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General recombination-generation laws for charge transport equations (**)

1 - Introduction

There is a strict relation in representation of ion electro-diffusion and the phenomenological theory of the electron and hole transport in semiconductors. The basic equations [**20**] are essentially the same, the main difference being in the generation-recombination term which may have different analytical expressions [**13**], [**23**]. In this paper we present a unified treatment for a quite general class of generation-recombination laws. Moreover, since heat production plays an important role in many applications, we assume the diffusion coefficients to be given functions of the temperature and couple the system of diffusion with the heat flow equation. Let p and n denote the concentration of positive and negative charges respectively, φ the electric potential and $\boldsymbol{E} = -\nabla\varphi$ the corresponding electric field. The flux densities are given by the relations

$$\boldsymbol{J}_p = -B(\theta)\,\nabla p - \mu p \nabla\varphi \tag{1.1}$$

$$\boldsymbol{J}_n = -D(\theta)\,\nabla n + \nu n \nabla\varphi\ , \tag{1.2}$$

where $B(\theta)$ and $D(\theta)$ are the coefficients of diffusion, μ and ν the ionic mobilities, θ the temperature and

$$\boldsymbol{I} = \boldsymbol{J}_p - \boldsymbol{J}_n \tag{1.3}$$

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the total current. Throughout this paper ionic mobilities are supposed to be constant. This is inherent to the method and, as pointed out by the referee, excludes consideration of, e.g., Ga-As devices, see [3]. We shall study the system

$$p_t + \nabla \cdot \boldsymbol{J}_p = R(p, n) \tag{1.4}$$

$$n_t + \nabla \cdot \boldsymbol{J}_n = R(p, n) \tag{1.5}$$

$$-\varepsilon \Delta \varphi = p - n \tag{1.6}$$

$$\theta_t - \nabla \cdot (\kappa \nabla \theta) = -\boldsymbol{I} \cdot \nabla \varphi , \tag{1.7}$$

where κ, a positive constant, is the coefficient of thermal diffusion. Crucial in the model is the form of the generation-recombination law $R(p, n)$. The law, typical of the ionic conduction, for example in a salt

$$R(p, n) = N_0^2 - \alpha p n ,$$

is often inadequate in other situations. It is well-known that other mechanisms are more important, especially in semiconductor devices. A more realistic model is the Shockley-Read-Hall recombination-generation rate which reads

$$R = \frac{N_0^2 - \alpha p n}{r_1 + r_2 p + r_3 n} , \qquad r_i > 0 \qquad i = 1, 2, 3$$

and take into account the essentially quantistic nature of the processes involved. In this paper we make an attempt to present a unified treatment for a quite large class of recombination-generation law satisfying the geometric condition (D) of Section **2**, which permits to prove the existence of an invariant region for the concentrations and the unique solubility of the related initial-boundary value problem. Hypothesis (D) also implies the existence of at least one electro-neutral stationary solution $(\overline{p}, \overline{n}, \overline{\varphi}, \overline{\theta})$, $\overline{p} = \overline{n}$. In certain cases (see Section **3**), the time-dependent solution tends, as $t \to \infty$, to $(\overline{p}, \overline{n}, \overline{\varphi}, \overline{\theta})$. The heat production term in the right-hand side of the heat flow equation (1.7) is also a major modelization problem. The simplest form $W_1 = (\boldsymbol{J}_p - \boldsymbol{J}_n) \cdot \boldsymbol{E} = \boldsymbol{I} \cdot \boldsymbol{E}$ proposed in [8] is the same of the metallic conduction. However, in a semiconductor, where there are two types of carriers and where the individual carrier fluxes are not conserved, there is an additional production-consumption of heat through recombination-generation. To take into account this peculiar phenomenon, a different formulation has been proposed in [**1**] i.e.: $W_2 = \nabla \cdot (E_v \boldsymbol{J}_p - E_c - \boldsymbol{J}_n)$, where E_c and E_v are the conduction band edge energy and the valence band edge energy respectively. For non degenerate materials W_2 simplifies to $W_3 = (\boldsymbol{J}_p - \boldsymbol{J}_n)\cdot \boldsymbol{E} + R(p, n)(E_c - E_v)$, where the dependence on the recombination law becomes nicely apparent. On the other hand, even more general formulations for heat generation have been proposed, see e.g.: [**23**]. In Section **4** we consider as heat generation term a generic function $W(\theta, p, n, \nabla\varphi, \nabla p, \nabla n)$ with minimal assumption on W. Clearly in this way there is no hope of proving for the stationary problem a theorem of existence and uniqueness for arbitrarily large data. However, for small data this can be obtained using the implicit function theorem in Banach spaces.

Time dependent solutions for carrier transport equations have attracted great interest from the point of view of nonlinear PDE's. We quote the pioneering paper of M. S. Mock [**16**], the contribution of H. Gajewski and K. Gröger [**7**], and the works of T. Seidman and G. Troianiello [**21**], [**22**] where the temperature coupling is considered. Most of these works are concerned with special recombination-generation laws.

2 - Existence and uniqueness of a classical solution

Let Ω be an open and bounded subset of $\mathbb{R}^3$ with a regular boundary $\partial\Omega$. We shall use the Hölder space $C^{k,\gamma}(\overline{\Omega})$ and the Sobolev spaces $W^{k,p}(\Omega)$, $W_0^{k,p}(\Omega)$ and $L^q(0, T; W^{k,p}(\Omega))$. For definitions and properties we refer to [**1**]. The L^2-norm in Ω is denoted $\| \, \|$, moreover we define

$$Q = \Omega \times (0, T), \qquad \Gamma = \Omega \times \{0\} \cup \partial\Omega \times (0, T], \qquad S = \partial\Omega \times [0, T].$$

Let $u(x, t)$ be defined in $\overline{Q}$ and let $0 < \alpha < 1$. We set

$$\|u\|_0 = \sup_Q |u|, \qquad \|u\|_\alpha = \|u\|_0 + \sup_{P_1, P_2 \in Q} \frac{|u(P_1) - u(P_2)|}{d(P_1, P_2)^\alpha},$$

$$\|u\|_{1+\alpha} = \|u\|_\alpha + \sum_{i=1}^{3} \left\| \frac{\partial u}{\partial x_i} \right\|_\alpha, \qquad \|u\|_{2+\alpha} = \|u\|_{1+\alpha} + \sum_{i=1}^{3} \left| \frac{\partial u}{\partial x_i} \right|_{1+\alpha} + \left| \frac{\partial v}{\partial t} \right|_\alpha,$$

where $d(P_1, P_2)$ is the usual parabolic distance i.e. $d(P_1, P_2) = (|x' - x''|^2 + |t' - t''|)^{1/2}$, $P_1 = (x', t')$, $P_2 = (x'', t'')$ $x \in \Omega$. We say that $u \in \mathcal{C}^q(Q)$, $(q = 0, \alpha, 1+\alpha, 2+\alpha)$, if $\|u\|_q$ is finite (1). Crucial in proving existence will be the following a priori estimates proved by A. Friedman [**5**], [**6**].

(1) Note that $\mathcal{C}^q(Q) \neq C^q(Q)$.

