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R. NIVAS and S. ALI (*)

On certain structures in the tangent bundle (**)

Introduction

Almost contact and almost complex structures in the tangent bundle have been studied by K. Yano and S. Ishihara [4]. *GF*-structure manifolds also play an important role in the theory of structures on manifolds. In this paper we have studied generalised contact structure and *GF*-structure in the tangent bundle. Some interesting results related to Nijenhuis tensor and Lie-dervative have also been obtained.

1 - Preliminaries

Let M be an n-dimensional differentiable manifold of class C^{∞} and let $T_p(M)$ be the tangent space of M at a point p of M. Then set

$$T(M) = \bigcup_{p \varepsilon M} T_p(M)$$

is called the tangenet bundle over the manifold M. For any point $\tilde{p}$ of $T(M)$, correspondence $\tilde{p} \rightarrow p$ determines the bundle projection $\pi : T(M) \rightarrow M$, thus $\pi(\tilde{p}) = p$, where $\pi : T(M) \rightarrow M$ defines the bundle projection of $T(M)$ over M. The set $\pi^{-1}(p)$ is called the fibre over $p \varepsilon M$ and M the base space.

Suppose that the base space M is covered by a system of coordinate neighbourhoods $\{U; x^h\}$, where (x^h) is a system of local coordinates defined in the neighbourhood U of M. The open set $\pi^{-1}(U) \subset T(M)$ is naturally differentiably

(*) Dept. of Maths and Astronomy, Lucknow University, Lucknow 226007, India.
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homeomorphic to the direct product $U \times R^n$, R^n being the n-dimensional vector space over the real field R, in such a way that a point $\tilde{p} \varepsilon T_p(M)(p \varepsilon U)$ is represented by an ordered pair (P, X) of the point $p \varepsilon U$, and a vector $X \varepsilon R^n$ whose components are given by the cartesian coordinates (y^h) of $\tilde{p}$ in the tangent space $Tp(M)$ with respect to the natural base $\{\partial_h\}$ where $\partial_h = \dfrac{\partial}{\partial x^h}$. Denoting by (x^h) the coordinates of $p = \pi(\tilde{p})$ in U and establishing the correspondence $(x^h, y^h) \rightarrow \tilde{p} \varepsilon \pi^{-1}(U)$, we can introduce a system of local coordinates (x^h, y^h) in the open set $\pi^{-1}(U) \subset T(M)$. Here we call (x^h, y^h) the coordinates in $\pi^{-1}(U)$ induced from (x^h) or simply, the induced coordinates in $\pi^{-1}(U)$ [**4**].

We denote by $\Im_s^r(M)$ the set of all tensor fields of class C^∞ and of type (r, s) in M. We now put $\Im(M) = \sum_{r,s=0}^{\infty} \Im_s^r(M)$, which is the set of all tensor fields in M. Similarly, we denote by $\Im_s^r(T(M))$ and $\Im(T(M))$ respectively the corresponding sets of tensor fields in the tangent bundle $T(M)$.

Vertical lifts

If f is a function in M, we write f^V for the function in $T(M)$ obtained by forming the composition of $\pi : T(M) \rightarrow M$ and $f : M \rightarrow R$, so that

$$f^V = f \circ \pi .$$

Thus, if a point $\tilde{p} \varepsilon \pi^{-1}(U)$ has induced coordinates (x^h, y^h), then

$$f^V(\tilde{p}) = f^V(x, y) = f \circ \pi(\tilde{p}) = f(p) = f(x).$$

Thus the value of $f^V(\tilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^V the vertical lift of the function f.

