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PIROSKA LAKATOS (*)

On a number theoretical application of Coxeter transformations (**)

1 - Introduction and Notation

Recall that a real algebraic integer $\alpha > 1$ is called *PV* (Pisot-Vijayaraghavan) *number* if all its conjugates lie inside the unit circle and α is called a *Salem number* if it has all but one of its conjugates on the unit circle. The monic irreducible polynomial over $\boldsymbol{Q}$ having a Salem number as a zero is called a *Salem polynomial*.

Salem in [8] has shown that each PV-number is the limit of Salem numbers. We show that the spectral radii of the Coxeter transformation of wild stars are Salem numbers and their suitable limits are PV-numbers. The link between these limits and PV-numbers is the polynomial $F(x) = x^{k+1} - (s-1)\dfrac{x^{k+1}-1}{x-1}$. The fact that the largest positive zero η of F is a PV-number follows from the theory of the first derived set of PV-numbers given in Chapter 6 in [**1**]. We give an *elementary proof* of this result and also show how η can be located:

$$s - s^{-k} < \eta < s - s^{-k-1}$$

where $s+1$ is the number of all arms of the wild star and k is the length of the arm that remains fixed during the limiting process.

(*) Institute of Mathematics and Informatics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary.

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From this it follows that each integer is a limit of a sequence of PV-numbers and the elements of this sequence (of PV-numbers) are limits of special Salem numbers, namely spectral radii of wild stars.

Let Δ be a tree, i.e. a finite non-oriented connected graph without cycles (multiple edges are allowed); let $\{1, 2, \dots, n\}$ be the set of its vertices (with fixed order!). The *adjacency matrix* of Δ is the matrix $A = A(\Delta) = (a_{ij})$ where a_{ij} is the number of edges between the vertices i and j.

Denote the *spectral radius* of Δ (i.e. the largest eigenvalue of A) by $\varrho(\Delta)$. The transformation $\mathcal{C}_\Delta$: $\boldsymbol{C}^n \to \boldsymbol{C}^n$ is called *Coxeter transformation* with respect to the standard basis if it is defined by the matrix $\Phi = -(I + A^+)^{-1}(I + A^+)^{tr}$, where A^+ is the upper triangular part of the adjacency matrix A and I is the identity matrix. The characteristic polynomial of Φ is called the *Coxeter polynomial* of $\mathcal{C}_\Delta$. The *spectrum* $\mathrm{Spec}(\mathcal{C}_\Delta)$ is the set of all eigenvalues of Φ and the *spectral radius* of $\mathcal{C}_\Delta$ is

$$\varrho(\mathcal{C}_\Delta) = max\,\{|\lambda| : \lambda \in \mathrm{Spec}(\mathcal{C}_\Delta)\}.$$

Generally, the definition of the Coxeter transformation of a graph depends on its orientation. It is well-known (see [3]) that the characteristic polynomial of the Coxeter transformation is reciprocal and it does not depend on the orientation of the tree – this allows us to speak about the Coxeter polynomial of a (non-oriented) tree.

Let $p = (p_1, p_2, \dots, p_s)$, be an s-tuple ($s \geq 3$) of positive integers p_i ($1 \leq i \leq s$) and let $n = \sum_{i=1}^{s} p_i + 1$. A star is a tree with simple edges i.e. a star consists of paths with one common endpoint. The star is called *wild star* if its adjacency matrix has at most one eigenvalue greater than 2. Denote by $\Delta[p_1, p_2, \dots, p_s]$ the wild star consisting of s paths of length $p_1, p_2, \dots p_s$, and denote by $\varrho(\mathcal{C}_{[p_1, p_2, \dots, p_s]})$ the spectral radius of $\mathcal{C}_{\Delta[p_1, p_2, \dots, p_s]}$.

The following theorem is the core of the link between the Coxeter transformation and the Salem numbers.

Corollary 1. *The Coxeter polynomial of a wild star is a Salem polynomial.*

This is an immediate consequence of

Theorem [5]. *The Coxeter polynomial of a wild star has exactly two real zeros and one irreducible non-cyclotomic factor.*

2 - Spectrum of Coxeter transformation and PV-numbers

The smallest known Salem number is the spectral radius $\varrho(\mathcal{C}_{[1,2,6]})(\sim 1.3241)$ (of the wild star $\Delta[1, 2, 6]$). The $\lim_{m\to\infty} \varrho(\mathcal{C}_{[1,2,m]})$ = the only real zero of the polynomial $x^3 - x - 1$, which happens to be also the smallest PV-number.