Theorem 2.1. Let $u(x, t) \in \mathcal{C}^{2+\alpha}(\overline{Q})$ be a solution of the equation

$$\sum_{ij=1}^{3} a_{ij}(x, t)\, u_{x_i x_j} + \sum_{i=1}^{3} b_i(x, t)\, u_{x_i} + c(x, t)\, u - u_t = f(x, t) \tag{2.1}$$

in the cylinder Q and $u|_\Gamma = h$. Assume that there exists a function $\Psi(x, t)$ such that Ψ_{x_i}, $\Psi_{x_i x_j}$, Ψ_t are continuous in $\overline{Q}$ and which coincides with h on Γ. Let the domain $\Omega \in C^{2,\alpha}$ and the coefficients in (2.1) be continuous and satisfy the conditions

$$\sum_{i,j=1}^{3} a_{i_j}(x, t)\, \xi_i \xi_j \geq H_0 \sum_{i=1}^{3} \xi_i^2 \tag{2.2}$$

$$\sum_{i,j=1}^{3} \|a_{i_j}\|_\alpha + \sum_{j=1}^{3} \|b_i\|_0 + \|c\|_0 \leq H_1, \qquad \sum_{ij=1}^{3} \|a_{ij}\|_1^S \leq H_2 \tag{2.3}$$

where

$$\|a_{ij}\|_1^S = \sup_{P \in S} \|a_{ij}(P)\| + \sup_{P_1, P_2 \in S} \frac{|a_{ij}(P_1) - a_{ij}(P_2)|}{|x' - x''| + |t' - t''|}\,. \tag{2.4}$$

Then, for any positive $\sigma < 1$, there exists a constant k depending only on σ, H_0, H_1, H_2 and Q such that

$$\|u\|_{1+\delta} \leq k \|f\|_0 \|\Psi\|_2 . \tag{2.5}$$

We shall study the following initial-boundary value problem (EP) which is obtained inserting (1.1) and (1.2) into (1.4) and (1.5) and plugging (1.3) in (1.8)

$$p_t - \nabla \cdot (B(\theta)\, \nabla p) - \mu \nabla \cdot (p \nabla \varphi) = R(p, n) \tag{2.6}$$

$$n_t - \nabla \cdot (D(\theta)\, \nabla n) + \nu \nabla \cdot (n \nabla \varphi) = R(p, n) \tag{2.7}$$

$$-\varepsilon \Delta \varphi = p - n \tag{2.8}$$

$$\theta_t - \nabla \cdot (\kappa \nabla \theta) = (\mu p + \nu n)\, |\nabla \varphi|^2 + [B(\theta)\, \nabla p - D(\theta)\, \nabla n] \cdot \nabla \varphi \tag{2.9}$$

$$p(x, 0) = p_0(x), \qquad n(x, 0) = n_0(x), \qquad \theta(x, 0) = \theta_0(x), \qquad x \in \Omega \tag{2.10}$$

$$p = p_b, \qquad n = n_b, \qquad \theta = \theta_b \qquad \text{on} \qquad S = \partial\Omega \times [0, T] \tag{2.11}$$

$$\varphi = \varphi_b \qquad \text{on} \qquad S = \partial\Omega \times [0, T]. \tag{2.12}$$

Equations (2.6) and (2.7) can be rewritten, by (2.8), as:

$$p_t - B(\theta)\,\Delta p - [B'(\theta)\,\nabla\theta + \mu\nabla\varphi]\cdot\nabla p = -\frac{\mu}{\varepsilon}p(p-n) + R(p,n) \tag{2.13}$$

$$n_t - D(\theta)\,\Delta n + [\nu\nabla\varphi - D'(\theta)\,\nabla\theta]\cdot\nabla n = \frac{\nu}{\varepsilon}n(p-n) + R(p,n)\,. \tag{2.14}$$

We make on the data the following hypotheses.

(A) There exist functions $\Psi_k(x,t)$, $k=1,2,3$ of class $C^{2,\alpha}(\overline{Q})$ which coincide on Γ respectively with p_0, p_b, n_0, n_b and θ_0, θ_b. Moreover

$$p_0 \geq 0\,, \qquad p_b \geq 0\,, \qquad n_0 \geq 0\,, \qquad n_b \geq 0\,. \tag{2.15}$$

(B) There exists a function $\Lambda \in C^{2,\alpha}(\overline{Q})$ which reduces to φ_b on S.

(C) $B(\theta), D(\theta) \in C^1(\mathbb{R}^1)$ satisfy

$$B(\theta) \geq B_0 > 0\,, \qquad D(\theta) \geq D_0 > 0 \tag{2.16}$$

$$|B'(\theta)| \leq B_1, \qquad |D'(\theta)| \leq D_1. \tag{2.17}$$

(D) $R(p,n)$ is a function locally lipschitz in $\mathbb{R}^2$. Moreover, let $\boldsymbol{F} = (f, g)$ be the vector field

$$\begin{cases} f(p,n) = -\dfrac{\mu}{\varepsilon}p(p-n) + R(p,n) \\[2ex] g(p,n) = \dfrac{\nu}{\varepsilon}n(p-n) + R(p,n)\,. \end{cases}$$

Let us define $\Sigma_r = \{(p,n) \in \mathbb{R}^2;\ 0 < p < r,\ 0 < n < r\}$, $r > 0$ and denote $\partial\Sigma_r$ the boundary of Σ_r. We assume that there exists $\overline{r} > 0$ such that, for all $r \geq \overline{r}$, $\boldsymbol{F}|_{\partial\Sigma_r}$ does not vanish and points toward the interior of Σ_r.

Remark 2.1. If Hypothesis (D) holds, at least one stationary solution of electroneutrality, i.e. $p = n$, exists. In fact, we have $R(0,0) > 0$ and $R(\overline{r}, \overline{r}) < 0$. Let $\eta > 0$ be any solution of $R(p,p) = 0$. It is easy to verify that $(p, n, \varphi, \theta) = (\eta, \eta, \overline{\varphi}, \overline{\theta})$ is a stationary solution of (1.4)-(1.8) if $\Delta\overline{\varphi} = 0$ and $\overline{\theta}$ solves the equation

$$-\nabla\cdot(\kappa\nabla\theta) = (\mu\eta + \nu\eta)\,|\nabla\overline{\varphi}|^2.$$

An example of a dissociation-recombination law to which the theory applies is given by

$$R(p, n) = N_0^\beta - \alpha p^\beta n^\beta, \qquad \beta > 1, \qquad \alpha > 0 .$$

Define $\Gamma_1 = \{(p,n);\ n=0,\ 0<p<r\}$, $\Gamma_2 = \{(p,n);\ p=r,\ 0<n<r\}$, $\Gamma_3 = \{(p,n);$ $n = r,\ 0 < p < r\}$, $\Gamma_4 = \{(p, n);\ p = 0,\ 0 < n < r\}$ and let $\boldsymbol{n}$ be the exterior unit normal to $\partial\Sigma_r$.

Let $\omega(n) = \boldsymbol{F}\cdot\boldsymbol{n}\,|_{\Gamma_2} = -\frac{\mu}{\varepsilon} r(r-n) + N_0^2 - \alpha r^\beta n^\beta$. It is easily seen that

$$\omega(n) \leqslant -\frac{\mu}{\varepsilon} r^2 + \frac{\mu}{\varepsilon}\left(\frac{\mu}{\alpha\beta\varepsilon}\right)^{\frac{1}{\beta-1}} + N_0^2 - \alpha\left(\frac{\mu}{\alpha\beta\varepsilon}\right)^{\frac{\beta}{\beta-1}} .$$

Hence, if $\bar r$ is sufficiently large we have $\boldsymbol{F}\cdot\boldsymbol{n}\,|_{\Gamma_2} < 0$ when $0 < n < r$ and $r \geqslant \bar r$. Eventually increasing $\bar r$ we find, on the whole boundary of Σ_r, $\boldsymbol{F}\cdot\boldsymbol{n} < 0$ when $r \geqslant \bar r$, except for the corner's point on which condition (D) can be verified directly. Another recombination term which satisfies (D) is of the form

$$R(p, n) = \frac{N_0^2 - \alpha pn}{r(p, n)}$$

with a positive function $r(p,n)$. When $r(p,n) = (r_1 + r_2 p + r_3 n)$, $r_i > 0$, $i = 1,2,3$ we have the well-known [**23**] Shockley-Read-Hall law.

Theorem 2.2. Let (A)-(D) be satisfied. Then problem (EP) has one and only one classical solution.