Let $\tilde{X} \varepsilon \Im_0^1(T(M))$ be such that $\tilde{X} f^V = 0$ for all $f \varepsilon \Im_0^0(M)$. Then we say that $\tilde{X}$ is a vertical vector field. Let $\begin{pmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{pmatrix}$ be components of $\tilde{X}$ with respect to the induced coordinates. Then $\tilde{X}$ is vertical if and only if its components in $\pi^{-1}(U)$ satisfy

$$\begin{pmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{X}^{\bar{h}} \end{pmatrix}.$$

Suppose that $X \varepsilon \Im_0^1(M)$, so that X is a vector field in M. We define a vector field

X^V in $T(M)$ by

$$X^V(\iota\omega) = (\omega(X))^V,$$

ω being an arbitrary 1-form in M. We call X^V the vertical lift of X.

Let $\tilde{\omega}\,\varepsilon\Im_1^0(T(M))$ be such that $\tilde{\omega}(X^V) = 0$ for all $X\varepsilon\Im_0^1(M)$. Then we say that $\tilde{\omega}$ is a vertical 1-form in $T(M)$. We define the vertical lift ω^V of the 1-form ω by

$$\omega^V = (\omega_i)^V (dx^i)^V$$

in each open set $\pi^{-1}(U)$, where $\{U; x^h\}$ is a coordinate neighbourhood in M and ω is given by $\omega = \omega_i dx^i$ in U. The vertical lift ω^V of ω with local expression $\omega = \omega_i dx^i$ has components of the form

$$w^V : (w^i, O)$$

with respect to the induced coordinates in $T(M)$.

Vertical lifts to a unique algebraic isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients by the conditions

$$(P \otimes Q)^V = P^V \otimes Q^V, (P + R)^V = P^V + R^V$$

P, Q and R being arbitrary elements of $\Im(M)$. The vertical lifts F^V of an element $F\varepsilon\Im_1^1(M)$ with local components F_i^h has components of the form

$$F^V : \begin{pmatrix} 0 & 0 \\ F_i^h & 0 \end{pmatrix}.$$

Complete lifts

If f is a function in M, we write f^C for the function in $T(M)$ defined by

$$F^C = \iota(df)$$

and call f^C the complete lift of the function f. The complete lift f^C of a function f has the local expression

$$f^C = y^i \partial_i f = \partial f$$

with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $X\varepsilon\Im_0^1(M)$. We define a vector field X^C in $T(M)$ by

$$X^C f^C = (Xf)^C,$$

f being an arbitrary function in M and call X^C the complete lift of X in $T(M)$. The complete lift X^C of X with components x^h in M has components

$$X^C:\begin{pmatrix} x^h \\ \partial x^h \end{pmatrix},$$

with respect to the induced coordinates in $T(M)$.

Suppose that $\omega \varepsilon \Im_1^0(M)$. Then a 1-form ω^c in $T(M)$ defined by

$$\omega^C(X^C) = (\omega(X))^C,$$

X being an arbitrary vector field in M. We call ω^C the complete lift of ω. The complete lift ω^C of ω with components ω_i in M has componets of the form

$$\omega^C:(\partial\omega_i,\ \omega_i)$$

with respect to the induced coordinates in $T(M)$.

The complete lifts to a unique algebra isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients, by the conditions

$$(P\otimes Q)^C = P^C\otimes Q^V + P^V\otimes Q^C, (P+R)^C = P^C + R^C,$$

P, Q and R being arbitrary elements of $\Im(M)$.

The complete lifts F^C of an element F of $\Im_1^1(M)$ with local components F_i^h has components of the form

$$F^C:\begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{pmatrix}.$$

Horizontal lifts

The horizontal lift f^H of $f\Im_0^0(M)$ to the tangent bundle $T(M)$ by

$$f^H = f^C - \nabla_\gamma f$$

where

$$\nabla_\gamma f = {}_\gamma(\nabla f).$$

Let $X\varepsilon\Im_0^l(M)$. Then the horizontal lift X^H of X defined by

$$X^H = X^C - \nabla_\gamma X,$$

in $T(M)$, where

$$\nabla_\gamma X = {}_\gamma(\nabla X).$$

The horizontal lift X^H of X has the components

$$X^H : \begin{pmatrix} x^h \\ -\Gamma_i^h x^i \end{pmatrix}$$

with respect to the induced coordinates in $T(M)$, where

$$\Gamma_i^h = y^j \Gamma_{ji}^h.$$

Let $\omega \varepsilon \Im_1^0(M)$ with affine connection ∇. Then the horizontal lift ω^H of ω is defined by

