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On new subclasses of analytic and multivalent functions (**)

1 - Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N = \{1, 2, \ldots\}) \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in A(p)$ is called p-valent starlike of order α if it satisfies the conditions

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.2}$$

and

$$\int_0^{2\pi} Re\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2p\pi \tag{1.3}$$

for $0 \leqslant \alpha < p$, $p \in N$, and $z \in U$. We denote by $S(p, \alpha)$ the class of all p-valent stalike functions of order α. Also a function $f(z) \in A(p)$ is called p $-$valent convex of order α if $f(z)$ satisfies the conditions

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{1.4}$$

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and

$$\int_0^{2\pi} Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta = 2p\pi \tag{1.5}$$

for $0 \leqslant \alpha < p$, $p \in N$ and $z \in U$. We denote by $K(p, \alpha)$ the class of all p-valent convex functions of order α. We note that

$$f(z) \in K(p, \alpha) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in S(p, \alpha) \tag{1.6}$$

for $0 \leqslant \alpha < p$.

The class $S(p, \alpha)$ was introduced by Patil and Thakare [**7**], and the class $K(p, \alpha)$ was introduced by Owa [**5**].

Many essentially equivalent definitions of fractional calculus (that is fractional derivatives and fractional integrals) have been given in the literature (cf. e.g. [**1**] Chapter 13, [**2**], [**3**], [**8**], [**9**], [**11**] p. 28 et seq., and [**13**]). We find it to be convenient to recall here the following definitions which were used earlier by Owa [**4**] (and by Srivastava and Owa [**12**]).

Definition 1. The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta (\lambda > 0), \tag{1.7}$$

where $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$

Definition 2. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \qquad (0 \leqslant \lambda < 1), \tag{1.8}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 1.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $n+\lambda$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^\lambda f(z) (0 \leqslant \lambda < 1;\ n \in N_0 = N \cup \{0\}). \tag{1.9}$$

Let $S^*(p, \alpha, \lambda)$ denote the class of all functions $f(z)$ in $A(p)$ satisfying the inequality

$$Re\left\{ \frac{\dfrac{\Gamma(p-\lambda)\, z^{1+\lambda} D_z^{1+\lambda} f(z)}{\Gamma(p+1)}}{f(z)} \right\} > \frac{\alpha}{p} \qquad (z \in U) \tag{1.10}$$

for $\lambda < 1$ and $0 \leqslant \alpha < p$, $p \in N$. Also let $K(p, \alpha, \lambda)$ denote the class of all functions $f(z)$ in $A(p)$ such that

$$\frac{\Gamma(p-\lambda)}{\Gamma(p+1)} z^{1+\lambda} D_z^{1+\lambda} f(z) \in S^*(p, \alpha, \lambda). \tag{1.11}$$

for $\lambda < 1$ and $0 \leqslant \alpha < p$, $p \in N$. Clearly,

$$S^*(p, \alpha, 0) \equiv S^*(p, \alpha) \quad \text{and} \quad K(p, \alpha, 0) \equiv K(p, \alpha),$$

where we have set $\lambda = 0$. Thus $S^*(p, \alpha, \lambda)$ and $K(p, \alpha, \lambda)$ are generalizations of the classes $S^*(p, \alpha)$ and $K(p, \alpha)$, respectively.

Let $T(p)$ denote the subclass of $A(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (a_{p+n} \geqslant 0;\ p \in N). \tag{1.12}$$

We denote by $T^*(p, \alpha, \lambda)$ and $C(p, \alpha, \lambda)$ the classes obtained by taking intersections, respectively, of the classes $S^*(p, \alpha, \lambda)$ and $K(p, \alpha, \lambda)$ with $T(p)$, that is

$$T^*(p, \alpha, \lambda) = S^*(P, \alpha, \lambda) \cap T(P)$$

and

$$C(p, \alpha, \lambda) = K(p, \alpha, \lambda) \cap T(p).$$

Furthermore, by specializing the parameters p and λ, we obtain the following subclasses studied by various authors:

(i) $T^*(1, \alpha, \lambda) = T^*(\alpha, \lambda)$ and $C(1, \alpha, \lambda) = C(\alpha, \lambda)$ (Owa [**6**]);
(ii) $T^*(p, \alpha, 0) = T^*(p, \alpha)$ and $C(p, \alpha, 0) = C(p, \alpha)$ (Owa [**5**]);
(iii) $T^*(1, \alpha, 0) = T^*(\alpha)$ and $C(1, \alpha, 0) = C(\alpha)$ (Silverman [**10**]).

In this paper, we prove several intersting results for functions belonging to the general classes $T^*(p, \alpha, \lambda)$ and $C(p, \alpha, \lambda)$.