Using Coxeter transformation we can construct new families of Salem and PV-numbers. About the calculation of particular Coxeter polynomials and spectral radii we refer to [**2**] and to the Maple program for generating Coxeter polynomials for a large class of oriented graphs developed by Boldt [**2**].

Proposition. [**9**]. *If the tree Δ is neither of Dynkin nor of Euclidean type, then $\mu_0 \leqslant \varrho(\mathcal{C}_\Delta)$; where μ_0 is the largest (real) zero of the polynomial*

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1\,.$$

Take now a sequence of wild stars with $s+1$ arms $s \geqslant 2$ of lengths $k, p_1(t), p_2(t), \ldots, p_s(t)$ $(t \in \boldsymbol{N})$ where k *is fixed.* Then we have

Theorem 1. *If $\{\Delta[k, p_1(t), p_2(t), \ldots, p_s(t)] \,|\, t \in \boldsymbol{N}\}$ is a sequence of wild stars with $\lim_{t\to\infty} p_i(t) = \infty$ $(i = 1, 2, \ldots, s;\ s \geqslant 2)$, then for the limit*

$$\eta := \lim_{t\to\infty} \varrho(\mathcal{C}_{[k,\, p_1(t),\, p_2(t),\, \ldots,\, p_s(t)]})$$

we have

$$s - 1 \leqslant \eta < s\,; \tag{1}$$

further, η is the largest positive real zero of the polynomial

$$F(x) = x^{k+1} - (s-1)\,x^k - \ldots - (s-1)\,x - (s-1)\,. \tag{2}$$

Proof. First we remark that the limit in (1) exists, since the spectral radius of the Coxeter transformation of a proper subgraph of a graph is not greater than that of the graph.

The statement concerning the upper bound in (1) and the second statement follows from Prop. 2.7 of [**6**] (with $s-1$ replaced by s). The lower bound in (1) is a consequence of Theorem 2.2 of [**5**] with $s-1$ replaced by s and setting $p_1 = k$, $p_2 = p_1(t), \ldots p_{s+1} = p_s(t)$ and taking the limit $t \to \infty$. ■

Theorem 1 relates the polynomial F to the limit of the spectral radii of Coxeter transformation of wild stars. But this polynomial also turned up in number theory.

The fact that the largest positive zero of $F(x)$ is a PV-number follows from the basic theory of the first derived set of PV-numbers given in Chapter 6 in [**1**].

In the next theorem, which is our main result, we give a *new elementary proof* of this result and also show how η can be quite precisely located by help of the number of all arms $s+1$ of the wild star and of the length k of the arm that remains fixed during the limiting process.

Theorem 2. *If* $\{\Delta[k, p_1(t), p_2(t), \dots, p_s(t)] \mid t \in \mathbf{N}\}$ *is a sequence of wild stars with* $\lim_{t\to\infty} p_i(t) = \infty$ $(i = 1, 2, \dots, s;\ s \geq 2,)$ *then* $\eta = \lim_{t\to\infty} \varrho(\mathcal{C}_{[k,\, p_1(t),\, p_2(t),\, \dots,\, p_s(t)]})$ *is a PV-number and*

$$s - s^{-k} < \eta < s - s^{-k-1}.$$

Proof. By Theorem 1 η is the largest real zero of the polynomial F. Let $F(x) = (x-\eta)\, f(x)$ and $f(x) = x^k + \alpha_{k-1} x^{k-1} + \dots + \alpha_1 x + \alpha_0$. Comparing the coefficients we get

$$\begin{aligned} -\eta\alpha_0 &= -(s-1), \\ \alpha_0 - \eta\alpha_1 &= -(s-1), \\ &\ \vdots \\ \alpha_{k-2} - \eta\alpha_{k-1} &= -(s-1), \\ \alpha_{k-1} - \eta &= -(s-1). \end{aligned}$$

Hence

$$\begin{aligned} \alpha_0 &= \frac{s-1}{\eta}, \quad \alpha_1 = \frac{(s-1)(\eta+1)}{\eta^2}, \dots, \\ \alpha_{k-2} &= \frac{(s-1)(\eta^{k-2} + \eta^{k-1} + \dots + \eta + 1)}{\eta^{k-1}}, \\ \alpha_{k-1} &= \frac{(s-1)(\eta^{k-1} + \eta^{k-2} + \dots + \eta + 1)}{\eta^k} = \eta - (s-1). \end{aligned}$$