$$\omega^H = \omega^C - \nabla_\gamma \omega$$

in $T(M)$, where $\nabla_\gamma \omega = {}_\gamma(\nabla\omega)$. The horizontal lift ω^H of ω has component of the form

$$\omega^H : (\Gamma_i^h \omega_h, \omega_i)$$

with respect to the induced coordinates in $T(M)$.

Suppose there is given a tensor field

$$S = S_{k\ldots j}{}^{i\ldots h} \frac{\partial}{\partial x^i} \otimes \ldots \otimes \frac{\partial}{\partial x^h} \otimes dx^k \otimes \ldots \otimes dx^j$$

in M with affine connection ∇, and in $T(M)$ a tensor field $\nabla_\gamma S$ defined by

$$\nabla_\gamma S = (y^l \nabla_l S_{k\ldots j}^{i\ldots h}) \frac{\partial}{\partial y^i} \otimes \ldots \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes \ldots \otimes dx^j$$

with respect to the induced coordinates (x^h, y^h) in $\pi^{-1}(U)$.

The horizontal lift S^H of a tensor field S of arbitrary type in M to $T(M)$ is defined by

$$S^H = S^C - \nabla_\gamma S .$$

For any P, $Q \varepsilon \Im(M)$. We have

$$\nabla_\gamma (P \otimes Q) = (\nabla_\gamma P) \otimes Q^V + P^V \otimes (\nabla_\gamma Q) \quad \text{or} \quad (P \otimes Q)^H = P^H \otimes Q^V + P^V \otimes Q^H.$$

GF-structure

Let M an n-dimensional differentiable manifold of class C^∞. Suppose there exists on M a tensor field $F(\neq 0)$ of type (1,1) satisfying

$$F^2 = a^2 1$$

where $a(\neq 0)$ is a complex number and 1 denotes the unit tensor field. We say that the manifold M endowed with GF-structure.

2 - Complete lifts of *GF*-structures and generalised contact structures in the tangent bundle

Let M be an n-dimensional differentiable manifold of class C^∞ and $T(M)$ denotes the tangent bundle of M. Suppose that there is given in M, a tensor field F of type (1,1), a vector field U and a 1-form ω satisfying

$$F^2 = a^2 I_n + U \otimes \omega \tag{2.1}$$

and

$$\begin{aligned} &\text{(i)} \quad FU = 0 \\ &\text{(ii)} \quad \omega \circ F = 0 \\ &\text{(iii)} \quad \omega(U) = -a^2 . \end{aligned} \tag{2.2}$$

Then the manifold M satisfying conditions (2.1) and (2.2) will be said to possess a generalised contact manifold and the structure $\{F, U, \omega, a\}$ is called *generalised contact structure* (g.c.s.) on M.
From (2.1) and (2.2) we have

$$(F^C)^2 = a^2 I_n + U^V \otimes \omega^C + U^C \otimes \omega^V \tag{2.3}$$

and

$$\begin{aligned} &\text{(i)} \quad F^C U^V = 0 \qquad F^C U^C = 0 \\ &\text{(ii)} \quad \omega^V \circ F^C = 0 , \quad \omega^C \circ F^V = 0 , \ \omega^C \circ F^C = 0 , \\ &\text{(iii)} \quad \omega^V(U^V) = 0 , \quad \omega^V(U^C) = -a^2, \ \omega^C(U^V) = -a^2, \ \omega^C(U^C) = 0 . \end{aligned} \tag{2.4}$$

