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Inequalities for solutions to advection-diffusion problems (**)

dedicated to the memory of Giulio Di Cola

1 - Basic assumptions and equations

Let a passive tracer be released istantaneously at a given point of a semi-enclosed marine basin and subsequently dispersed by sea motions. It is assumed that the time and space distribution $u(t, \boldsymbol{x})$ of the tracer satisfies an advection-diffusion problem. This paper is addressed to derive some inequalities for the distribution $u(t, \boldsymbol{x})$, satisfying homogeneous boundary conditions expressed in terms of the total flux of the tracer. An inequality of Friedrich's second type ([**1**] p. 124, [**7**] p. 20) is obtained, which holds for any time t (see equation (11)). From this inequality we get an upper bound of the total quantity of the tracer in the basin in terms of a negative exponential in t (see equations (12) and (17)). These inequalities are used in particular in proving that the residence time ([**2**], [**3**]) of the tracer in the basin has a finite value.

Let a semi-enclosed water basin be represented by a bounded, open, connected set $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ (figure 1). Let $\Gamma = \Gamma_s \cup \Gamma_f$ be the boundary of Ω, assumed sufficiently smooth, where Γ_s and Γ_f are the solid and fluid boundaries respectively. Let $u(t, \boldsymbol{x})$ be the distribution of the passive tracer. The evolution

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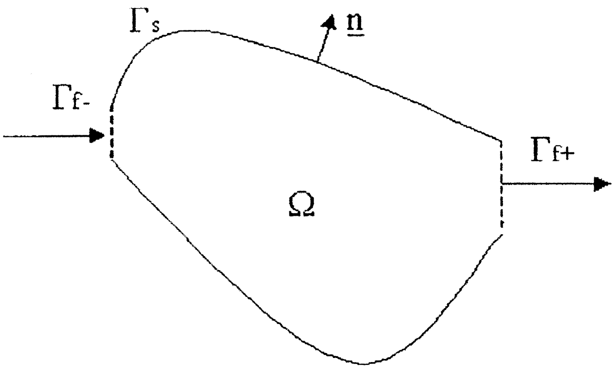

FIGURE 1 - Scheme of an horizontal section of a basin

of the dispersion process is described by the following initial boundary value problem:

$$(1)\qquad \begin{cases} \dfrac{\partial hu}{\partial t} + \nabla\cdot h(-A\nabla + \boldsymbol{b})\, u = 0 & \text{in } \Omega_T = (0, T] \times \Omega \\ (-A\nabla u + \boldsymbol{b}u)\cdot\boldsymbol{n} = b^* u & \text{on } \Gamma_T = (0, T] \times \Gamma \\ u(0, \boldsymbol{x}) = u_0(\boldsymbol{x}) \geqslant 0 & \text{in } \overline{\Omega} = \Omega \cup \Gamma \end{cases}$$

where $h(t, \boldsymbol{x})$ depends on the type of problem (see section **3** and **4**) and satisfies

$$(2)\qquad 0 < h_{\min} \leqslant h(t, \boldsymbol{x}) \leqslant h_{\max},$$

$A(\boldsymbol{x})$ is the eddy diffusivity matrix, assumed diagonal with $\operatorname{diag} A(\boldsymbol{x}) = (a_1(\boldsymbol{x}), \ldots, a_d(\boldsymbol{x}))$ and $a_i(\boldsymbol{x}) > 0$, $\boldsymbol{b}(t, \boldsymbol{x}) = (b_1(t, \boldsymbol{x}), \ldots, b_d(t, \boldsymbol{x}))$ is the large scale mean velocity field, with $\boldsymbol{b}\cdot\boldsymbol{n} = 0$ on Γ_s, $b^*(t, \boldsymbol{x})$ is a parameter regulating the flux of the tracer between the basin and the open sea. Here $\boldsymbol{n}$ denotes the outward unit normal vector on the boundary Γ.

In the following we will consider dispersion processes in two different hydrodynamics scenarios: basins with unidirectional flows, for which the sign of $\boldsymbol{b}\cdot\boldsymbol{n}$ on Γ_f is time independent, and flows forced by the tidal motion, for which the the sign of $\boldsymbol{b}\cdot\boldsymbol{n}$ on a part of Γ_f is periodical in time. Let $\Gamma_f = \Gamma_{f-} \cup \Gamma_{f+}$ where:

(i) Γ_{f-} is the inflow part of Γ_f, $\boldsymbol{b}\cdot\boldsymbol{n} < 0$ on Γ_{f-};

(ii) Γ_{f+} is the outflow part of Γ_f, $\boldsymbol{b}\cdot\boldsymbol{n} > 0$ on Γ_{f+}, for unidirectional flows;

(iii) Γ_{f+} is the inflow-outflow part of Γ_f for flows forced by tidal motion.

