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Roger Yue Chi Ming (*)

On injectivity and p-injectivity, III (**)

Introduction

In 1974, we introduced p-injective modules [**15**] to study von Neumann regular rings, V-rings and their generalizations.

M. Auslander characterized von Neumann regular rings as absolutely flat rings in the sense that all modules (left or right) are flat. Similarly, we may say that a von Neumann regular ring A is absolutely p-injective since all A-modules (left or right) are p-injective [**15**]. Flatness and p-injectivity are distinct concepts (cf. [**20**], Example). As an analogy to the study of injective modules over non-semisimple Artinian rings, it is interesting to consider p-injective A-modules when A is not von Neumann regular. In particular, the class of p-injective rings which include von Neumann regular rings, quasi-Frobeniusean rings, pseudo-Frobeniusean rings and the maximal quotient rings of non-singular rings. In [**17**], pseudo-Frobeniusean rings are characterized in term of p-injective rings. Our purpose here is to characterize quasi-Frobeniusean rings in term of p-injectivity. Note that both left pseudo-Frobeniusean and quasi-Frobeniusean rings are left and right p-injective rings.

Throughout, A denote an associative ring with identity and A-modules are unital. Recall that:

(*) Université Paris VII-Denis Diderot, UFR de Mathematiques, UMR 9994 CNRS, 2, Place Jussieu, 75251 Paris cedex 05, France.

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(i) A left A-module M is divisible if for every non-zero-divisor c of A, $M=cM$.

(ii) A ring Q is called a classical left (resp. right) quotient ring of A if: (a) $A \subseteq Q$; (b) every non-zero divisor in A is invertible in Q; (c) every element q of Q is of the form $q = b^{-1}a$ (resp. ab^{-1}), where $a, b \in A$, b being a non-zero-divisor. As usual, an ideal of A will always mean a two-sided ideal of A. As in [**15**], [**16**], a left A-module M is called p-injective (resp. f-injective) if for any principal (resp. finitely generated) left ideal I of A, every left A-homomorphism of I into M extends to one of ${}_AA$ into M. ${}_AM$ is p-injective if for every principal left ideal P of A, $\mathrm{Ext}_A^1(A/p, M) = 0$. Since several years, p-injectivity (also noted principal injectivity) has drawn the attention of many authors (cf. for example, [**1**]-[**3**], [**5**]$\rightarrow$[**21**]).

A is called left p-injective if ${}_AA$ is p-injective.

Proposition 1. *Let* A *be a left p-injective ring.*

Then

(1) *Every non-zero-divisor is invertible in A*;

(2) *Every left (right) A-module is divisible.*

Proof. (1) Let c be a non-zero-divisor of A. Define a map $f : Ac \rightarrow A$ by $f(ac) = a$ for all $a \in A$. Then f is a well-defined left A-homomorphism and since ${}_AA$ is p-injective, there exist $d \in A$ such that $f(ac) = acd$ for all $a \in A$. In particular, $f(c) = cd$ which yields $cd = 1$. Then $cdc = c$ implies $c(dc - 1) = 0$, whence $dc = 1$. Then c is invertible in A.

(2) For any left A-module M and any non-zero-divisor c of A, $M = AM = cAM = cM$. Similarly, every right A-module is divisible.

The next corollary then follows immediately.

Corollary 2. *If A is left p-injective, then the classical left (or right) quotient ring of A coincides with A.*

A left A-module M is called FP-injective if for every finitely presented left A-module F, $\mathrm{Ext}_A^1(F, M) = 0$. A is called left (self) FP injective if ${}_AA$ is FP-injective. If ${}_AM$ is FP-injective, then ${}_AM$ is f-injective [5], p. 121 and consequently ${}_AM$ is p-injective.

Let A be now commutative for the next two results. A is called *FSFPI* (fractionally self FP-injective) if for every ideal I of A, the classical quotient ring of A/I, noted $Q(A/I)$, is self FP-injective [5], p. 124. Similarly, A is called fractionally p-injective if for every ideal I of A, $Q(A/I)$ is p-injective.

Proposition 3. *The following conditions are equivalent for a commutative ring A*:

(1) *For every ideal I of A, the factor ring A/I is self FP-injective*;
(2) *For every ideal I of A, the factor ring A/I is f-injective*;
(3) *For every ideal I of A, the factor ring A/I is p-injective.*

Proof. Obviously, $(1)\Rightarrow(2)\Rightarrow(3)$.