2 - Coefficient inequalities

Theorem 1. *Let the function* $f(z)$ *defined by* (1.1). *If*

$$\sum_{n=1}^{\infty}\left\{\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha\right\}|a_{p+n}| < p-\alpha \tag{2.1}$$

for $\lambda < 1$ *and* $0 \leqslant \alpha < p$, $p \in N$, *then* $f(z) \in S^*(p, \alpha, \lambda)$. *The result* (2.1) *is sharp.*

Proof. We need only prove that (2.1) implies (1.10). In order to prove (2.1), it suffices to show that

$$\left|\frac{\dfrac{\Gamma(p-\lambda)\, z^{1+\lambda} D_z^{1+\lambda} f(z)}{\Gamma(p+1)}}{f(z)} - 1\right| < 1 - \frac{\alpha}{p} \qquad (z \in U). \tag{2.2}$$

Note that

$$D_z^{1+\lambda} f(z) = \frac{\Gamma(p+\lambda)}{\Gamma(p-\lambda)} z^{p-\lambda-1} + \sum_{n=1}^{\infty} \frac{\Gamma(p+n+1)}{\Gamma(p+n-\lambda)} a_{p+n} z^{p+n-\lambda-1}, \tag{2.3}$$

which readily yields

$$\left| \frac{\dfrac{\Gamma(p-\lambda)\, z^{1+\lambda} D_z^{1+\lambda} f(z)}{\Gamma(p+1)}}{f(z)} - 1 \right|$$

$$(2.4) \qquad = \left| \frac{\displaystyle\sum_{n=1}^{\infty} \left\{ \frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p+1)\,\Gamma(p+n-\lambda)} - 1 \right\} a_{p+n} z^{p+n}}{z^p + \displaystyle\sum_{n=1}^{\infty} a_{p+n} z^{p+n}} \right|$$

$$< \frac{\displaystyle\sum_{n=1}^{\infty} \left\{ \frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p+1)\,\Gamma(p+n-\lambda)} - 1 \right\} |a_{p+n}|}{1 - \displaystyle\sum_{n=1}^{\infty} |a_{p+n}|}$$

$$\leqslant 1 - \frac{\alpha}{p}$$

provided that

$$(2.5) \qquad \sum_{n=1}^{\infty} \left\{ \frac{\Gamma(p+n-1)\,\Gamma(p-\lambda)}{\Gamma(p+1)\,\Gamma(p+n-\lambda)} - 1 \right\} |a_{p+n}| \leqslant \left(1 - \sum_{n=1}^{\infty} |a_{p+n}| \right) \left(1 - \frac{\alpha}{p}\right).$$

We note that (2.5) is equivalent to (2.1). Further, the result (2.1) is sharp for the function

$$(2.6) \qquad f(z) = z^p + \frac{p-\alpha}{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} - \alpha} z^{p+n} \qquad (n \in N).$$

Thus we complete the proof of Theorem 1.

Remark 1. (1) Letting $\lambda = 0$ in Theorem 1, we obtain the corresponding result for the class $S^*(p, \alpha, 0)$ due to Owa [**5**].

(2) Letting $p = 1$ in Theorem 1, we obtain the corresponding result for the class $S^*(1, \alpha, \lambda)$ due to Owa [**6**].

(3) Letting $P = 1$ and $\lambda = 0$ in Theorem 1, we obtain the corresponding result for the class $S^*(1, \alpha, 0)$ due to Silverman [**10**].

Theorem 2. *Let the function* $f(z)$ *be defined by* (1.1). *If*

$$\sum_{n=1}^{\infty} \frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} \left\{ \frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} - \alpha \right\} |a_{p+n}| \leqslant p(p-\alpha) \tag{2.7}$$

for $\lambda < 1$ *and* $0 \leqslant \alpha < p$, $p \in N$, *then* $f(z) \in K(p, \alpha, \lambda)$. *The result* (2.7) *is sharp.*

Proof. Note that $f(z) \in K(p, \alpha, \lambda)$ if and only if

$$\frac{\Gamma(p-\lambda)}{\Gamma(p+1)} z^{1+\lambda} D_z^{1+\lambda} f(z) \in S^*(p, \alpha, \lambda).$$

Therefore, on replacing a_{p+n} by $\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p+1)\,\Gamma(p+n-\lambda)} a_{p+n}$ in Theorem 1. we have Theorem 2. Further, the result (2.7) is sharp for the function

$$f(z) = z^p + \frac{p-\alpha}{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} \left\{ \dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} - \alpha \right\}} z^{p+n} \quad (n \in N). \tag{2.8}$$

Remark 2. (1) Letting $\lambda = 0$ in Theorem 2, we obtain the corresponding result for the class $K(p, \alpha, 0)$ given by Owa [**5**].

(2) Letting $p = 1$ in Theorem 2, we obtain the corresponding result for the class $K(1, \alpha, \lambda)$ given by Owa [**6**].