Let

$$\beta_j = \begin{cases} \alpha_j/\alpha_{j+1}, & \text{if } 0 \leq j \leq k-2 \\ \alpha_j, & \text{if } j = k-1\,. \end{cases}$$

Then we have

(4) $$\beta_{k-2} > \beta_{k-3} > \dots \beta_1 > \beta_0,$$

since $\dfrac{x}{x+1}$ is an increasing function of x for $x > 0$ and

$$\beta_j = \frac{\eta^{j+1} + \eta^j + \dots + \eta}{\eta^{j+1} + \eta^j + \dots + \eta + 1} \quad (0 \leqslant j \leqslant k-2).$$

<u>*Case* 1</u>: $s \geqslant 3$.

By (1) we have $\eta \geqslant s - 1 \geqslant 2$ and we show that

(5) $$\beta_{k-1} > \beta_{k-2}.$$

Using the definition of β_j this is equivalent to

$$\alpha_{k-1} > \frac{\alpha_{k-2}}{\alpha_{k-1}}$$

or, using the expressions of α_{k-1}, α_{k-2} by help of η we can rewrite this as

$$\frac{s-1}{\eta^k} \frac{\eta^k - 1}{\eta - 1} > \frac{\eta^{k-1} + \eta^{k-2} + \dots + \eta}{\eta^{k-1} + \eta^{k-2} + \dots + \eta + 1} = \frac{\eta^k - \eta}{\eta^k - 1}$$

or

$$s - 1 > \frac{(\eta - 1)(\eta^{2k} - \eta^{k+1})}{(\eta^k - 1)^2}.$$

This is true since by $\eta \geqslant 2$ we have

$$(\eta^k - 1)^2 - (\eta^{2k} - \eta^{k+1}) = \eta^{k+1} - 2\eta^k + 1 = \eta^k(\eta - 2) + 1 > 0,$$

therefore (using (1) too)

$$s - 1 > \eta - 1 > \frac{(\eta - 1)(\eta^{2k} - \eta^{k+1})}{(\eta^k - 1)^2}.$$

Let $x = \alpha_{k-1} y$ and $g(y) = f(\alpha_{k-1} y)$. Then we have

$$g(y) = \alpha_{k-1}^k y^k + \alpha_{k-1}^k y^{k-1} + \alpha_{k-2} \alpha_{k-1}^{k-2} y^{k-2} + \dots + \alpha_1 \alpha_{k-1} y + \alpha_0.$$

From (4) and (5) we have

(6) $$\alpha_{k-1}^{k} > \alpha_{k-2}\alpha_{k-1}^{k-2} > \ldots > \alpha_1 \alpha_{k-1} > \alpha_0 > 0 .$$

To continue the proof we need the following result of Eneström and Kakeya (see [7]).

Proposition [7]. *Let*

$$f(x) = b_i x^i + b_{i-1} x^{i-1} + \ldots + b_1 x + b_0$$

be a polynomial whose coefficients satisfy the inequalities $b_i \geqslant b_{i-1} \geqslant \ldots \geqslant b_1 \geqslant b_0 > 0$. *Then no zero of* f *has absolute value greater than* 1.

Equation (6) shows that the above result is applicable for $g(y)$, thus the absolute values of the zeros of $g(y)$ are all less than or equal to one. Hence the absolute values of zeros of $f(x)$ are all $\leqslant \alpha_{k-1} = \eta - (s-1) < 1$. This implies that η is a PV-number.

Case 2: $s = 2$.

In this case f satisfies the conditions of Proposition [7] since

$$1 \geqslant \alpha_{k-1} \geqslant \alpha_{k-2} \geqslant \ldots \geqslant \alpha_0 > 0$$

holds. The first of these inequalities $1 \geqslant \alpha_{k-1} = \eta - 1$ is a consequence of $s = 2 \geqslant \eta \geqslant 1$. The other inequalities

$$\frac{\eta^{k-i+1}-1}{\eta - 1} \frac{1}{\eta^{k-i+1}} = \alpha_{k-i} \geqslant \alpha_{k-i-1} = \frac{\eta^{k-i}-1}{\eta - 1} \frac{1}{\eta^{k-i}} \quad (k-1 \geqslant i \geqslant 2)$$

also easily follow from $\eta \geqslant 1$. Hence all the zeros of f have absolute value $\leqslant 1$.

Next we show that f has no zero on the unit circle. This, together with the previous statement shows that η is a PV-number.

