assume that $R' = k[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_{n-1}, x_{n-1}^{-1}]$ is a d-simple ring, then we must show that $R = R'[x_n, x_n^{-1}]$ is also a d-simple ring.

For this, let I be a non zero d-ideal of R, then $I' = I \cap R'[x_n]$ is obviously a non zero d-ideal of $R'[x_n]$. As in the proof of Lemma 3.1 I' contains a polynomial of the form $f = x_n'^t + \sum_{i=0}^{t-1} g_i x_n^i$, with g_i in R' for each I and $df = ta_n f$. Then, calculating df, we get $ta_n g_i = dg_i + ia_n g_i$ or $dg_i = (t-i)\, a_n g_i$ (1), for each i.

Thus $g_i R'$ is a d-ideal of R' and therefore is either $g_i = 0$ or g_i is a unit of R'.

If $g_i = 0$ for each i, then $f = x_n^t$ is in I' and therefore $I = R$, i.e. R is a d-simple ring.

Assume therefore that g_i is a unit of R' for some $i < t$, then we shall have $g_i = bx_1^{t_1} x_2^{t_2} \ldots x_{n-1}^{t_{n-1}}$ with b in k and t_j in Z for each $j = 1, 2, \ldots, n-1$.

Then $dg_i = bd(x_1^{t_1})\, x_2^{t_2} \ldots x_{n-1}^{t_{n-1}} + bx_1^{t_1} d(x_2^{t_2}) \ldots x_{n-1}^{t_{n-1}} + \ldots + x_1^{t_1} x_2^{t_2} \ldots d(x_{n-1}^{t_{n-1}})$ $= g_i \sum_{j=1}^{n-1} a_j t_j$ and therefore relation (1) gives that $(i-t)\, a_n + \sum_{j=1}^{n-1} a_j t_j = 0$.

Thus $i - t = t_1 = t_2 = \ldots = t_{n-1} = 0$, i.e. $i = t$, which is absurd.

Conversely assume that R is a d-simple ring and that the a_i's are linearly dependent over Z; then we can find $t_1, t_2, \ldots, t_n$ in Z, not all zero, such that $\sum_{i=1}^{n} a_i t_i = 0$. Consider the Laurent polynomial $f = x_1^{t_1} x_2^{t_2} \ldots x_n^{t_n} + 1$, which is obviously a non unit of R, then $df = (f-1) \sum_{i=1}^{n} a_i t_i = 0$. Therefore fR is a proper, non zero, d-ideal of R, which is a contradiction.

The following corollary is an immediate consequence of the previous theorem:

Corollary 3.6. *Let R and d be as in the previous theorem. Assume further that the dimension of k (as a vector space) over the field Q of rationals is less than n. Then R is not a d-simple ring.*

We shall close by giving a generalization of the definition of a d-simple ring. Let D be a set of derivations of a ring R, then an ideal I of R is called a D-ideal if $dI \subseteq I$ for each derivation d of D, and R is called a D-simple ring if it has no proper non zero D-ideals.

It becomes evident that, if R is d-simple for some d in D, then it is also a D-simple ring, but the converse is not true in general. For example, by Theorem 2.3, the formal power series ring $R = k[\![x_1, x_2]\!]$ over a field k is not $\frac{\partial}{\partial x_1}$-simple, but it is easy to check that R is a $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\}$-simple ring.

Finally notice that the D-simplicity of a ring R is connected with the simplicity of the skew polynomial ring $R[x, D]$ (cf. [**12**] and [**13**]).

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Abstract

In the present paper the differential simplicity of a commutative ring is studied with respect to its dimension. It is shown that a d-simple ring of prime characteristic is 0*-dimensional. In the case of characteristic zero a necessary and sufficient condition is given for the d-simplicity of a* 1*-dimensional finitely generated algebra over a field k and examples are presented of rings with dimension greater than* 1 *and even of infinite dimension (polynomial rings in finitely many and infinitely many indeterminates and Laurent polynomial rings).*

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