(3) Letting $P = 1$ and $\lambda = 0$ in Theorem 2, we obtain the corresponding result for the class $K(1, \alpha, 0)$ given by Silverman [**10**].

Theorem 3. *Let the function* $f(z)$ *be defined by* (1.12). *Then* $f(z)$ *belongs to the class* $T^*(p, \alpha, \lambda)$ *if and only if*

$$\sum_{n=1}^{\infty} \left\{ \frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} - \alpha \right\} a_{p+n} \leqslant p-\alpha \tag{2.9}$$

for $\lambda < 1$, $0 \leqslant \alpha < p$ *and* $p \in N$. *The result* (2.9) *is sharp.*

Proof. By means of Theorem 1, (2.9) implies that $f(z) \in T^*(p, \alpha, \lambda)$. Suppose that $f(z)$ is the class $T^*(p, \alpha, \lambda)$. Then

$$(2.10)\quad Re\left\{\frac{\Gamma(p-\lambda)\,z^{1+\lambda}D_z^{1+\lambda}f(z)}{\Gamma(p+1)f(z)}\right\}=Re\left\{\frac{1-\sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\Gamma(p-\lambda)}{\Gamma(p+1)\Gamma(p+n-\lambda)}a_{p+n}z^n}{1-\sum_{n=1}^{\infty}a_{p+n}z^n}\right\}>\frac{\alpha}{p}$$

for $\lambda < 1$ and $0 \leqslant \alpha < p$, $p \in N$ and $z \in U$. Choose values of z on real axis so that $\frac{\Gamma(p-\lambda)\,z^{1+\lambda}D_z^{1+\lambda}f(z)}{\Gamma(p+1)\,f(z)}$ is real. Upon clearing the denominator in (2.10) and letting $z \to 1^-$, we obtain

$$(2.11)\qquad p-\sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}a_{p+n} \geqslant \alpha\left(1-\sum_{n=1}^{\infty}a_{p+n}\right)$$

which implies (2.9). The result (2.9) is sharp for the functions

$$(2.12)\qquad f(z)=z^p-\frac{(p-\alpha)}{\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha}z^{p+n}(n\in N).$$

This completes the proof of Theorem 3.

Corollary 1. *Let the function $f(z)$ defined by* (1.12) *belongs to the class $T^*(p, \alpha, \lambda)$. Then*

$$(2.13)\qquad a_{p+n} \leqslant \frac{(p-\alpha)}{\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha}\qquad (n\in N).$$

The result (2.13) *is sharp for the functions $f(z)$ given by* (2.12).

Next we have

Theorem 4. *Let the function $f(z)$ be defined by* (1.12). *Then $f(z)$ belongs to the class $C(p, \alpha, \lambda)$ if and only if*

$$(2.14)\qquad \sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}\left\{\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha\right\}a_{p+n} < p(p-\alpha)$$

for $\lambda < 1$ *and* $0 \leqslant \alpha < p$, $p \in N$. *The result* (2.14) *is sharp for the function*

$$(2.15) \qquad f(z) = z^p - \frac{p(p-\alpha)}{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}\left\{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} - \alpha\right\}}\, z^{p+n} \quad (n \in N).$$

Corollary 2. *Let the function* $f(z)$ *defined by* (1.12) *belongs to the class* $C(p, \alpha, \lambda)$. *Then*

$$(2.16) \qquad a_{p+n} \leqslant \frac{p(p-\alpha)}{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}\left\{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} - \alpha\right\}} \qquad (n \in N).$$

The result (2.16) *is sharp for the functions* $f(z)$ *given by* (2.15).

3 - Distortion theorems

By virtue of the coefficient inequalities, we can prove the following distorsion theorems for functions $f(z)$ belonging to the classes $T^*(p, \alpha, \lambda)$ and $C(p, \alpha, \lambda)$.

Lemma 1. *The class* $T^*(p, \alpha, \lambda)$ *is closed under linear combinations.*

Proof. Let the functions

$$(3.1) \qquad f_i(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,\,i}\, z^{p+n} \qquad (a_{p+n,\,i} \geqslant 0;\ i = 1, 2)$$

be in the class $T^*(p, \alpha, \lambda)$. Then we need only prove that the function

$$F(z) = \mu f_1(z) + (1-\mu)\, f_2(z) \qquad (0 \leqslant \mu \leqslant 1)$$

is also in the class $T^*(p, \alpha, \lambda)$. In fact, we have

$$(3.2) \qquad F(z) = \mu f_1(z) + (1-\mu) f_2(z) = z^p - \sum_{n=1}^{\infty} \{\mu a_{p+n,\,1} + (1-\mu)\, a_{p+n,\,2}\}\, z^{p+n}.$$

Hence, with the aid of Theorem 3, we have

$$\sum_{n=1}^{\infty}\left\{\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha\right\}\{\mu a_{p+n,1}+(1-\mu)\,a_{p+n,2}\} \leq \mu(p-\alpha)+(1-\mu)(p-\alpha)=p-\alpha\,, \tag{3.3}$$

which completes the proof of Lemma 1.