Let us define an element $\tilde{J}$ of $\Im_1^1(T(M))$ by

$$\tilde{J} = F^C + a^* \{ U^V \otimes \omega^V + U^C \otimes \omega^C \} , \tag{2.5}$$

where

$$a^* = \frac{1}{a} . \tag{2.6}$$

Then in view of the equations (2.3), (2.4) and (2.5), we can easily show

$$\tilde{J}^2 = a^2 I . \tag{2.7}$$

Thus $\tilde{J}$ defines a GF-structure in $T(M)$. Hence, we have

Theorem 2.1. *Let M be a differentiable manifold endowed with the generalised contact structure $\{F, U, \omega, a)$. Let also $\tilde{J}$ be as in* (2.5). *Then $\tilde{J}$ is GF-structure on $T(M)$.*

Now in view of the equation (2.5), we have

$$\tilde{J}X^V = (FX)^V + a^*(\omega(X))^V U^C. \tag{2.8}$$

$$\tilde{J}X^C = (FX)^C + a^*\{(\omega(X))^V U^V + (\omega(X))^C U^C\} \tag{2.9}$$

for any $X \varepsilon \Im_0^1(M)$. In particular, we have

$$\tilde{J}X^V = (FX)^V, \tilde{J}X^C = (FX)^C \tag{2.10}$$

$$\tilde{J}U^V = -aU^C, \tilde{J}U^C = -aU^V, \tag{2.11}$$

X being an arbitrary vector field in M such that $\omega(X) = 0$.

Theorem 2.2. *Let the tangent bundle $T(M)$ of the manifold M admits the GF-structure tensor defined by* (2.5). *Then for $X, Y \varepsilon \Im_0^1(M)$ such that $\omega(Y) = 0$, we have*

$$\begin{aligned}
&\text{(i)} \ (\mathfrak{L}_{X^V}\tilde{J})\,Y^V = 0\\
&\text{(ii)} \ (\mathfrak{L}_{X^V}\tilde{J})\,Y^C = ((\mathfrak{L}_X F)\,Y)^V - a^*((\mathfrak{L}_X\omega)\,Y)^V U^C\\
&\text{(iii)} \ (\mathfrak{L}_{X^V}\tilde{J})\,U^V = -a(\mathfrak{L}_X U)^V\\
&\text{(iv)} \ (\mathfrak{L}_{X^V}\tilde{J})\,U^C = ((\mathfrak{L}_X F)\,U)^V + a^*((\mathfrak{L}_X\omega)(U))^V U^C
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&\text{(i)} \ (\mathfrak{L}_{X^C}\tilde{J})\,Y^V = ((\mathfrak{L}_X F)\,Y)^V - a^*((\mathfrak{L}_X\omega)\,Y)^V U^C\\
&\text{(ii)} \ (\mathfrak{L}_{X^C}\tilde{J})\,Y^C = ((\mathfrak{L}_X F)\,Y)^C + a^*\{((\mathfrak{L}_X\omega)(Y)^V)\,U^V + (\mathfrak{L}_X\omega)(Y))^C U^C\}\\
&\text{(iii)} \ (\mathfrak{L}_{X^C}\tilde{J})\,U^V = ((\mathfrak{L}_X F)\,U^V + a^*((\mathfrak{L}_X\omega)(U))^V U^C - a[X, U]^C\\
&\text{(iv)} \ (\mathfrak{L}_{X^C}\tilde{J})\,U^C = ((\mathfrak{L}_X F)\,U)^C\\
&\qquad + a^*\{((\mathfrak{L}_X\omega)(U))^V U^V + ((\mathfrak{L}_X\omega)(U))^C U^C\} - a[X, U]^V.
\end{aligned} \tag{2.13}$$

Proof. The proof follows from (2.4), (2.8), (2.9), (2.11), (2.12) and pages 20, 23 and 49 in [4].