Proof. We apply the Leray-Schauder principle, referring for a precise statement to the book [**6**], pag. 293. Let $\mathcal{B} = (\mathcal{C}^{1+\alpha}(\overline{Q}))^3$ and $w = (p, n, \theta) \in \mathcal{B}$. Define the map $\Phi : \mathcal{B} \times [0, 1] \to \mathcal{B}$,

$$\tilde w = \Phi(w, \lambda), \qquad \tilde w = (\tilde p, \tilde n, \tilde\theta), \qquad \lambda \in [0, 1], \tag{2.18}$$

via the following chains of linear problems:

$$-\varepsilon\Delta\varphi = p - n, \qquad \varphi = \varphi_b \quad \text{on} \quad \partial\Omega \tag{2.19}$$

$$\begin{aligned} &\tilde p_t - [(1-\lambda) + \lambda B(\theta)]\,\Delta\tilde p - \lambda[\mu\nabla\varphi + B'(\theta)\nabla\theta]\cdot\nabla\tilde p \\ &\qquad = \lambda\left[R(p, n) - \frac{\mu}{\varepsilon} p(p-n)\right] \end{aligned} \tag{2.20}$$

$$\tilde p(x, 0) = p_0(x), \qquad \tilde p = p_b \quad \text{on} \quad S, \tag{2.21}$$

$$\tilde{n}_t - [(1-\lambda) + \lambda D(\theta)]\,\Delta\tilde{n} + \lambda[\nu\nabla\varphi - D'(\theta)\,\nabla\theta]\cdot\nabla\tilde{n} = \lambda\left[R(p,\,n) + \frac{\nu}{\varepsilon}\,n(p-n)\right] \tag{2.22}$$

$$\tilde{n}(x,\,0) = n_0(x), \qquad \tilde{n} = n_b \quad \text{on} \quad S \tag{2.23}$$

$$\tilde{\theta}_t - \nabla\cdot(\kappa\nabla\tilde{\theta}) = \lambda[(\mu p + \nu n)\,|\nabla\varphi|^2 + (B(\theta)\,\nabla p - D(\theta)\,\nabla n)\cdot\nabla\varphi] \tag{2.24}$$

$$\tilde{\theta}(x,\,0) = \theta_0(x), \qquad \tilde{\theta} = \theta_b \quad \text{on} \quad S\,. \tag{2.25}$$

By elliptic regularization we have

$$\varphi(x,\,t) \in \mathcal{C}^{2+\alpha}(\overline{Q})\,, \tag{2.26}$$

and, by Theorem 2.1,

$$\tilde{p}(x,\,t),\, \tilde{n}(x,\,t) \in \mathcal{C}^{1+\delta}(\overline{Q}), \qquad 0 < \alpha < \delta < 1 \tag{2.27}$$

$$\tilde{\theta}(x,\,t) \in \mathcal{C}^{1+\delta}(\overline{Q})\,. \tag{2.28}$$

Thus Φ is well-defined and compact. We claim that the solutions of the functional equation

$$\Phi(w,\,\lambda) = w\,, \qquad \lambda \in [0,\,1] \tag{2.29}$$

are a priori bounded in the $\mathcal{B}$-norm by a constant which depends only on the data. Equation (2.29) is equivalent to the system

$$p_t - [(1-\lambda) + \lambda B(\theta)]\,\Delta p - \lambda[\mu\nabla\varphi + B'(\theta)\,\nabla\theta]\cdot\nabla p = \lambda\left[-\frac{\mu}{\varepsilon}\,p(p-n) + R(p,\,n)\right] \tag{2.30}$$

$$p(x,\,0) = p_0(x)\,, \qquad p = p_b \quad \text{on} \quad S \tag{2.31}$$

$$n_t - [(1-\lambda) + D(\theta)]\,\Delta n + \lambda[\nu\nabla\varphi - D'(\theta)\,\nabla\theta]\cdot\nabla n = \lambda\left[\frac{\nu}{\varepsilon}\,n(p-n) + R(p,\,n)\right] \tag{2.32}$$

$$n(x,\,0) = n_0(x)\,, \qquad n = n_b \quad \text{on} \quad S \tag{2.33}$$

$$-\varepsilon\Delta\varphi = p - n \tag{2.34}$$

$$\varphi = \varphi_b \quad \text{on} \quad S \tag{2.35}$$

$$\theta_t - \nabla \cdot (\kappa \nabla \theta) = \lambda[(\mu p + \nu n)\,|\nabla \varphi|^2 + (D(\theta)\,\nabla p - D(\theta)\,\nabla n) \cdot \nabla \varphi] \tag{2.36}$$

$$\theta(x, 0) = \theta_0(x), \qquad \theta = \theta_b \quad \text{on} \quad S\,. \tag{2.37}$$

We first find an a priori bound in the $C^0(\overline{Q})$-norm using assumption (D). Take $r > 0$ such that $r \geqslant \bar{r}$ and

$$(p_0(x), n_0(x)) \in \Sigma_r, \qquad \forall x \in \overline{\Omega} \tag{2.38}$$

$$(p_b(x, t), n_b(x, t)) \in \Sigma_r, \qquad \forall (x, t) \in S\,. \tag{2.39}$$

We prove that, if $\lambda \in (0, 1]$ and $w = (p, n, \theta)$ is a solution of (2.29), then

$$(p(x, t), n(x, t)) \in \Sigma_r, \qquad \forall (x, t) \in \overline{Q}\,. \tag{2.40}$$

Let, by contradiction,

$$(\bar{x}, \bar{t}) \in \overline{Q} \quad \text{and} \quad (p(\bar{x}, \bar{t}), n(\bar{x}, \bar{t})) \notin \Sigma_r.$$

Recalling (2.38) and (2.39), there exists, by continuity, $t^* \in (0, T]$ and $x^* \in \Omega$ such that

$$(p(x, t), n(x, t)) \in \Sigma_r, \qquad \forall t \in [0, t^*), \qquad \forall x \in \Omega \tag{2.41}$$

$$(p(x^*, t^*), n(x^*, t^*)) \in \partial \Sigma_r. \tag{2.42}$$

Assume, e.g.

$$(p(x^*, t^*), n(x^*, t^*)) \in \Gamma_2. \tag{2.43}$$

Then

$$p(x, t^*) \leqslant p(x^*, t^*), \qquad \forall x \in \Omega \tag{2.44}$$

$$p(x^*, t) \leqslant p(x^*, t^*), \qquad \forall t \in [0, t^*]. \tag{2.45}$$

Hence

$$\nabla p(x^*, t^*) = 0, \qquad \Delta p(x^*, t^*) \leqslant 0, \qquad p_t(x^*, t^*) \geqslant 0\,. \tag{2.46}$$

Furthermore, we have

$$-\frac{\mu}{\varepsilon} p(x^*, t^*)[p(x^*, t^*) - n(x^*, t^*)] + R(p(x^*, t^*), n(x^*, t^*)) < 0\,, \tag{2.47}$$

since the vector field $\boldsymbol{F}$ defined in assumption (D) points toward the interior of Σ_r. Evaluating (2.30) for $(x, t) = (x^*, t^*)$ and $\lambda \in (0, 1]$, we obtain that (2.43) is impossible in view of (2.46) and (2.47). If $(p(x^*, t^*), n(x^*, t^*))$ belongs to other parts of the boundary of Σ_r the proof is similar. Hence (2.40) follows and we have

$$(2.48) \quad 0 \leq p(x, t) \leq r, \qquad 0 \leq n(x, t) \leq r, \qquad \forall (x, t) \in \overline{Q}, \qquad \forall \lambda \in [0, 1].$$

From (2.34) we infer, with a constant C depending only on the data,

$$(2.49) \qquad \|\nabla \varphi\|_{\mathcal{C}^{\alpha}(Q)} \leq C .$$

Multiplying (2.30), i.e.

$$(2.50) \qquad p_t - (1-\lambda)\Delta p - \lambda \nabla \cdot (B(\theta)\nabla p) - \lambda \nabla \cdot (\mu p \nabla \varphi) = \lambda R(p, n),$$

by $p - \Psi$ and integrating by parts over Ω, we deduce, using the Cauchy-Schwartz inequality and recalling assumption (B) and (2.48), (2.49),

$$(2.51) \qquad \frac{d}{dt}\int_{\Omega} p^2\, dx + \int_{\Omega} |\nabla p|^2\, dx \leq C$$

for all $0 \leq t \leq T$ and with C an appropriate constant.