Suppose, on the contrary, that $e^{i\varphi}$ $(0 \leqslant \varphi < 2\pi)$ is a zero of f. Then it is also a zero of F. But $F(1) = 1 - (k+1) = -k \neq 0$, therefore $\varphi > 0$. The equation $F(e^{i\varphi}) = 0$ can be rewritten as

$$e^{i(k+1)\varphi} = \frac{e^{i(k+1)\varphi} - 1}{e^{i\varphi} - 1} .$$

Multiplying by $e^{i\varphi} - 1$ and separating the real and imaginary parts we get

$$\cos(k+2)\varphi = \cos(k+1)\varphi - 1, \qquad \sin(k+2)\varphi = \sin(k+1)\varphi .$$

Adding the squares of these equations we conclude that $\cos(k+1)\varphi = 1$, thus also $\cos(k+2)\varphi = 2-1 = 1$. From this

$$0 = \cos(k+2)\varphi - \cos(k+1)\varphi = -2\sin\frac{2k+3}{2}\varphi\sin\frac{\varphi}{2}.$$

$\sin\frac{\varphi}{2} \neq 0$ since $\frac{\varphi}{2} \in (0, \pi)$. Therefore $\sin\frac{2k+3}{2}\varphi = 0$, $\frac{2k+3}{2}\varphi = n\pi$ where n is an integer with $0 < n < 2k+3$. But then we have

$$\cos(k+1)\varphi = \cos\left(n\pi - \frac{n\pi}{2k+3}\right) = (-1)^n\cos\frac{n\pi}{2k+3}.$$

The conditions for k imply that $\left|\cos\frac{n\pi}{2k+3}\right| < 1$, thus $|\cos(k+1)\varphi| < 1$ which is a contradiction.

Therefore in *Case* 2 the number η is a PV-number too.

Let

$$Q(x) = (x-1)\,F(x) = x^{k+2} - sx^{k+1} + (s-1).$$

We claim that

$$Q(s - s^{-k}) < 0 \tag{7}$$

and

$$Q(s - s^{-k-1}) > 0. \tag{8}$$

We have

$$Q(s - s^{-k}) = s - 1 - s(1 - s^{-k-1})^{k+1} < s - 1 - s(1 - (k+1)s^{-k-1})$$
$$= (k+1)s^{-k} - 1$$

since by Bernoulli's inequality

$$(1 - s^{-k-1})^{k+1} > 1 - (k+1)\,s^{-k-1}.$$

Let $G(z, s) = (z+1)\,s^{-z} - 1$ $(z \geq 1,\ s \geq 2)$ then

$$\frac{\partial}{\partial z}G(z, s) = s^{-z} + (z+1)\,s^{-z}(-1)\ln s$$
$$= s^{-z}(1 - (z+1)\ln s) \leq s^{-z}(1 - 2\ln 2) < 0,$$

therefore

$$G(z,s) \leqslant G(1,s) = \frac{2}{s} - 1 \leqslant 0 \qquad \text{for } z \geqslant 1,\ s \geqslant 2$$

and we can complete the proof of (7) by

$$Q(s - s^{-k}) < (k+1)\, s^{-k} - 1 = G(k,s) \leqslant 0\,.$$

(8) follows from

$$Q(s - s^{-k-1}) = s - 1 - (1 - s^{-k-2})^{k+1} > s - 1 > 0$$

since $0 < (1 - s^{-k-2})^{k+1} < 1$. The inequalities (7) and (8) imply that $F(x)$ has a zero ξ between $s - s^{-k}$ and $s - s^{-k-1}$. We have proved that all zeros of $F(x)$ but η have absolute value less than 1. Therefore $\xi = \eta$ and the proof of (3) and that of Theorem 2 is complete. ■

Remark 1. From (3) if k tends to infinity then $\eta = \eta_{k,s}$ tends to $s \geqslant 2$ which is an integer. Thus *every integer* $\geqslant 2$ can be obtained as an element of the second derived set of spectral radii of Coxeter transformations.

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Abstract

We show that the spectral radii of the Coxeter transformation of wild stars are Salem numbers and their suitable limits are PV-numbers. The link between these limits and PV-numbers is the polynomial $F(x) = x^{k+1} - (s-1)\dfrac{x^{k+1}-1}{x-1}$. *The fact that the largest positive zero* η *of* F *is a PV-number follows from the theory of the first derived set of PV-numbers given in Chapter 6 in* [**1**]. *We give an* elementary proof *of this result and also show how* η *can be located:*

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