The boundary condition in (1) is illustrated in [2], [3]; here we assume

$$b^*(t, \boldsymbol{x}) = 0 \text{ on } \Gamma_s \cup \Gamma_{f-}, \quad b^*(t, \boldsymbol{x}) \geq \max(0, \boldsymbol{b}(t, \boldsymbol{x})\cdot \boldsymbol{n}) \text{ on } \Gamma_{f+}. \tag{3}$$

Some basic assumptions are necessary to assure the existence, uniqueness, positivity, boundedness and regularity of the solution of (1). We assume that, for $\alpha \in (0, 1)$, Γ is of class $\mathcal{C}^{2+\alpha}$, $u_0(\boldsymbol{x}) \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$, $a_i \in \mathcal{C}^{1+\alpha}(\overline{\Omega})$, $h(t, \boldsymbol{x}) \in \mathcal{C}^{1+\alpha/2, 2+\alpha}([0, T]\times\overline{\Omega})$, $b_i(t, \boldsymbol{x}) \in \mathcal{C}^{1/2+\alpha/2, 1+\alpha}([0, T]\times\overline{\Omega})$, $b^* \in \mathcal{C}^{1/2+\alpha/2, 1+\alpha}([0, T] \times \overline{\Omega})$. Moreover we must assume that the compatibility condition holds: $(-A\nabla u_0 + \boldsymbol{b}u_0)\cdot\boldsymbol{n} = b^* u_0$ for $t = 0$ and $\boldsymbol{x} \in \partial\Omega$. Under these assumptions, we can state that problem (1) has a unique solution, in the sense that u has continuous partial derivatives u_t, u_{x_i} and $u_{x_i x_j}$ and satisfies equation (1) for every $(t, \boldsymbol{x}) \in \Omega_T$. The boundary and initial conditions are also satisfied in the pointwise sense. Hence the solution necessarily lies in $\mathcal{S} = \mathcal{C}^{1+\alpha/2, 2+\alpha}([0, T] \times \overline{\Omega})$ ([**10**]). More smooth properties of the solution can be obtained by assuming additional conditions on the coefficients of the differential operator and the given data. Since $M = \max_{\boldsymbol{x}\in\overline{\Omega}} u_0(\boldsymbol{x}) > 0$ and 0 are ordered upper and lower solutions for (1), the solution u is positive and bounded, in particular holds: $M \geq u \geq 0$ (for more details see [**9**]).

2 - Global balance equations

For any solution $u(t, \boldsymbol{x}) \in \mathcal{S}$ let we define

$$\begin{aligned} f_i(t) &= \int_\Omega h u^i \, d\Omega, \qquad i = 1, 2, \\ g_1(t) &= \int_\Gamma h(-A\nabla u + \boldsymbol{b}u)\cdot\boldsymbol{n}\, d\Gamma, \\ g_2(t) &= 2\int_\Omega h(A\nabla u\cdot\nabla u)\, d\Omega + 2\int_\Gamma hu\left(-A\nabla u + \frac{1}{2}\boldsymbol{b}u\right)\cdot\boldsymbol{n}\, d\Gamma. \end{aligned} \tag{4}$$

Taking into account the boundary conditions in (1) and the second equation in (3) we obtain

$$(5)\qquad \begin{aligned} g_1(t) &= \int_{\Gamma_f} hb^* u\, d\Gamma\,, \\ g_2(t) &= 2\int_{\Omega} h(A\nabla u\cdot\nabla u)\, d\Omega + 2\int_{\Gamma_f} hu^2\Big(b^* - \frac{1}{2}\boldsymbol{b}\cdot\boldsymbol{n}\Big)\, d\Gamma\,. \end{aligned}$$

From (2), (3) and the positivity of u it follows that $g_i(t) \geqslant 0$.