Let us define a tensor field S of type $(1,2)$ by

$$S(X, Y) = N(X, Y) + (X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]))\, U \tag{2.14}$$

for any X, $Y \varepsilon \Im_0^1(M)$, M being the Nijenhuis tensor of F.

Theorem 2.3. *If $\omega(X) = 0$, $\omega(Y) = 0$, then we have*

$$\begin{aligned} &\text{(i)}\ S(X, Y) = [FX, FY] + a^2[X, Y] - F[FX, Y] - F[X, FY] \\ &\text{(ii)}\ (S(X, U))^V = a^2[X, U]^V - (F[FX, U])^V \\ &\text{(iii)}\ (S(X, U))^C = a^2[X, U]^C - (F[FX, U])^C. \end{aligned} \tag{2.15}$$

Proof. The proof follows easily by virtute of equations (2.4), (2.7), (2.8), (2.9), (2.10), (2.11) and pages 16 and 23 in [4].

Theorem 2.4. *Let us define tensor fields S_1, S_2 and S_3 by*

$$\begin{aligned} &\text{(i)}\ S_1(X, Y) = \frac{1}{a}\,\omega([FX, Y] + [X, FY]) \\ &\text{(ii)}\ S_2(X) = a\{[U, FX] - F[U, X]\} \\ &\text{(iii)}\ S_3(X) = -\omega[U, X] \end{aligned} \tag{2.16}$$

for any X, $Y \varepsilon \Im_0^1(M)$. Let $\widetilde{H}$ be the Nijenhuis tensor of $\tilde{J}$ defined by (2.5). *Then for any X, $Y \varepsilon \Im_0^1(M)$ such that $\omega(X) = \omega(Y) = 0$, we have*

$$\begin{aligned} &\text{(i)}\ \widetilde{H}(X^V, Y^V) = 0\,, \\ &\text{(ii)}\ \widetilde{H}(X^V, Y^C) = (S(X, Y))^V - (S_1(X, Y))^V U^C \\ &\text{(iii)}\ \widetilde{H}(X^C, Y^C) = (S(X, Y))^C - (S_1(X, Y))^V U^V - (S_1(X, Y))^C U^C \\ &\text{(iv)}\ \widetilde{H}(X^V, U^V) = (S_2(X))^V - (S_3(X))^V U^C \\ &\text{(v)}\ \widetilde{H}(X^V, U^C) = (S(X, U))^V - (S_1(X, U))^V U^C \\ &\text{(vi)}\ \widetilde{H}(X^C, U^V) = (S_2(X))^C + (S(X, U))^V + (S_3(X))^V U^C - ((S_1(X, U))^V \\ &\qquad - (S_3(X))^C)\, U^V \\ &\text{(vii)}\ \widetilde{H}(X^C, U^C) = (S(X, U))^C + (S_2(X))^V - (S_1(X, U))^V U^V - (S_1(X, U))^C \\ &\qquad - (S_3(X))^V)\, U^C \\ &\text{(viii)}\ \widetilde{H}(U^V, U^C) = 0\,. \end{aligned} \tag{2.17}$$

Proof. The proof can be obtained from equations (2.4), (2.7), (2.8), (2.9), (2.10), (2.11), (2.15) and pages 16, 20 and 23 in [4].

3 - Horizontal lifts of *GF*-structures in the tangent bundle

Let M be a manifold with an affine connection ∇. We define a tensor field $\tilde{F}$ of type (1,1) in $T(M)$ by

$$\tilde{F}X^H = aX^V \quad \text{and} \quad \tilde{F}X^v = aX^H \tag{3.1}$$

for any $X \varepsilon \Im_0^1(M)$. Then we have

$$\tilde{F}^2 X^H = \tilde{F}(\tilde{F}X^H) = \tilde{F}(aX^V) = a\tilde{F}(X^V) = a^2 X^H$$

and

$$\tilde{F}^2 X^V = \tilde{F}(\tilde{F}X^V) = \tilde{F}(aX^H) = a\tilde{F}(X^H) = a^2 X^V$$

which implies

$$\tilde{F}^2 = a^2 I\ . \tag{3.2}$$

Hence we have

Theorem 3.1. *Let M be a differentiable manifold with an affine connection ∇. Then there exists a GF-structure $\tilde{F}$ in $T(M)$ defined by the equation* (3.1).