By means of Lemma 1, we know that $T^*(p, \alpha, \lambda)$ is convex, and further that $T^*(p, \alpha, \lambda)$ has some extreme points.

Lemma 2. *Let*

$$f_p(z)=z^p \tag{3.4}$$

and

$$f_{p+n}(z)=z^p-\frac{(p-\alpha)}{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha}z^{p+n}\qquad(n\in N). \tag{3.5}$$

Then $f(z)$ is in the class $T^*(p, \alpha, \lambda)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=0}^{\infty}\mu_{p+n}f_{p+n}(z) \tag{3.6}$$

where $\mu_{p+n}\geq 0$ and $\sum_{n=0}^{\infty}\mu_{p+n}=1$.

Proof. We assume that $f(z)$ has the form (3.6). Then

$$f(z)=z^p-\sum_{n=1}^{\infty}\frac{(p-\alpha)\,\mu_{p+n}}{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha}z^{p+n}. \tag{3.7}$$

Consequently, we obtain

$$\sum_{n=1}^{\infty}\left\{\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha\right\}-\frac{(p-\alpha)\,\mu_{p+n}}{\left\{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha\right\}} \tag{3.8}$$
$$=(p-\alpha)\sum_{n=1}^{\infty}\mu_{p+n}=(p-\alpha)(1-\mu_p)\leqslant p-\alpha\,.$$

which implies that $f(z)\in T^*(p,\alpha,\lambda)$.

For the converse, we assume that $f(z)$ is in the class $T^*(p,\alpha,\lambda)$. Then by setting

$$\mu_{p+n}=\frac{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha}{p-\alpha}\,a_{p+n}\qquad(n\in N) \tag{3.9}$$

and

$$\mu_p=1-\sum_{n=1}^{\infty}\mu_{p+n} \tag{3.10}$$

we have (3.6).

Theorem 5. *The extreme points of the class* $T^*(p,\alpha,\lambda)$ *are the functions* $f_{p+n}(z)(n\geqslant 0)$ *given by* (3.5) *and* (3.6), *respectively.*

Similarly, we have

Theorem 6. *The extreme points of the class* $C(p,\alpha,\lambda)$ *are the functions*

$$f_p(z)=z^p \tag{3.11}$$

and

$$f_{p+n}(z)=z^p-\frac{p(p-\alpha)}{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}\left\{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha\right\}}\,z^{p+n}\;\;(n\in N). \tag{3.12}$$

Theorem 7. *Let the function* $f(z)$ *defined by* (1.12) *belong to the class* $T^*(p, \alpha, \lambda)$, *then*

$$|f(z)| \geqslant |z|^P - \frac{(p-\alpha)(p-\lambda)}{p(p+1)-\alpha(p-\lambda)} |z|^{P+1} \tag{3.13}$$

and

$$|f(z)| \leqslant |z|^P + \frac{(p-\alpha)(p-\lambda)}{p(p+1)-\alpha(p-\lambda)} |z|^{P+1} \tag{3.14}$$

for $z \in U$. *Furthermore, if* $0 \leqslant \lambda < 1$,

$$|f'(z)| \geqslant p|z|^{P-1} - \frac{(p+1)(p-\alpha)(p-\lambda)}{p(p+1)-\alpha(p-\lambda)} |z|^P \tag{3.15}$$

and

$$|f'(z)| \leqslant p|z|^{P-1} + \frac{(p+1)(p-\alpha)(p-\lambda)}{p(p+1)-\alpha(p-\lambda)} |z|^P \tag{3.16}$$

for $z \in U$. *The bounds* (3.13) *to* (3.16) *are sharp.*

Proof. Note that the extremal function is one of the extreme points. Therefore, we have

$$|f(z)| \geqslant |z|^P - \max_{n \geqslant 1} \left\{ \frac{p-\alpha}{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} - \alpha} |z|^{P+n} \right\} \tag{3.17}$$

and

$$|f(z)| \leqslant |z|^P + \max_{n \geqslant 1} \left\{ \frac{p-\alpha}{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} - \alpha} |z|^{P+n} \right\}. \tag{3.18}$$

Now define

$$\xi(\lambda,\alpha,n,p,|z|)=\frac{\Gamma(p)\,\Gamma(p+n-\lambda)\,|z|^{p+n}}{\Gamma(p+n+1)\,\Gamma(p-\lambda)-\alpha\Gamma(p)\,\Gamma(p+n-\lambda)} \tag{3.19}$$

for $-1\leqslant\lambda<1$, $0\leqslant\alpha<p$, $p\in N$, $n\in N$ and $z\in U$.