A similar inequality holds for n. The Gronwall's inequality yields the estimates:

$$(2.52) \quad p, n \quad \text{are bounded in} \quad L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)).$$

$$(2.53) \qquad \nabla p,\ \nabla n \quad \text{are bounded in} \quad L^2(0, T; L^2(\Omega)).$$

As the right hand side of (2.36) is bounded in $L^2(Q)$, by (2.49) and (2.53) we have

$$(2.54) \qquad \theta \quad \text{is bounded in} \quad W^{1,2}(0, T; W^{2,2}(\Omega)).$$

Applying a result of O. A. Ladyzhenskaia (see [**12**] page 443), we obtain from (2.30)-(2.33) that

$$(2.55) \qquad \nabla p,\ \nabla n \quad \text{are bounded in} \quad C^0(\overline{Q}).$$

By Theorem 2.1 applied to (2.30)-(2.37) we also conclude that:

$$(2.56) \qquad p, n \quad \text{and} \quad \theta \quad \text{are bounded in} \quad \mathcal{C}^{1+\alpha}(\overline{Q}),$$

by constants depending only on Ψ_i, $(i=1,2,3)$, B_0, D_0, r and T. Thus all solutions of (2.29) remain in a bounded set of $\mathcal{B}$. To apply the Leray-Schauder principle we need to show that the map Φ is uniformly continuous with respect to $\lambda \in [0,1]$. Let $w=(p,n,\theta)\in\mathcal{B}, \lambda_1, \lambda_2 \in [0,1]$ and $\widetilde{w}_1 = \Phi(w,\lambda_1)$, $\widetilde{w}_2=\Phi(w,\lambda_2)$, where $\widetilde{w}_1 = (\tilde{p}_1, \tilde{n}_1, \tilde{\theta}_1)$, $\widetilde{w}_2 = (\tilde{p}_2, \tilde{n}_2, \tilde{\theta}_2)$. Let $\widetilde{P} = \tilde{p}_1 - \tilde{p}_2$, $\widetilde{N} = \tilde{n}_1 - \tilde{n}_2$, $\widetilde{\Theta} = \tilde{\theta}_1 - \tilde{\theta}_2$. By difference from $(2.30)_{\lambda=\lambda_1}$, $(2.30)_{\lambda=\lambda_2}$, we obtain that $\widetilde{P}$ satisfies

$$\widetilde{P}_t - a(x,t;\lambda_1)\,\Delta\widetilde{P} - \lambda_1 \boldsymbol{b}(x,t)\cdot\nabla\widetilde{P} = (\lambda_2-\lambda_1)\,\mathcal{F}(x,t), \tag{2.57}$$

where $\boldsymbol{b}(x,t)$ and $\mathcal{F}(x,t)$ are easily computed and do not depend on λ. Owing to Theorem 2.1, we have

$$\|\widetilde{P}\|_{\mathcal{C}^{1+\alpha}(\overline{Q})} \leq k|\lambda_2 - \lambda_1|. \tag{2.58}$$

In a similar way

$$\|\widetilde{N}\|_{\mathcal{C}^{1+\alpha}(\overline{Q})} \leq k|\lambda_2-\lambda_1|, \qquad |\widetilde{\Theta}\|_{\mathcal{C}^{1+\alpha}(\overline{Q})} \leq k|\lambda_2-\lambda_1|. \tag{2.59}$$

Hence the uniform continuity follows and problem (EP) has at least one solution. A straightforward application of the Gronwall's inequality implies uniqueness. Let $(p_i, n_i, \varphi_i, \theta_i)$, $i=1,2$ be two solutions of problem (EP) and define

$$P = p_1 - p_2, \quad N = n_1 - n_2, \quad \phi = \varphi_1 - \varphi_2, \quad \Theta = \theta_1 - \theta_2.$$

By difference from (2.6) we have

$$\begin{gathered} P_t - \nabla\cdot(B(\theta_1)\,\nabla P) - \nabla\cdot[(B(\theta_1)-B(\theta_2))\,\nabla p_2] \\ -\mu\nabla\cdot(p_1\nabla\phi) - \mu\nabla\cdot(P\nabla\varphi_2) = R(p_1,n_1) - R(p_2,n_2). \end{gathered} \tag{2.60}$$

Multiplying this equation by P and integrating by parts we obtain, using assumption (B),

$$\begin{gathered} \frac{1}{2}\frac{d}{dt}\int_\Omega P^2\,dx + B_0\int_\Omega |\nabla P|^2\,dx \leq \int_\Omega |\Theta|\,|\nabla p_2|\,|\nabla P|\,dx \\ +\mu\int_\Omega |p_1|\,|\nabla\phi|\,|\nabla P|\,dx + \mu\int_\Omega |P|\,|\nabla\varphi_2|\,|\nabla P|\,dx + L_R\int_\Omega (P^2 + |N|\,|P|)\,dx. \end{gathered} \tag{2.61}$$

In the same way, having (2.7) multiplied by N and (2.9) multiplied by Θ, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega N^2\,dx + D_0\int_\Omega |\nabla N|^2\,dx \leq L_D\int_\Omega |\Theta|\,|\nabla n_2|\,|\nabla N|\,dx\\ +\nu\int_\Omega |n_1|\,|\nabla\phi|\,|\nabla N|\,dx + \nu\int_\Omega |N|\,|\nabla\varphi_2|\,|\nabla N|\,dx + L_R\int_\Omega (N^2 + |N|\,|P|)\,dx\,.\end{aligned} \tag{2.62}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Theta|^2\,dx + \kappa\int_\Omega |\nabla\Theta|^2\,dx\\ &\leq \int_\Omega (\mu|p_1| + \nu|n_1|)(|\nabla\varphi_1| + |\nabla\varphi_2|)\,|\nabla\varphi|\,|\Theta|\,dx\\ &+\int_\Omega (\mu|P| + \nu|N|)\,|\Theta|\,|\nabla\varphi_2|^2\,dx + \int_\Omega B(\theta_1)\,|\nabla p_1|\,|\nabla\phi|\,|\Theta|\,dx\\ &+\int_\Omega B(\theta_1)\,|\nabla P|\,|\nabla\varphi_2|\,|\Theta|\,dx + L_D\int_\Omega |\nabla p_2|\,|\nabla\varphi_2|\,|\Theta|^2\,dx\,.\end{aligned} \tag{2.63}$$

Moreover, from (2.8) we easily deduce

$$\|\nabla\phi\| \leq C_1(\|P\| + \|N\|)\,. \tag{2.64}$$

Adding (2.61), (2.62) and (2.63), using the elementary inequality $|a|\,|b| \leq \frac{1}{2\eta}a^2 + \frac{\eta}{2}b^2$, $\eta > 0$, and the Cauchy-Schwartz inequality, we obtain, recalling (2.48) and (2.64),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|P\|^2 + \|N\|^2 + \|\Theta\|^2) + (1 - \eta C_2)(\|\nabla P\|^2\\ + \|\nabla N\|^2 + \|\nabla\Theta\|^2) \leq C_3(\|P\|^2 + \|N\|^2 + \|\Theta\|^2)\,,\end{aligned} \tag{2.65}$$

where C_2 and C_3 are constants easily computed. If we take $\eta = \frac{1}{2C_2}$ and apply the Gronwall's inequality, uniqueness follows. ■

3 - Asymptotic behaviour of solutions

In this section we treat the recombination-generation term typical of the ionic transport [**13**], i.e.

$$R(p, n) = N_0^2 - \alpha pn\,, \qquad \alpha > 0\,, \tag{3.1}$$

and assume

$$\mu = \nu > 0, \qquad D(\theta) = B(\theta), \qquad D_0 \leqslant D(\theta) \leqslant d\,. \tag{3.2}$$