If $\tilde{N}$ be the Nijenhuis tensor of $\tilde{F}$, we have

$$\tilde{N}(\tilde{X}, \tilde{Y}) = [\tilde{F}\tilde{X}, \tilde{F}\tilde{Y}] + \tilde{F}^2[\tilde{X}, \tilde{Y}] - \tilde{F}[\tilde{F}X, \tilde{Y}] - \tilde{F}[\tilde{X}, \tilde{F}\tilde{Y}] \tag{3.3}$$

for any $\tilde{X}, \tilde{Y} \varepsilon \Im_0^1(T(M))$. Then we have

$$\begin{aligned} &\text{(i)}\ \ \tilde{N}(X^V, Y^V) = a^2\{T(X, Y)^H - {}_\gamma\hat{R}(X, Y)\} \\ &\text{(ii)}\ \ \tilde{N}(X^V, Y^H) = -a^2 T(X, Y)^V + a\tilde{F}_\gamma\hat{R}(X, Y) \\ &\text{(iii)}\ \ \tilde{N}(X^H, Y^H) = a^2\{T(X, Y)^H - {}_\gamma\hat{R}(X, Y)\} \end{aligned} \tag{3.4}$$

for any $X, Y \varepsilon \Im_0^1(M)$, where T is the torsion tensor of ∇ and $\tilde{R}$ the curvature tensor of the affine connection $\hat{\nabla}$ defined by (on page 88 in [4])

$$\hat{\nabla}_X Y = \nabla_Y X + [X, Y] \tag{3.5}$$

for any $X, Y \varepsilon \Im_0^1(M)$.

Proof. (i) From (2.3), we have

$$\tilde{N}(X^V, Y^V) = [\tilde{F}X^V, \tilde{F}Y^V] + \tilde{F}^2[X^V, Y^V] - \tilde{F}[\tilde{F}X^V, Y^V] - \tilde{F}[X^V, \tilde{F}Y^V]\ .$$

Using (3.1) and (3.2), we get

$$\widetilde{N}(X^V, Y^V) = a^2[X^H, Y^H] + a^2[X^V, Y^V] - a\widetilde{F}[X^H, Y^V] - a\widetilde{F}[X^V, Y^H]$$

$$= a^2[X, Y]^H - a^2_{\gamma}\widehat{R}(X, Y) - a\widetilde{F}[X^H, Y^V] - a\widetilde{F}[X^V, Y^H],$$

(by pages 16 and 90 in [4])

$$= a^2[X, Y]^H - a^2_{\gamma}\widehat{R}(X, Y) + a\widetilde{F}[Y, X]^V - a\widetilde{F}(\nabla_Y X)^V - a\widetilde{F}[X, Y]^V + a\widetilde{F}(\nabla_X Y)^V.$$

Using (3.1) we get the result (i).

(ii) From (3.1), (3.2) and (3.3), we have

$$\widetilde{N}(X^V, Y^V) = a^2[X^H, Y^H] + a^2[X^V, Y^H] - a\widetilde{F}[X^H, Y^H] - a\widetilde{F}[X^V, Y^V]$$

$$= a^2[X, Y]^V + a^2(\nabla_Y X)^V + a^2[X, Y]^V - a^2(\nabla_X Y)^V - a\widetilde{F}[X, Y]^H + a\widetilde{F}_{\gamma}\widehat{R}(X, Y)$$

(by pages 16 and 90 in [4]).