Assume (3). For any ideal I of A, $Q(A/I)=A/I$ by Corollary 2.

Thus $Q(A/I)$ is p-injective and therefore A is fractionally p-injective. Then A is a FSFPI ring [5], p. 310, whence A/I is self FP-injective. Thus (3) implies (1).

Recall that quasi-Frobeniusean rings are left and right Artinian, left and right self-injective rings whose left (right) ideals are annihilators.

An example of a non-trivial quasi-Frobeniusean ring.

Let K be a field, R the commutative K-algebra with the basis 1, a, b, c and the multiplication $1r=r1=r$ for $r\in R$, $ab=ba=0$, $a^2=b^2=c$, $ac=ca=bc=cb=c^2=0$. If J denotes the Jacobson radical of R, then $J^2=\mathrm{Soc}(R)=cR$ and R is a quasi-Frobeniusean ring but the factor ring R/J^2 is not quasi-Frobeniusean.

(cf. F. KASCH, *Modules and rings*, London Math. Soc. Monograph **17** (1982), 362-363).

We know that a commutative ring A is quasi-Frobeniusean iff A is f-injective with maximum condition on annihilators [16], Corollary 2.

The next result then follows.

Corollary 4. *The following conditions are equivalent for a commutative ring A*:

(1) *Every factor ring of A is quasi-Frobeniusean*;
(2) *Every factor ring of A is p-injective with maximum condition on annihilators.*

Corollary 4 is not a surprising result since for the class of rings considered therein, injectivity coincides with p-injectivity ([4], Proposition 25.4.6B).

We mention another result on factor rings.

A is called fully idempotent if every ideal of A is idempotent. As before, the ring A is called a left (resp. right) MI-ring if A contains an injective maximal left (resp. right) ideal ([**19**]). Left MI-rings need not be right MI.

PROPOSITION 5. *The following conditions are equivalent for a ring A*:

(1) *Every factor ring of A is a left self-injective regular ring with non-zero socle.*

(2) *Every factor ring of A is semi-prime left MI.*

PROOF. (1) implies (2) evidently.

Assume (2). Then A is a fully idempotent ring. Let B be a prime factor ring of A. Since B is prime left MI, then B is left self-injective by [20], Theorem 6 and has non-zero socle. Then $Z(B)$, the left singular ideal of B, must be zero (otherwise, $Z(B)$ would contain a non-zero idempotent which is impossible). Since B is left self-injective, then B is von Neumann regular. Therefore A is fully idempotent such that every prime factor ring is von Neumann regular which implies A von Neumann regular by a result of J. W. Fisher-R. L. Snider. Now every factor ring is regular left MI and hence (2) implies (1) by [20], Theorem 6.

A left p-injective left Noetherian ring is left perfect and hence left Artinian. But left p-injective right Noetherian rings need not be right Artinian. However, the following result holds.

PROPOSITION 6. *If A is a right Noetherian ring whose factor rings are left p-injective, then A is right Artinian.*

PROOF. Let B be a prime factor ring of A. Then B is a left p-injective right Noetherian ring. Since B is left p-injective, by Corollary 2, $Q(B)$, the classical right quotient ring of B, coincides with B. But $Q(B)$ is simple Artinian by a well-known theorem of A.W Goldie. Therefore B is simple Artinian.

If A is prime, then A is simple Artinian as just shown. If A is not prime, then every proper prime factor ring of A is simple Artinian and by [5], Theorem 2.19B, A is right Artinian. In any case, A is right Artinian.

QUESTION. Is a right Noetherian left f-injective ring right Artinian?

This paper is motivated by a letter from V. A. Hiremath about classical quotient rings and p-injectivity. In this direction, we may add a last remark inspired by [5], Theorem 5.43: A is quasi-Frobeniusean iff A is a right p-injective right FPF ring with maximum condition on right annihilators.

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Summary

In this sequel to [**18**], [**19**], *it is shown that the following conditions are equivalent for a commutative ring A:* (1) *Every factor ring of A is quasi-Frobeniusean;* (2) *Every factor ring of A is a p-injective ring with maximum condition on annihilators. Also, the following conditions are equivalent for any ring A:* (1) *Every factor ring of A is left self-injective regular with non-zero socle;* (2) *Every factor ring of A is a semi-prime ring with an injective maximal left ideal. A right Noetherian ring whose factor rings are left p-injective must be right Artinian.*

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