Let $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$, $\partial\Omega_2 \neq \emptyset$. We consider equations (2.6), (2.7), (2.8) and (2.9) with the boundary conditions

$$p = n = \frac{N_0}{\sqrt{\alpha}} \quad \text{on} \quad \partial\Omega_1 \tag{3.3}$$

$$p = p_b, \qquad n = n_b \quad \text{on} \quad \partial\Omega_2 \tag{3.4}$$

$$\varphi = \varphi_b \quad \text{on} \quad \partial\Omega_1 \tag{3.5}$$

$$\nabla\varphi \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega_2, \tag{3.6}$$

where $\boldsymbol{n}$ is the unit outward normal vector along $\partial\Omega_2$. All other assumptions remain the same. We prove the following asymptotic estimate:

$$\begin{aligned} &\int_t^{t+1} \left(\left\| p(\cdot,\tau) - \frac{N_0}{\sqrt{\alpha}} \right\|^2 + \left\| n(\cdot,\tau) - \frac{N_0}{\sqrt{\alpha}} \right\|^2 \right) d\tau \\ &\leqslant \int_0^1 \left(\left\| p_0 - \frac{N_0}{\sqrt{\alpha}} \right\|^2 + \left\| n_0 - \frac{N_0}{\sqrt{\alpha}} \right\|^2 \right) d\tau \; e^{-t} + C(2d^{1/4} + d^{1/2})\,. \end{aligned} \tag{3.7}$$

If $\partial\Omega_2 = \emptyset$, we have the stronger result

$$\left\| p(\cdot,t) - \frac{N_0}{\sqrt{\alpha}} \right\| \to 0, \qquad \left\| n(\cdot,t) - \frac{N_0}{\sqrt{\alpha}} \right\| \to 0, \qquad \text{as } t \to \infty\,. \tag{3.8}$$

A similar problem, but with different boundary conditions, is studied in [**16**] and [**17**]. To rescale the problem in a more convenient form we define in this section

$$P = \frac{\sqrt{\alpha}}{N_0} p - 1, \quad N = \frac{\sqrt{\alpha}}{N_0} n - 1, \quad \tilde{t} = \frac{t}{\sqrt{\alpha} N_0}, \quad \tilde{\mu} = \frac{\mu}{\sqrt{\alpha} N_0}, \quad \tilde{\varphi} = \frac{\sqrt{\alpha}}{N_0} \varphi\,. \tag{3.9}$$

In terms of these new quantities, equations (2.6), (2.7) and (2.8) become, after suppressing the tildas,

$$P_t - \nabla\cdot(B(\theta)\nabla P) - \mu\nabla P\cdot\nabla\varphi + \left(1-\frac{\mu}{\varepsilon}\right)N + \left(1+\frac{\mu}{\varepsilon}\right)P + \left(1-\frac{\mu}{\varepsilon}\right)PN + \frac{\mu}{\varepsilon}P^2 = 0 \tag{3.10}$$

$$N_t - \nabla\cdot(D(\theta)\nabla N) + \mu\nabla N\cdot\nabla\varphi + \left(1-\frac{\mu}{\varepsilon}\right)P + \left(1+\frac{\mu}{\varepsilon}\right)N + \left(1-\frac{\mu}{\varepsilon}\right)PN + \frac{\mu}{\varepsilon}N^2 = 0 \tag{3.11}$$

$$-\varepsilon\Delta\varphi = P - N \tag{3.12}$$

$$P = N = 0 \quad \text{on} \quad \partial\Omega_1 \tag{3.13}$$

$$\varphi = \varphi_b \quad \text{on} \quad \partial\Omega_1 \tag{3.14}$$

$$P = P_b, \qquad N = N_b \quad \text{on} \quad \partial\Omega_2 \tag{3.15}$$

$$\nabla\varphi\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega_2. \tag{3.16}$$

Let $\zeta(x) \in C^\infty(\overline{\Omega})$ such that

$$\lim_{x\to\partial\Omega_2} \frac{\zeta(x)}{\operatorname{dist}(x, \partial\Omega_2)} = 1 \tag{3.17}$$

and define $\zeta_\eta = \min\{\zeta, \eta\}\,\eta^{-1}$, η a positive constant. If we multiply (3.10) by $P\zeta_\eta$ and (3.11) by $N\zeta_\eta$ we obtain, using (3.12) and adding the resulting equations,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (P^2+N^2)\,\zeta_\eta\,dx + \int_\Omega D(\theta)(|\nabla P|^2 + |\nabla N|^2)\,\zeta_\eta\,dx \\ &+ \int_\Omega G(P, N, \beta)\,\zeta_\eta\,dx = -\int_\Omega D(\theta)(P\nabla P + N\nabla N)\cdot\nabla\zeta_\eta\,dx \\ &\qquad + \frac{\mu}{2}\int_\Omega (N^2 - P^2)\,\nabla\varphi\cdot\nabla\zeta_\eta\,dx\,, \end{aligned} \tag{3.18}$$

where $\beta = \dfrac{\mu}{\varepsilon}$, and

$$G(P, N, \beta) = \frac{\beta}{2}(P^3 + N^3) + \left(1 - \frac{\beta}{2}\right)(P^2 N + N^2 P) + (1+\beta)(P^2 + N^2) + 2(1-\beta)\, PN\,. \tag{3.19}$$

We relay on the following elementary

Lemma 3.1. If

$$\beta > \overline{\beta} = \frac{5 - 2\sqrt{21}}{4}\,, \tag{3.20}$$

there exists $m > 0$ such that

$$G(P, N, \beta) \geqslant m(P^2 + N^2) \quad \text{for all} \quad P \geqslant -1,\ N \geqslant -1\,. \tag{3.21}$$

Proof. We have

$$G(P, N, \beta) = \frac{\beta}{2}(P-N)^2(N+P) + \beta(P-N)^2 + P^2(N+1) + N^2(P+1) + 2PN\,.$$

Consequently, if $P \geqslant 0$, $N \geqslant 0$, we get $G(P, N, \beta) \geqslant P^2 + N^2$. Let $-1 \leqslant P \leqslant 0$. We have

$$G(P, N, \beta) \geqslant -\frac{\beta}{2} + \frac{\beta}{2}N^3 - \left|1 - \frac{\beta}{2}\right|(N + N^2) + (1+\beta)\, N^2 - 2\,|1-\beta|N\,.$$

Hence we can find $C > 0$ such that $G(P, N, \beta) \geqslant C(P^2 + N^2)$ if $-1 \leqslant P \leqslant 0$ and $N \geqslant \overline{N}$ with $\overline{N} > 0$ sufficiently large. Since $G(P, N, \beta) = G(N, P, \beta)$, it remains to examine the behaviour of G in the set

$$A = \{(P, n);\ 0 > P > -1,\ -1 < N < \overline{N}\} \cup \{(P, N);\ 0 > N > -1,\ -1 < P < \overline{N}\}\,.$$

If (3.20) holds it is easy to check that $G(P, N, \beta) > 0$ when $(P, N) \in \partial A \backslash \{(0, 0)\}$. We claim that $G(P, N, \beta) > 0$ in A; if this is not true, there is a point $(P_C, N_C) \in A$ such that $\nabla G(P_C, N_C, \beta) = 0$. Now the critical points of G are $(0, 0)$ and $\left(-\dfrac{4}{3}, -\dfrac{4}{3}\right)$, for all $\beta > 0$, and $(\alpha(\beta), \gamma(\beta))$, $(\gamma(\beta), \alpha(\beta))$ where

$$\alpha(\beta) = \frac{2\beta - 4\beta^2 - 2\sqrt{2\beta^3 - 5\beta^2 + 2\beta}}{4\beta^2 - 4\beta + 1}\,,$$

$$\gamma(\beta) = \frac{2\beta - 4\beta^2 + \sqrt{2\beta^3 - 5\beta^2 + 2\beta}}{4\beta^2 - 4\beta + 1}\,.$$

If (3.20) holds, the roots $(\alpha(\beta),\ \gamma(\beta)), (\gamma(\beta),\ \alpha(\beta))$ are either complex coniugate or $\alpha(\beta) < -1$ (in particular $\alpha(\overline{\beta}) = -1$). Since G has no critical points in A, we infer the validity of (3.21). ■