Again using (3.1), we get

$$\widetilde{N}(X^V, Y^H) = -a^2 T(X, Y)^V + a\widetilde{F}_{\gamma}\widehat{R}(X, Y).$$

(iii) In a similar way from (3.1), (3.2) and (3.3) and pages 16 and 90 in [4], we get the result.

Hence we have

Theorem 3.2. *The GF-structure* $\widetilde{F}$ *defined by* (3.1) *is integrable* i.e. $\widetilde{N} = 0$, *if and only if* $\widehat{R} = 0$ *and* $T = 0$, *where* T *is the torsion tensor of* ∇ *and* $\widehat{R}$ *the curvature tensor of the affine connection* $\widehat{\nabla}$ *defined by* (3.5).

Theorem 3.3. *We have*

$$\text{(3.6)} \quad \begin{aligned} &\text{(i)} \ (\mathfrak{L}_{Y^V}\widetilde{F})\, X^H = a(\widehat{\nabla}_X Y)^H \\ &\text{(ii)} \ (\mathfrak{L}_{Y^V}\widetilde{F})\, X^V = -a(\widehat{\nabla}_X Y)^V \end{aligned}$$

$$\text{(3.7)} \quad \begin{aligned} &\text{(i)} \ (\mathfrak{L}_{Y^H}\widetilde{F})\, X^H = a\{(\nabla_X Y)^V + \widetilde{F}_{\gamma}\widehat{R}(Y, X)\} \\ &\text{(ii)} \ (\mathfrak{L}_{Y^H}\widetilde{F})\, X^V = -a\{(\nabla_X Y)^H + {}_{\gamma}\widehat{R}(Y, X)\} \end{aligned}$$

for any $X, Y \varepsilon \mathfrak{I}^1_0(M)$.

Proof. Operating $\mathcal{L}_{Y^V}$ on both sides of (3.1), we get

$$\mathcal{L}_{Y^V}(\tilde{F}X^H) = \mathcal{L}_{Y^V}(aX^V)$$

$$\Rightarrow \quad (\mathcal{L}_{Y^V}\tilde{F})\, X^H + \tilde{F}\mathcal{L}_{Y^V}X^H = a\,\mathcal{L}_{Y^V}X^V = 0, \quad \text{(by page 49 in [4])}$$

$$\Rightarrow \quad (\mathcal{L}_{Y^V}\tilde{F})\, X^H = -\tilde{F}[Y^V, X^H] = -\tilde{F}(-(\hat{\nabla}_X Y)^V), \quad \text{(by page 90 in [4])}.$$

Using (3.1), we get the result (3.6) (i). Similarly we can prove the other results.

Thus from (3.6) and (3.7) we have

Theorem 3.4. *Let $Y = \Im_0^1(M)$. Then Y^V is an almost analytic vector field with respect to $\tilde{F}$ defined by* (3.1), i.e., $\mathcal{L}_{Y^V}\tilde{F} = 0$, *if and only if $\hat{\nabla} Y = 0$, $\hat{\nabla}$ being defined by* (3.5). *The horizontal lift Y^H is almost analytic with respect to $\tilde{F}$ if and only if $\nabla Y = 0$, $\hat{R}(X, Y) = 0$, X being an arbitrary element of $\Im_0^1(M)$, where $\hat{R}$ is the curvature tensor of $\hat{\nabla}$.*