If

$$\xi(\lambda,\alpha,n,p,|z|)\geqslant\xi(\lambda,\alpha,n+1,p,|z|) \tag{3.20}$$

for $|z|\neq 0$, then we have the first half of the theorem. Set

$$\begin{aligned}\xi_1(\lambda,\alpha,n,p,|z|)&=\Gamma(p+n+1)\,\Gamma(p-\lambda)\{(p+n+1)-(p+n-\lambda)|z|\}\\&\quad-\alpha\Gamma(p)\,\Gamma(p+n+1-\lambda)(1-|z|).\end{aligned} \tag{3.21}$$

If $\xi_1(\lambda,\alpha,n,p,|z|)\geqslant 0$, then we have (3.20). We note that $\xi_1(\lambda,\alpha,n,p,|z|)$ is a decreasing function of α. This implies that

$$\begin{aligned}\xi_1(\lambda,\alpha,n,p,|z|)&\geqslant\xi_1(\lambda,p,n,p,|z|)\\&=\{\Gamma(p+n+2)\,\Gamma(p-\lambda)-\Gamma(p)\,\Gamma(p+n+1-\lambda)\}\\&\quad-\{(p+n-\lambda)\,\Gamma(p+n+1)\,\Gamma(p-\lambda)-\Gamma(p)\,\Gamma(p+n+1-\lambda)\}\,|z|.\end{aligned} \tag{3.22}$$

Since $\xi_1(\lambda,\alpha,n,p,|z|)$ is a decreasing function of $|z|$, we also have

$$\xi_1(\lambda,\alpha,n,p,|z|)\geqslant\xi_1(\lambda,p,n,p,1)=(1+\lambda)\,\Gamma(P+n+1)\,\Gamma(p-\lambda)\geqslant 0 \tag{3.23}$$

for $-1\leqslant\lambda<1$, $n\in N$ and $p\in N$.

In order to estabilsh the second half of Theorem 7, we note that

$$|f'(z)|\geqslant p|z|^{P-1}-\max_{n\in N}\left\{\frac{(p+n)(p-\alpha)}{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha}\,|z|^{P+n-1}\right\} \tag{3.24}$$

and

$$|f'(z)|\leqslant p|z|^{P-1}+\max_{n\in N}\left\{\frac{(p+n)(p-\alpha)}{\dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha}\,|z|^{P+n-1}\right\}. \tag{3.25}$$

Define

$$F(\lambda, \alpha, n, p, |z|) = \frac{(p+n)\,\Gamma(p)\,\Gamma(p+n-\lambda)\,|z|^{p+n-1}}{\Gamma(p+n+1)\,\Gamma(p-\lambda) - \alpha\Gamma(p)\,\Gamma(p+n-\lambda)} \tag{3.26}$$

and

$$\begin{aligned} F_1(\lambda, \alpha, n, p, |z|) &= \Gamma(p+n+2)\,\Gamma(p-\lambda)\{(p+n) - (p+n-\lambda)\,|z|\} \\ &\quad - \alpha\Gamma(p)\,\Gamma(p+n+1-\lambda)\{(p+n) - (p+n+1)\,|z|\} \end{aligned} \tag{3.27}$$

for $0 \leqslant \lambda < 1$, $0 \leqslant \alpha < p$, $p \in N$, $n \in N$, and $z \in U$. Since $F_1(\lambda, \alpha, n, p, |z|)$ is a decreasing function of α,

$$\begin{aligned} &F_1(\lambda, \alpha, n, p, |z|) \geqslant F_1(\lambda, p, n, p, |z|) \\ &= (p+n)\{\Gamma(p+n+2)\Gamma(p-\lambda) - \Gamma(p)\,\Gamma(p+n+1-\lambda)\} \\ &- (p+n-\lambda)\{\Gamma(p+n+2)\,\Gamma(p-\lambda) - \Gamma(p)(p+n+1)\,\Gamma(p+n-\lambda)\}\,|z|. \end{aligned} \tag{3.28}$$

Further, since $F_1(\lambda, p, n, p, |z|)$ is a decreasing function of $|z|$, we get

$$\begin{aligned} &F_1(\lambda, \alpha, n, p, |z|) \geqslant F_1(\lambda, p, n, p, 1) \geqslant \\ &\lambda\Gamma(p+n+2)\,\Gamma(p-\lambda) + \Gamma(p)\,\Gamma(p+n+1-\lambda) \geqslant 0 \end{aligned} \tag{3.29}$$

for $0 \leqslant \lambda < 1$, $n \in N$, and $p \in N$. Thus we have (3.15) and (3.16).