By integrating the balance equation in (1) we obtain

$$(6)\qquad \frac{df_1}{dt} + g_1 = 0\,.$$

By integrating the balance equation in (1) multiplied by u we obtain

$$(7)\qquad \frac{df_2}{dt} + g_2 + q = 0\,,$$

where

$$(8)\qquad q(t) = \int_{\Omega} u^2\Big(\frac{\partial h}{\partial t} + \nabla\cdot h\boldsymbol{b}\Big)\, d\Omega\,.$$

For the velocity fields considered in this paper, see sections **3** and **4**, we have that $q(t) = 0$. In these sections we obtain a lower bound for $g_2(t)$ in terms of $f_2(t)$; thus we are able to obtain upper bounds for $f_1(t)$ and $f_2(t)$. We recall that the residence time of the tracer in the basin is defined by $\int_0^\infty f_1\, dt$ ([**2**], [**3**]). Thus, an upper bound of $f_1(t)$ can be used to estimate an upper bound of the residence time.

3 - Unidirectional flows at the fluid boundaries

In the three dimensional case, $d = 3$, we use the rigid lid approximation for the air-sea interface ([**5**], p. 7); thus, this surface is included in the boundary Γ_s. In this section the flow is characterized by the following properties:

– the sign of $\boldsymbol{b}\cdot\boldsymbol{n}$ on Γ_f is time independent;
– the field $\boldsymbol{b}$ is divergence free,

$$(9)\qquad \nabla\cdot\boldsymbol{b} = 0 \quad \text{in } \Omega$$

so that $\int_{\Gamma_f} \boldsymbol{b}\cdot\boldsymbol{n}\, d\Gamma = 0$.

In the case of unidirectional flows, when (9) holds true, the function h is constant; therefore, from (8), (9) it follows $q(t) = 0$.

Theorem 1. *Assume*

$$\frac{1}{2}\,|\boldsymbol{b}(t, \boldsymbol{x})\cdot \boldsymbol{n}| \geq \beta(\boldsymbol{x}) \geq 0 \quad on\ \Gamma, \tag{10}$$

where $\beta = 0$ *on* Γ_s *and* $\beta > 0$ *on* Γ_f. *Then, for any solution* $u(t, \boldsymbol{x})$ *to* (1) *there exist positive constants* λ_0, c_i, μ_i, $i = 1, 2$, *such that the following inequalities hold:*

$$g_2(t) \geq 2\lambda_0 f_2(t), \tag{11}$$

$$f_i(t) \leq c_i e^{-\mu_i t}, \quad i = 1, 2. \tag{12}$$

Proof. Let us define the functional

$$p(v) = \int_\Omega A\nabla v\cdot \nabla v\, d\Omega + \int_\Gamma \beta v^2\, d\Gamma, \qquad v \in H^1(\Omega).$$

We have that

$$p(v) \geq \lambda_0 \int_\Omega v^2\, d\Omega \tag{13}$$

where $\lambda_0 > 0$ is the minimum eigenvalue of the selfadjoint problem:

$$\begin{cases} -\nabla\cdot(A\nabla v) = \lambda v & \text{in } \Omega \\ (-A\nabla v)\cdot \boldsymbol{n} = \beta v & \text{on } \Gamma. \end{cases} \tag{14}$$

From (3) and (10)

$$b^* - \frac{1}{2}\boldsymbol{b}\cdot\boldsymbol{n} \geq \frac{1}{2}\,|\boldsymbol{b}\cdot\boldsymbol{n}| \geq \beta. \tag{15}$$

Since for any fixed t any solution $u(t, \boldsymbol{x})$ to (1) belongs to $\mathcal{C}^{2+\alpha}(\overline{\Omega}) \subset H^1(\Omega)$, from (5), (15) we have that

$$g_2(t) \geq 2p(u(t, \cdot)).$$

Thus, by the monotonicity principle ([**11**], p. 62) it follows (11).

From (6), (7), (11) and taking into account that

$$f_1(t) \leqslant \sqrt{\left[f_2(t) \int_\Omega d\Omega \right]}$$

we obtain (12) with

$$c_1 = \sqrt{\int_\Omega d\Omega \int_\Omega u_0^2 \, d\Omega}\,, \tag{16}$$
$$c_2 = \int_\Omega u_0^2 \, d\Omega\,, \ \mu_1 = \lambda_0, \ \mu_2 = 2\lambda_0 .$$

4 - Flows forced by the tidal motion

The analysis of this situation is performed by assuming the shallow water approximation, thus $d = 2$ ([4]). Here $h(t, \boldsymbol{x})$ represents the total depth of the basin and it is defined by $h(t, \boldsymbol{x}) = h_0(\boldsymbol{x}) + \eta(t, \boldsymbol{x})$ where $z = -h_0(\boldsymbol{x})$ represents the bottom surface and $z = \eta(t, \boldsymbol{x})$ defines the sea surface with respect to the same horizontal reference level. We assume that $|\eta(t, \boldsymbol{x})| \ll h_0(\boldsymbol{x})$; thus, the inequalities (2) hold. In the shallow water approximation the velocity field $\boldsymbol{b}$ represents averaged values over the depth of the basin $(-h_0, \eta)$ and h satisfies the continuity equation ([8], p. 45)

$$\frac{\partial h}{\partial t} + \nabla \cdot h\boldsymbol{b} = 0 \quad \text{in } \Omega .$$

Thus, from (8) it follows $q(t) = 0$.