Assuming hereafter (3.20), we have from (3.18), since $P \geqslant -1$, $N \geqslant -1$,

$$\tag{3.22} \begin{aligned} &\frac{d}{dt}\int_\Omega (P^2 + N^2)\,\zeta_\eta\,dx + m\int_\Omega (P^2 + N^2)\,\zeta_\eta\,dx \\ &\leqslant C\left(d\int_\Omega |\nabla P\cdot\nabla\zeta_\eta|\,dx + d\int_\Omega |\nabla N\cdot\nabla\zeta_\eta|\,dx + \int_\Omega |\nabla\zeta_\eta\cdot\nabla\varphi|\,dx\right). \end{aligned}$$

We integrate (3.22) with respect to t over $[t,\,t+1]$ to obtain

$$\tag{3.23} \begin{aligned} &\int_\Omega [P^2(t+1) + N^2(t+1)]\,\zeta_\eta\,dx - \int_\Omega [P^2(t) + N^2(t)]\,\zeta_\eta\,dx \\ &+ m\int_t^{t+1}\int_\Omega (P^2 + N^2)\,\zeta_\eta\,dx\,d\tau \leqslant C\left(d\int_t^{t+1}\int_\Omega |\nabla P\cdot\nabla\zeta_\eta|\,dx\,d\tau + \right.\\ &\left. + d\int_t^{t+1}\int_\Omega |\nabla N\cdot\nabla\zeta_\eta|\,dx\,d\tau + \int_t^{t+1}\int_\Omega |\nabla\varphi\cdot\nabla\zeta_\eta|\,dx\,d\tau\right). \end{aligned}$$

To estimate the terms on the right hand side we integrate (2.51) and the analogous inequality for n over $[t,\,t+1]$ to get

$$\tag{3.24} d\sup_{t\geqslant 0}\int_t^{t+1}\int_\Omega |\nabla P|^2\,dx\,d\tau \leqslant C$$

$$\tag{3.25} d\sup_{t\geqslant 0}\int_t^{t+1}\int_\Omega |\nabla N|^2\,dx\,d\tau \leqslant C\,.$$

Hence

$$\int_t^{t+1}\int_\Omega |\nabla P\cdot\nabla\zeta_\eta|\,dx\,d\tau \leqslant \left(\int_t^{t+1}\int_\Omega |\nabla P|^2\,dx\,d\tau\right)^{\frac{1}{2}}$$

$$\cdot\left(\int_t^{t+1}\int_\Omega |\nabla\zeta_\eta|^2\,dx\,d\tau\right)^{\frac{1}{2}} \leqslant \frac{C}{\eta d^{1/2}}\left(\int_{\mathcal{U}(\eta,\,\partial\Omega_2)} |\nabla\zeta|^2\,dx\right)^{\frac{1}{2}},$$

where $\mathcal{U}(\eta, \partial\Omega_2) = \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega_2) < \eta\}$, (the various C denote positive constants generally different). Since $|\nabla\zeta|$ is bounded and meas $\{\mathcal{U}(\eta, \partial\Omega_2)\} < C_\eta$, we deduce

$$d\int_t^{t+1}\int_\Omega |\nabla P\cdot\nabla\zeta_\eta|\,dx\,d\tau \leqslant C\sqrt{\frac{d}{\eta}} \tag{3.26}$$

and, in a similar way,

$$d\int_t^{t+1}\int_\Omega |\nabla N\cdot\nabla\zeta_\eta|\,dxd\tau \leqslant C\sqrt{\frac{d}{\eta}}\,. \tag{3.27}$$

On the other hand,

$$\begin{aligned}\int_\Omega |\nabla\varphi\cdot\nabla\zeta_\eta|\,dx &\leqslant \frac{1}{\eta}\int_{\mathcal{U}(\eta,\,\partial\Omega_2)} |\nabla\varphi\cdot\nabla\zeta|\,dx\\ &\leqslant \frac{1}{\eta}\left(\int_{\mathcal{U}(\eta,\,\partial\Omega_2)} \frac{|\nabla\varphi\cdot\nabla\zeta|^2}{\zeta^2}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}(\eta,\,\partial\Omega_2)} \zeta^2\,dx\right)^{\frac{1}{2}}\\ &\leqslant \eta^{\frac{1}{2}}\left(\int_\Omega \frac{|\nabla\varphi\cdot\nabla\zeta|^2}{\zeta^2}\,dx\right)^{\frac{1}{2}}.\end{aligned} \tag{3.28}$$

Since $\nabla\varphi\cdot\nabla\zeta$ belongs to $W^{1,2}(\Omega)$ and vanishes on $\partial\Omega_2$ we have, by (3.16),

$$\int_\Omega \frac{|\nabla\varphi\cdot\nabla\zeta|^2}{\zeta^2}\,dx \leqslant C\|\nabla\varphi\|^2_{(W^{1,2}(\Omega))^3} \tag{3.29}$$

(see [**9**] page 65). Collecting the estimates from (3.27) to (3.29), we rewrite (3.23) as

$$\begin{aligned}&\int_\Omega (P^2+N^2)(t+1)\,\zeta_\eta\,dx - \int_\Omega (P^2+N^2)(t)\,\zeta_\eta\,dx\\ &+m\int_t^{t+1}\int_\Omega (P^2+N^2)\,\zeta_\eta\,dx\,dt \leqslant C\left(\frac{\sqrt{d}}{\sqrt{\eta}}+\sqrt{\eta}\right).\end{aligned} \tag{3.30}$$

Owing to the fact that

$$\left| \int_\Omega (P^2+N^2)\,dx - \int_\Omega (P^2+N^2)\,\zeta_\eta\,dx \right| \leq C\eta\,,$$

inequality (3.30) turns into

$$\begin{aligned}&\int_\Omega (P^2+N^2)(t+1)\,dx - \int_\Omega (P^2+N^2)(t)\,dx \\ &+ \int_t^{t+1}\int_\Omega (P^2+N^2)\,dx\,d\tau \leq C\left(\frac{\sqrt{d}}{\sqrt{\eta}} + \sqrt{\eta}+\eta\right).\end{aligned} \tag{3.31}$$

If $\eta = d^{1/2}$ and

$$\lambda(t) = \int_t^{t+1}\int_\Omega (P^2+N^2)\,dx\,d\tau\,,$$

(3.31) becomes

$$\lambda'(t) + \lambda(t) \leq C(2d^{1/4} + d^{1/2}) \tag{3.32}$$

and the Gronwall's inequality implies (3.7).

To prove (3.8) when $\partial\Omega_2 = \emptyset$, we simply multiply (3.10) and (3.11) by P and N to find, again by Lemma 3.1,

$$\frac{1}{2}\frac{1}{dt}\int_\Omega (P^2+N^2)\,dx + d\int_\Omega (|\nabla P|^2 + |\nabla N|^2)\,dx + m\int_\Omega (P^2+N^2)\,dx \leq 0 \tag{3.33}$$

which implies (3.8).