4 - Horizontal lifts of generalised contact structure

Let $\{F, U, \omega, a\}$ be a generalised contact structure in M with an affine connection ∇. Then from (1.1) and (1.2) we have

$$\begin{aligned} &(F^H)^2 = (a^2 I + U \otimes \omega)^H \\ \text{(4.1)} \qquad \Rightarrow \quad &(F^H)^2 = a^2 I + (U \otimes \omega)^H \\ \Rightarrow \quad &(F^H)^2 = a^2 I + U \otimes \omega^V + U^V \otimes \omega^H, \ \text{(by page 94 in [4])} \end{aligned}$$

and with the help of page 93 and 96 in [4], we get

$$\text{(4.2)} \qquad \begin{aligned} &\text{(i)} \ F^H U^H = 0\,, \quad F^H U^V = 0 \\ &\text{(ii)} \ \omega^H(U^V) = 0\,, \quad \omega^H(U^V) = -a^2, \ \omega^V(U^H) = -a^2 \\ &\text{(iii)} \ \omega^H \circ F^H = 0\,, \quad \omega^V \circ F^H = 0\,. \end{aligned}$$

Let us define a tensor field $\tilde{J}^*$ of type (1,1) in $T(M)$ by

$$\text{(4.3)} \qquad \tilde{J}^* = F^H + a^*\{U^V \otimes \omega^V + U^H \otimes \omega^H\}, \quad \text{where } a^* = 1/a\,.$$

Then we can easily show that

$$\text{(4.4)} \qquad \tilde{J}^{*2} = a^2 I$$

that is $\tilde{J}^*$ is a GF-structure in $T(M)$. Thus we have

Theorem 4.1. *Let $\{F, U, \omega, a\}$ be a generalised contact structure in M with an affine connection ∇. Then there exists GF-structure $\tilde{J}^*$ defined by* (4.3) *in $T(M)$.*

From (1.4), (4.2), (4.3) and the help of pages 91, 93 and 96 in [4], we have

Theorem 4.2. *For any $X \varepsilon \Im_0^1(M)$*

$$(4.5)\quad \begin{aligned} &\text{(i)} \quad \tilde{J}^* X^H = (FX)^H + a^*(\omega(X))^V U^V \\ &\text{(ii)} \quad \tilde{J}^* X^V = (FX)^V + a^*(\omega(X))^V U^H \\ &\text{(iii)} \quad \tilde{J}^* X^C = (FX)^H + F^H(\nabla_\gamma X) + a^*(\omega(X))^V U^V + a^* \omega^C(\nabla_\gamma X)\, U^H \\ &\text{(iv)} \quad \tilde{J}^* X^C = (FX)^H + F^C(\nabla_\gamma X) + a^*(\omega(X))^V U^V + a^* \omega^C(\nabla_\gamma X)\, U^H. \end{aligned}$$

In particular, if X being an arbitrary vector field in M such that $\omega(X) = 0$, then

$$(4.6)\quad \begin{aligned} &\text{(i)} \quad \tilde{J}^* X^H = (FX)^H \\ &\text{(ii)} \quad \tilde{J}^* X^V = (FX)^V \quad \textit{and} \\ &\text{(iii)} \quad \tilde{J}^* X^C = (FX)^H + F^H(\nabla_\gamma X) + a^* \omega^C(\nabla_\gamma X)\, U^H. \end{aligned}$$

Also

$$(4.7)\quad \begin{aligned} &\text{(i)} \quad \tilde{J}^* U^H = -aU^V \\ &\text{(ii)} \quad \tilde{J}^* U^V = -aU^H \quad \textit{and} \\ &\text{(iii)} \quad \tilde{J}^* U^C = F^H(\nabla_\gamma U) - aU^V + a^* \omega^C(\nabla_\gamma U)\, U^H. \end{aligned}$$

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Summary

The present paper entitled «On certain structures in the tangent bundle» has been divided into four sections. First section contains the basic definition of tangent bundle and its lifts in a differentiable manifold. In section two, we have considered a generalised contact structure $\{F, U, w, a\}$ in M and then defined GF-structure in $T(M)$. Some results over this structure are also established in the same section. In section three, we have considered a manifold with an affine connection and then defined a GF-structure in $T(M)$. Some results related to Nijenhuis tensor have been established over this structure. Horizontal lifts of generalised contact structure $\{F, U, w, a\}$ have been considered in section four and also some more results have been established.

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