Finally, by taking the function

$$f(z) = z^p - \frac{(p-\alpha)(p-\lambda)}{p(p+1) - \alpha(p-\lambda)} z^{p+1} \tag{3.30}$$

we can show that all bounds given by Theorem 7 are sharp.

Corollary 3. *Let the function $f(z)$ defined by* (1.12) *be in the class $T^*(p, \alpha, \lambda)$ with $-1 \leqslant \lambda < 1$, $0 \leqslant \alpha < p$ and $p \in N$. Then the unit disc U is mapped into a domain that contains the disc*

$$|w| < \frac{p(1+\lambda)}{p(p+1) - \alpha(p-\lambda)}.$$

THEOREM 8. *Let the function $f(z)$ defined by* (1.12) *be in the class* $C(p, \alpha, \lambda)$. *Then, if* $-1 \leqslant \lambda < 1$,

$$|f(z)| \geqslant |z|^p - \frac{(p-\alpha)(p-\lambda)^2}{(p+1)\{p(p+1)-\alpha(p-\lambda)\}}|z|^{p+1} \tag{3.31}$$

and

$$|f(z)| \leqslant |z|^p + \frac{(p-\alpha)(p-\lambda)^2}{(p+1)\{p(p+1)-\alpha(p-\lambda)\}}|z|^{p+1} \tag{3.32}$$

for $z \in U$. *Furthermore, if* $0 \leqslant \lambda < 1$,

$$|f'(z)| \geqslant p|z|^{p-1} - \frac{(p-\alpha)(p-\lambda)^2}{p(p+1)-\alpha(p-\lambda)}|z|^p \tag{3.33}$$

and

$$|f'(z)| \leqslant p|z|^{p-1} + \frac{(p-\alpha)(p-\lambda)^2}{p(p+1)-\alpha(p-\lambda)}|z|^p \tag{3.34}$$

for $z \in U$. *The bounds* (3.31) *to* (3.34) *are sharp.*

PROOF By means of Theorem 6, we have

$$\begin{aligned} |f(z)| \geqslant\ & |z|^P \\ & - \max_{n \in N}\left\{\frac{(p-\alpha)}{\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p+1)\,\Gamma(p+n-\lambda)}\left\{\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha\right\}}\right\}|z|^{P+n} \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} |f(z)| \leqslant\ & |z|^P \\ & + \max_{n \in N}\left\{\frac{(p-\alpha)}{\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p+1)\,\Gamma(p+n-\lambda)}\left\{\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha\right\}}\right\}|z|^{P+n}. \end{aligned} \tag{3.36}$$

Let

$$H(\lambda, \alpha, n, p, |z|) = \left\{ \frac{(\Gamma(p+1)\,\Gamma(p+n-\lambda))^2\,|z|^{p+n}}{\Gamma(p+n+1)\,\Gamma(p-\lambda)\{p\Gamma(p+n+1)\,\Gamma(p-\lambda)-\alpha\Gamma(p+1)\,\Gamma(p+n-\lambda)\}} \right\} \tag{3.37}$$

and

$$\begin{aligned} H_1(\lambda,\alpha,n,p,|z|) &= \Gamma(p+n+1)\,\Gamma(p-\lambda)\{(p+n+1)^2-(p+n-\lambda)^2\,|z|\} \\ &\quad -\alpha\Gamma(p+1)\,\Gamma(p+n+1-\lambda)\{(p+n+1)-(p+n-\lambda)|z|\} \end{aligned} \tag{3.38}$$

for $-1 \leqslant \lambda < 1$, $0 \leqslant \alpha < p$, $n \in N$, $p \in N$ and $z \in U$. Then we know that $H(\lambda, \alpha, n, p, |z|)$ is a decreasing functions of n if

$$H_1((\lambda, \alpha, n, p, |z|) \geqslant 0\,,$$

in fact, since $H_1(\lambda, \alpha, n, p, |z|)$ is a decreasing function of α, $H_1(\lambda, p, n, p, |z|)$ is a decreasing function of $|z|$, we can prove that

$$\begin{aligned} &H_1(\lambda,\alpha,n,p,|z|) \geqslant H_1((\lambda,p,n,p,|z|) \geqslant H_1((\lambda,p,n,p,1) \\ &= p(1+\lambda)\{(2(p+n)+1-\lambda)\,\Gamma(p+n+1)\,\Gamma(p-\lambda)-\Gamma(p)\,\Gamma(p+n+1-\lambda)\} \geqslant 0 \end{aligned} \tag{3.39}$$

for $-1 \leqslant \lambda < 1$, $n \in N$ and $p \in N$. Consequently, we have the first half of Theorem 8.