4 - Existence and uniqueness for the stationary problem with general heat-production term

The heat generated and consumed in the process of generation-recombination is not taken into account in the energy equation (1.7). For this reason more comprehensive models have been proposed in [**2**] and [**23**], where as heat production term is taken

$$W = (\mu p + \nu n)\,|\nabla\varphi|^2 + \theta(\mu\nabla p - \nu\nabla n)\cdot\nabla\varphi + \frac{N_0^2 - \alpha p n}{r(p,\,n)}\left(a_1 - \frac{a_2\theta^2}{a_3+\theta}\right). \tag{4.1}$$

Besides, the Einstein relationship between mobility and diffusivity, i.e.

$$B(\theta) = \mu\theta , \qquad D(\theta) = \nu\theta , \tag{4.2}$$

θ absolute temperature, is not compatible, if μ and ν are constants, with assumption (C) of Section **2**. Here we present a local result of existence and uniqueness for the stationary problem under assumptions which are consistent with (4.1), (4.2) and also with temperature-dependent coefficients of thermal diffusion. Let $\overline{\varphi}(x)$ be the solution to the problem

$$\Delta\overline{\varphi} = 0, \qquad \overline{\varphi} = \varphi_b \quad \text{on} \quad \partial\Omega , \tag{4.3}$$

and put

$$\overline{p} = \frac{N_0}{\sqrt{\alpha}} , \qquad \overline{n} = \frac{N_0}{\sqrt{\alpha}} . \tag{4.4}$$

Suppose $\kappa(\theta) \in C^0(\mathbb{R}^1)$ and

$$\kappa(\theta) \geqslant \kappa_0 > 0 \quad \text{for all} \quad \theta \in \mathbb{R}^1. \tag{4.5}$$

Let $W(\theta, p, n, \nabla\varphi, \nabla p, \nabla n)$ be a general heat generation term and define

$$w = Z(\theta) = \int_0^\theta \kappa(\tau)\, d\tau, \qquad F(x, w) = W(Z^{-1}(w), \overline{p}, \overline{n}, \nabla\overline{\varphi}, 0, 0). \tag{4.6}$$

Assume $\theta_b \in C^{1,\alpha}(\partial\Omega)$,

$$\theta_b > 0 \tag{4.7}$$

$$F(x, w) \geqslant 0 . \tag{4.8}$$

$$\begin{gathered} F(x, w) \text{ is measurable in } x \text{ for all } w \\ \text{and continuous in } w \text{ for almost all } x \in \Omega . \end{gathered} \tag{4.9}$$

There exist two non-negative functions $C(x) \in L^\infty(\Omega)$, $C_1(x) \in L^2(\Omega)$ such that

$$|F(x, w)| \leqslant C(x)|w| + C_1(x). \tag{4.10}$$

$$\begin{gathered} F \text{ is monotone in } w, \text{ i.e.,} \\ [F(x, w_1) - F(x, w_2)](w_1 - w_2) \geqslant 0, \quad \forall w_1, w_2 \in \mathbb{R}^1, \quad \forall x \in \Omega . \end{gathered} \tag{4.11}$$

The temperature $\overline{\theta}(x)$ corresponding to electroneutrality is given by the problem

$$(4.12)\qquad -\nabla\cdot(\kappa(\theta)\nabla\theta) = W(\theta, \overline{p}, \overline{n}, \nabla\overline{\varphi}, 0, 0), \quad \theta = \theta_b \text{ on } \partial\Omega\,,$$

which is equivalent, by (4.6), to

$$(4.13)\qquad -\Delta w = F(x, w)\,, \quad w = Z(\theta_b) \quad \text{on} \quad \partial\Omega\,.$$

The theory of monotone operators applies to (4.13) in view of (4.9)-(4.11). If $\overline{w}(x)$ is the unique solution of (4.13), we have $\overline{\theta}(x) = Z^{-1}(\overline{w}(x))$. Moreover, the maximum principle implies, by (4.7) and (4.8),

$$(4.14)\qquad \overline{\theta}(x) > 0 \quad \text{in} \quad \overline{\Omega}\,.$$

Finally, we assume

$$(4.15)\qquad B(\overline{\theta}(x)) \geqslant B_0 > 0\,, \quad D(\overline{\theta}(x)) \geqslant D_0 > 0 \quad \text{in} \quad \overline{\Omega}\,.$$

If $D(\theta)$ and $B(\theta)$ are given by (4.2) and W by (4.1), all these hypotheses are satisfied. We consider the following problem (SP):

$$(4.16)\qquad -\nabla\cdot(B(\theta)\nabla p) - \nabla\cdot(\mu p\nabla\varphi) = \frac{N_0^2 - \alpha pn}{r(p, n)}$$

$$(4.17)\qquad p = \overline{p} + \xi_b \quad \text{on} \quad \partial\Omega$$

$$(4.18)\qquad -\nabla\cdot(D(\theta)\,\nabla n) + \nabla\cdot(\nu n\nabla\varphi) = \frac{N_0^2 - \alpha pn}{r(p, n)}$$

$$(4.19)\qquad n = \overline{n} + \zeta_b \quad \text{on} \quad \partial\Omega$$

$$(4.20)\qquad -\varepsilon\Delta\varphi = p - n$$

$$(4.21)\qquad \varphi = \varphi_b \quad \text{on} \quad \partial\Omega$$

$$(4.22)\qquad -\nabla\cdot(\kappa(\theta)\,\nabla(\theta) = W(\theta, p, n, \nabla\varphi, \nabla p, \nabla n)$$

$$(4.23)\qquad \theta = \theta_b \quad \text{on} \quad \partial\Omega\,,$$

where θ_b satisfies (4.7). If $\xi_b = \zeta_b = 0$, then $(p, n, \varphi, \theta) = (\overline{p}, \overline{n}, \overline{\varphi}, \overline{\theta})$ is a solution to problem (SP). Define

$$(4.24)\qquad \xi = p - \overline{p}, \quad \zeta = n - \overline{n}, \quad \psi = \varphi - \overline{\varphi}, \quad \eta = \theta - \overline{\theta}$$

and $\boldsymbol{u} = (\xi, \zeta, \psi, \eta)$. In term of this new variables (SP) becomes

$$M_1(\boldsymbol{u}) =: -\nabla\cdot(B(\overline{\theta}+\eta)\,\nabla\zeta) - \nabla\cdot[\mu(\overline{p}+\xi)\,\nabla(\overline{\varphi}+\psi)] + \frac{\alpha(\overline{p}\,\zeta+\overline{n}\,\xi+\xi\zeta)}{r(\overline{p}+\xi,\overline{n}+\zeta)} = 0 \tag{4.25}$$

$$\xi = \xi_b \quad \text{on} \quad \partial\Omega \tag{4.26}$$

$$M_2(\boldsymbol{u}) =: -\nabla\cdot(D(\overline{\theta}+\eta)\,\nabla\zeta) + \nabla\cdot[\nu(\overline{n}+\zeta)\,\nabla(\overline{\varphi}+\psi)] + \frac{\alpha(\overline{p}\,\zeta+\overline{n}\,\xi+\xi\zeta)}{r(\overline{p}+\xi,\overline{n}+\zeta)} = 0 \tag{4.27}$$

$$\zeta = \zeta_b \quad \text{on} \quad \partial\Omega \tag{4.28}$$

$$M_3(\boldsymbol{u}) =: -\varepsilon\Delta\psi - (\xi-\zeta) = 0 \tag{4.29}$$

$$\psi = 0 \quad \text{on} \quad \partial\Omega \tag{4.30}$$

$$M_4(\boldsymbol{u}) =: -\nabla\cdot[\kappa(\overline{\theta}+\eta)\,\nabla(\overline{\theta}+\eta)] - W(\overline{\theta}+\eta,\overline{p}+\xi,\overline{n}+\zeta,\nabla(\overline{\varphi}+\psi),\nabla\xi,\nabla\zeta) = 0 \tag{4.31}$$

$$\eta = 0 \quad \text{on} \quad \partial\Omega\,. \tag{4.32}$$

Theorem 4.1. There exists $\delta > 0$ such that, if

$$\|\xi_b\|_{C^\alpha} \leqslant \delta\,, \qquad \|\zeta\|_{C^\alpha} \leqslant \delta\,, \tag{4.33}$$

problem (SP) has one and only one solution.