In this section we consider a basin where the flow is mainly forced by the tidal motion through Γ_{f+} and by a possible inflow (e.g. a river) through Γ_{f-}. The velocity field at Γ_{f+} is periodic in time with period θ, and we assume that the tracer leaves the basin under the action of advection during the outflow periods (t_{2j+1}, t_{2j}), and its flux is zero during the inflow periods (t_{2j}, t_{2j+1}), where $t_j = t_0 + j\frac{\theta}{2}$, $j = 0, 1, 2, \ldots$. Thus, the boundary conditions on Γ_{f+} are written as

$$(\boldsymbol{b}u - A\nabla u)\cdot\boldsymbol{n} = 0 \ \text{ when } \boldsymbol{b}\cdot\boldsymbol{n} \leqslant 0, \quad t \in (t_{2j}, t_{2j+1});$$

$$-A\nabla u \cdot \boldsymbol{n} = 0 \ \text{ when } \boldsymbol{b}\cdot\boldsymbol{n} > 0, \quad t \in (t_{2j+1}, t_{2j+2}).$$

Problem (1) is solved in each internal (t_j, t_{j+1}) with the appropriate boundary conditions and assuming the continuity of $u(t, \boldsymbol{x})$ at times t_j. Note that, owing to this rule of changing the boundary condition on Γ_f, the time derivative of $u(t, \boldsymbol{x})$ is discontinuous at times t.

Theorem 2. *Assume that the velocity field can be written as*

$$\boldsymbol{b}(t, \boldsymbol{x}) = \overline{\boldsymbol{b}}(\boldsymbol{x}) + \boldsymbol{b}'(t, \boldsymbol{x})$$

where $\overline{\boldsymbol{b}}(\boldsymbol{x})$ *is the contribution of the stationary forcings and* $\boldsymbol{b}'(t, \boldsymbol{x})$ *is the field induced by the tide, which is periodic of period* θ.

Then,

(i) *the average* $\overline{u}(t, \boldsymbol{x})$ *of* $u(t, \boldsymbol{x})$ *over a period* θ,

$$\overline{u}(t, \boldsymbol{x}) = \frac{1}{\theta} \int_t^{t+\theta} u(t', \boldsymbol{x})\, dt',$$

satisfies a problem of type (1) *with an effective eddy diffusivity* A_{eff} [**6**], *the velocity field* $\overline{b}(\boldsymbol{x})$ *and a* $b^*_{\text{eff}}(\boldsymbol{x})$ *on* Γ_f *given by*

$$b^*_{\text{eff}}(\boldsymbol{x}) = \frac{1}{2} \text{ average of } b^*(t, \boldsymbol{x}) \text{ over the tide semiperiod of outflow};$$

(ii) $u(t, \boldsymbol{x})$ *satisfies the inequality*

$$f_1(t) \leqslant c_1 e^{\mu_1 \theta} e^{-\mu_1 t}. \tag{17}$$

Proof. For the proof of part (i) see [**4**], [**6**].

Part (ii). From theorem 1 and part (i) it follows that $\overline{u}(t, \boldsymbol{x})$ satisfies inequalities (12) with c_i, μ_i given by (16), where λ_0 is now the minimum eigenvalue of (14) with $\beta = b^*_{eff}$ on Γ.

Taking into account that $\frac{df_i}{dt} \leqslant 0$, we have that

$$c_1 e^{-\mu_1 t} \geqslant \int_\Omega \overline{u}(t, \boldsymbol{x})\, d\Omega = \frac{1}{\theta} \int_t^{t+\theta} f_i(t')\, dt' \geqslant f_1(t+\theta),$$

and then the inequality (17).

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Abstract

We consider the dispersion process of a tracer istantaneously released in a semi-enclosed water basin. It is assumed that the space and time distribution of the tracer is solution to an advection-diffusion problem. We derive some inequalities for this distribution, which allow to prove that the residence time of the tracer in the basin has a finite value.

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