Next, we note that

$$\begin{aligned} |f'(z)| &\geqslant p|z|^{P-1} \\ &- \max_{n\in N} \left\{ \frac{(p-\alpha)}{\dfrac{\Gamma(p+n)\,\Gamma(p-\lambda)}{\Gamma(p+1)\,\Gamma(p+n-\lambda)} \left\{ \dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} - \alpha \right\}} |z|^{P+n-1} \right\} \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} |f'(z)| &\leqslant p|z|^{P-1} \\ &+ \max_{n\in N} \left\{ \frac{(p-\alpha)}{\dfrac{\Gamma(p+n)\,\Gamma(p-\lambda)}{\Gamma(p+1)\,\Gamma(p+n-\lambda)} \left\{ \dfrac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} - \alpha \right\}} |z|^{P+n-1} \right\}. \end{aligned} \tag{3.41}$$

Define

$$G(\lambda, \alpha, n, p, |z|) = \frac{(\Gamma(p+1))^2(\Gamma(p+n-\lambda))^2 |z|^{p+n-1}}{\Gamma(p+n)\,\Gamma(p-\lambda)\{p\Gamma(p+n+1)\,\Gamma(p-\lambda) - \alpha\Gamma(p+1)\,\Gamma(p+n-\lambda)\}} \tag{3.42}$$

and

$$\begin{aligned} G_1(\lambda, \alpha, n, p, |z|) &= p\Gamma(p+n+1)\,\Gamma(p-\lambda)\{(p+n)(p+n+1) \\ &-(p+n-\lambda)^2|z|\} - \alpha\Gamma(p+1)\,\Gamma(p+n+1-\lambda)\{(p+n)-(p+n-\lambda)|z|\} \end{aligned} \tag{3.43}$$

for $0 \leqslant \lambda < 1$, $0 \leqslant \alpha < p$, $p \in N$, $n \in N$ and $z \in U$. Then it is sufficient to prove that

$$G_1(\lambda, \alpha, n, p, |z|)\, p \geqslant 0 \,.$$

Note that $G_1(\lambda, \alpha, n, p, |z|)$ is a decreasing function of α, and $G_1(\lambda, p, n, p, |z|)$ is a decreasing function of $|z|$. Thus we have

$$\begin{aligned} G_1(\lambda, \alpha, n, p, |z|) &\geqslant G_1(\lambda, p, n, p, |z|) \geqslant G_1(\lambda, p, n, p, 1) \\ &= p\{(p+n) + 2\lambda(p+n) - \lambda^2\}\,\Gamma(p+n+1)\,\Gamma(p-\lambda) \\ &\quad -\lambda\Gamma(p+1)\,\Gamma(p+n+1-\lambda) \geqslant 0 \end{aligned} \tag{3.44}$$

for $0 \leqslant \lambda < 1$, $n \in N$, and $p \in N$ which implies the second half of Theorem 8. Finally, all bounds asserted by Theorem 8 are sharp for the function

$$f(z) = z^p - \frac{(p-\alpha)(p-\lambda)^2}{(p+1)\{p(p+1) - \alpha(p-\lambda)\}}\, z^{p+1}. \tag{3.45}$$

Corollary 4. *Let the function $f(z)$ defined by* (1.12) *be in the class $C(p, \alpha, \lambda)$ with $-1 \leqslant \lambda < 1$, and $0 \leqslant \alpha < p$. Then the unit disc U is mapped into a domain that contains the disc $|w| < r_0$, where r_0 is given by*

$$r_0 = 1 - \frac{(p-\alpha)(p-\lambda)^2}{(p+1)\{p(p+1) - \alpha(p-\lambda)\}}\,. \tag{3.46}$$

4 - Starlikeness and convexity

Owa [5] proved the following lemmas.

Lemma 3. *Let the function $f(z)$ be defined by* (1.12). *Then $f(z)$ is p-valent starlike of order α if and only if*

$$\sum_{n=1}^{\infty}(p+n-\alpha)\,a_{p+n}\leqslant p-\alpha \tag{4.1}$$

for $0\leqslant\alpha<p$.

Lemma 4. *Let the function $f(z)$ be defined by* (1.12). *Then $f(z)$ is p-valent convex of order α if and only if*

$$\sum_{n=1}^{\infty}(p+n)(p+n-\alpha)\,a_{p+n}\leqslant p(p-\alpha) \tag{4.2}$$

for $0\leqslant\alpha<p$.

By applying the above lemmas, we now prove

Theorem 9. *Let the function $f(z)$ defined by* (1.12) *be in the class $T^*(p,\alpha,\lambda)$ with $0\leqslant\lambda<1$ and $0\leqslant\alpha<p$. Then $f(z)$ is p-valent starlike of order α.*

Proof. Note that

$$p+n\leqslant\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} \tag{4.3}$$

for $0\leqslant\lambda<1$, $0\leqslant\alpha<p$, $n\in N$ and $p\in N$. This shows that

$$\sum_{n=1}^{\infty}(p+n-\alpha)\,a_{p+n}\leqslant\sum_{n=1}^{\infty}\left\{\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)}-\alpha\right\}a_{p+n}\leqslant p-\alpha\,, \tag{4.4}$$

and we complete the proof of Theorem 9 in view of (4.1).