Proof. We apply the inverse function theorem in Banach spaces. Let

$$X = (C^{2,\alpha}(\overline{\Omega}))^2 \times \{v(x)\in C^{2,\alpha}(\overline{\Omega}),\ v=0\ \text{ on }\ \partial\Omega\}^2,$$

$$Y = (C^\alpha(\overline{\Omega}))^4 \times (C^{2,\alpha}(\partial\Omega))^2, \qquad u = (\xi,\zeta,\psi,\eta)\in X\,,$$

and define the operator $F: X \to Y$,

$$F(u) = (M_1(\boldsymbol{u}),\, M_2(\boldsymbol{u}),\, M_3(\boldsymbol{u}),\, M_4(\boldsymbol{u}),\ \xi|_{\partial\Omega},\, \zeta|_{\partial\Omega})\,.$$

We have $F(0)=0$. It is easy to check that $F\in C^1(X,Y)$ and $F'(0)(\boldsymbol{w})$,

$\boldsymbol{w} = (P, N, \phi, \Theta)$ is given by

$$F'(0)(\boldsymbol{w}) = \Big(-\nabla\cdot(B(\overline{\theta})\,\nabla P) - \mu\nabla\cdot(P\nabla\overline{\varphi}) - \mu\overline{p}\Delta\phi$$

$$+\gamma(P+N),\ -\nabla\cdot(D(\overline{\theta})\,\nabla N) + \nu\nabla\cdot(N\nabla\overline{\varphi}) + \nu\overline{n}\Delta\phi$$

$$+\gamma(P+N),\ -\varepsilon\Delta\phi - (P-N),\ -\nabla\cdot(\kappa(\overline{\theta})\nabla\Theta)$$

$$-\nabla\cdot(\kappa'(\overline{\theta})\,\Theta\nabla\overline{\theta}) - \overline{W}_\theta\,\Theta - \overline{W}_p P - \overline{W}_n N$$

$$-\sum_{i=1}^{3}(\overline{W}_{\varphi_{x_i}}\phi_{x_i} + \overline{W}_{p_{x_i}}P_{x_i} + \overline{W}_{n_{x_i}}N_{x_i}),\ P|_{\partial\Omega},\ N|_{\partial\Omega}\Big)$$

where $\gamma = \frac{1}{\beta}N_0\sqrt{\alpha}$, $\beta = r(\overline{p}, \overline{n})$ and the barred partial derivatives of W are computed in $(\theta, p, n, \nabla\varphi, \nabla p, \nabla n) = (\overline{\theta}, \overline{p}, \overline{n}, \nabla\overline{\varphi}, \nabla\overline{p}, \nabla\overline{n})$. Let $\boldsymbol{h} = (h_1, h_2, h_3, h_4, P_b, N_b) \in Y$. We prove that the linear functional equation

$$F'(0)(\boldsymbol{w}) = \boldsymbol{h} \tag{4.34}$$

has one and only one solution. This equation is equivalent to the boundary value problem:

$$-\nabla\cdot(B(\overline{\theta})\,\nabla P) - \mu\nabla\cdot(P\nabla\overline{\varphi}) - \mu\overline{p}\Delta\phi + \gamma(P+N) = h_1 \tag{4.35}$$

$$-\nabla\cdot(D(\overline{\theta})\,\nabla N) + \nu\nabla\cdot(N\nabla\overline{\varphi}) + \nu\overline{n}\Delta\phi + \gamma(P+N) = h_2 \tag{4.36}$$

$$-\varepsilon\Delta\phi - (P-N) = h_3 \tag{4.37}$$

$$P = P_b, \qquad N = N_b \qquad \text{on} \qquad \partial\Omega \tag{4.38}$$

$$\phi = 0 \qquad \text{on} \qquad \partial\Omega\,. \tag{4.39}$$

We do not write the equation in Θ which follows from (4.34) since it is uncoupled from the system (4.35)-(4.39). Plugging (4.37) in (4.35) and (4.36), we have

$$L_1(P, N) = -\nabla\cdot(B(\overline{\theta})\,\nabla P) - \mu\nabla\cdot(P\nabla\overline{\varphi}) + \gamma(aP + bN) = \overline{h}_1 \tag{4.40}$$

$$L_2(P, N) = -\nabla\cdot(D(\overline{\theta})\,\nabla N) + \nu\nabla\cdot(N\nabla\overline{\varphi}) + \gamma(cP + dN) = \overline{h}_2 \tag{4.41}$$

$$P = P_b, \qquad N = N_b \qquad \text{on} \qquad \partial\Omega\,, \tag{4.42}$$

where we used the notations

$$a = 1 + \tau, \qquad b = 1 - \tau, \qquad c = 1 - \chi, \qquad d = 1 + \chi\,, \tag{4.43}$$

$$\tau = \frac{\beta\mu}{\alpha\varepsilon}\,, \qquad \chi = \frac{\beta\nu}{\alpha\varepsilon} \tag{4.44}$$

$$\overline{h}_1 = h_1 - \frac{\mu}{\varepsilon}\,\overline{p}\,h_3, \qquad \overline{h}_2 = h_2 + \frac{\nu}{\varepsilon}\,\overline{n}\,h_3. \tag{4.45}$$

To rewrite (4.40)-(4.42) with homogeneous boundary conditions, let us define

$$u_1 = P - \widetilde{P}\,, \qquad u_2 = N - \widetilde{N}\,, \tag{4.46}$$

$$l_1(x) = \overline{h}_1 - L_1(\widetilde{P}, \widetilde{N}), \qquad l_2(x) = \overline{h}_2 - L_2(\widetilde{P}, \widetilde{N})\,, \tag{4.47}$$

where $\widetilde{P}$ and $\widetilde{N}$ are solutions to the problems

$$-\Delta\widetilde{P} = 0\,, \qquad \widetilde{P} = P_b \quad \text{on} \quad \partial\Omega \tag{4.48}$$

$$-\Delta\widetilde{N} = 0\,, \qquad \widetilde{N} = N_b \quad \text{on} \quad \partial\Omega\,. \tag{4.49}$$

The proof that (4.34) is an homeomorphism will follow from the existence and uniqueness for the problem

$$L_1(u_1, u_2) = l_1 \tag{4.50}$$

$$L_2(u_1, u_2) = l_2 \tag{4.51}$$

$$u_1 = u_2 = 0 \quad \text{on} \quad \partial\Omega\,. \tag{4.52}$$

Let us consider the bilinear forms in $(H_0^1(\Omega))^2$

$$a_1(u, v) = \int_\Omega [B(\overline{\theta})\,\nabla u_1\cdot\nabla v_1 + \mu u_1\nabla\overline{\varphi}\cdot\nabla v_1 + \gamma(au_1 + bu_2)\,v_1]\,dx\,, \tag{4.53}$$

$$a_2(u, v) = \int_\Omega [D(\overline{\theta})\,\nabla u_2\cdot\nabla v_2 - \nu u_2\nabla\overline{\varphi}\cdot\nabla v_2 + \gamma(cu_2 + du_2)\,v_2]\,dx\,. \tag{4.54}$$

Assume $c \neq 0$; we claim that there exists $\lambda = \overline{\lambda}$ such that $a(u, v) = a_1(u, v) + \lambda a_2(u, v)$ is coercive in $(H_0^1)^2$ i.e.:

$$a(u, u) \geq m\|u\|^2_{(H_0^1)^2} \quad \text{for all} \quad u \in (H_0^1)^2. \tag{4.55}$$

Recalling that $\Delta\overline{\varphi} = 0$ and integrating by parts, we have

$$a(u, u) = \int_{\Omega} [B(\overline{\theta}) \,|\nabla u_1|^2 + \lambda D(\overline{\theta}) \,|\nabla u_2|^2 + \gamma(a u_1^2 + (b - \lambda c)\, u_1 u_2 + \lambda d u_2^2)] \, dx \,.$$

If

$$\overline{\lambda} = \frac{3\tau + 3\tau\chi + 3\chi + 1}{(1-\chi)^2} \,,$$

the quadratic form

$$a u_1^2 + (b - \overline{\lambda} c)\, u_1 u_2 + \overline{\lambda} d u_2^2$$

is definite positive and (4.55) follows by (4.15). If $c = 0$ and $b \neq 0$ we consider the bilinear form $a(u, v) = \lambda a_1(u, v) + a_2(u, v)$ and redefine $\overline{\lambda}$ accordingly. When $c = b = 0$ of course no problem arises. By the Lax-Milgram lemma there is one and only one weak solution to problem (4.50)-(4.52) which can be regularized. This in turn implies that (4.34) can be uniquely solved with respect to w. Therefore, if (4.33) holds, problem (SP) has one and only one solution.

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Abstract

The nonlinear system of P.D.E. describing the electro-diffusion of ions is studied in both the evolution and stationary case. The crucial property for proving existence of solutions turns out to be the presence of an invariant region for the concentrations under quite general hypotheses on the recombination law.

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