Similarly, by using Lemma 4, we have

Theorem 10. *Let the function $f(z)$ defined by* (1.12) *be in the class $C(p,\alpha,\lambda)$ with $0\leqslant\lambda<1$ and $0\leqslant\alpha<p$. Then $f(z)$ is p-valent convex of order α.*

Remark 3. For $\lambda=0$, the classes $T^*(p,\alpha,\lambda)$ and $C(p,\alpha,\lambda)$ reduce to the class $T^*(p,\alpha)$ and $C(p,\alpha)$, respectively, which were introduced by Owa [5]. It follows that

$$T^*(p,\alpha,0)=T^*(p,\alpha) \tag{4.5}$$

and

$$C(p,\alpha,0)=C(p,\alpha)\,. \tag{4.6}$$

Hence, by means of Theorem 9 and Theorem 10, we have

$$T^*(p, \alpha, \lambda) \subset T^*(p, \alpha, 0) \quad (0 \leqq \lambda < 1) \tag{4.7}$$

and

$$C(p, \alpha, \lambda) \subset C(p, \alpha, 0) \quad (0 \leqq \lambda < 1). \tag{4.8}$$

Since

$$\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} - \alpha \leqq p + n - \alpha \tag{4.9}$$

and

$$\frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} \left\{ \frac{\Gamma(p+n+1)\,\Gamma(p-\lambda)}{\Gamma(p)\,\Gamma(p+n-\lambda)} - \alpha \right\} \leqq (p+n)(p+n-\alpha) \tag{4.10}$$

for $\lambda < 0$, $0 \leqq \alpha < p$, $p \in N$ and $n \in N$, we also have

$$T^*(p, \alpha, \lambda) \supset T^*(p, \alpha, 0) \quad (\lambda < 0) \tag{4.11}$$

and

$$C(p, \alpha, \lambda) \supset C(p, \alpha, 0) \quad (\lambda < 0). \tag{4.12}$$

References

[1] A. ERDE'LYI, W. MAGNUS, F. OBERHETTINGER and F. G. TRICOMI, *Tables of integral transforms* II, McGraw-Hill Book Co. New York, London, Toronto 1954.

[2] K. NISHIMOTO, *Fractional derivative and integral* I, J. College Engrg. Nihon Univ. Ser. B **17** (1976), 11-19.

[3] T. J. OSLER, *Leibniz rule for fractional derivatives generalized and an application to infinite series*, SIAM J. Appl. Math. **18** (1970), 658-674.

[4] S. OWA, *On the distortion theorems* I, Kyungpook Math. J. **18** (1978), 53-59.

[5] S. OWA, *On certain classes of p-valent functions with negative coefficients*, Simon Stevin **59** (1985), 385-402.

[6] S. OWA, *On new subclasses of analytic and univalent functions*, J. Korean Math. Soc. **22** (1985), 151-172.

[7] D. A. PATIL and N. K. THAKARE, *On convex hulls and extreme points of p-valent starlike and convex classes with applications*, Bull. Math. Soc. Sci. Math. R.S. Roumaine **27** (1983), 145-160.

[8] B. ROSS, *A brief history and exposition of the fundamental theory of fractional calculus* in *Fractional Calculus and Its Applications* (B. Ross ed., Springer-Verlag, Berlin, Heidelberg, New York 1975.

[9] M. SAIGO, *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. Kyushu Univ. **11** (1977/1978), 135-143.

[10] H. SILVERMAN, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109-116.

[11] H. M. SRIVASTAVA and R. G. BUSCHMAN, *Convolution integral equations with special function kernels*, J. Wiley and Sons, New York, Sydney 1977.

[12] H. M. SRIVASTAVA and S. OWA, *An application of the fractional derivative*, Math. Japon. **29** (1984), 383-389.

[13] H. M. SRIVASTAVA, S. OWA and K. NISHIMOTO, *A note on a certain class of fractional differintegrtal equations*, J. College Engrg. Nihon Univ. **25** (1984), 69-73.

[14] H. M. SRIVASTAVA, S. OWA and K. NISHIMOTO, *Some fractional differintegral equations*, J. Math. Anal. Appl. **106** (1985), 360-366.

Summary

The object of the present paper is to derive several interesting properties of the classes $T^*(p, \alpha, \lambda)$ *and* $C(p, \alpha, \lambda)$ *consisting of analytic and p-valent functions with negative coefficients.*

* * *