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The angle modulus of the deformation of a normed space (**)

Let $(X, \|.\|)$ be a real normed space, $S(X)$ the unit sphere in X and $B(X)$ the unit ball in X.

The functionals

$$\tau_{\pm}(x, y) := \lim_{t \to \pm 0} t^{-1}(\|x + ty\| - \|x\|),$$

$$g(x, y) := \frac{\|x\|}{2}(\tau_{-}(x, y) + \tau_{+}(x, y)) \qquad (x, y \in X),$$

always exist on X^2. The functional g has the following properties:

(1) $$g(x, x) = \|x\|^2 \qquad (x \in X),$$

(2) $$g(\alpha x, \beta y) = \alpha\beta g(x, y) \qquad (x, y \in X;\ \alpha, \beta \in \mathbb{R}),$$

(3) $$g(x, x + y) = \|x\|^2 + g(x, y) \qquad (x, y \in X),$$

(4) $$|g(x, y)| \leq \|x\| \|y\| \qquad (x, y \in X),$$

(see [**4**]).

If X is smooth, then g is linear in the second argument, and in this case $[y, x] := g(x, y)$ defines a semi-inner product in the sense of Lumer. If X is an inner product space (i.p. space) sense then $g(x, y)$ is the usual inner product of x and y.

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Definition 1 [**6**]. A normed space X in which the equality

$$\|x+y\|^4 - \|x-y\|^4 = 8\,(\|x\|^2 g(x,y) + \|y\|^2 g(y,x)) \qquad (x,y \in X), \tag{5}$$

holds is a *quasi-inner product space* (q.i.p. space).

If X is an i.p. space then (5) reduces to the parallelogram equality.

The space of sequences l^4 is q.i.p. space [**6**].

In accordance with (4), the angle between vector x and vector y can be defined in the following way.

Definition 2 [**5**]. For $x, y \in X \backslash \{0\}$, the number

$$\measuredangle(x,y) := \arccos \frac{g(x,y)+g(y,x)}{2\|x\|\,\|y\|}, \tag{6}$$

is called the *g-angle between x and y.*

In what follows we shall write $\cos(x,y)$ instead of $\cos\measuredangle(x,y)$.

Now we quote three known definitions.

Definition 3. *The modulus of convexity of X* is the function $\delta:[0,2]\to[0,1]$ defined by

$$\delta_X(\varepsilon) := \inf\left\{1 - \left\| \frac{x+y}{2} \right\| \,\middle|\, x, y \in B(X),\ \|x-y\| \geqslant \varepsilon\right\}.$$

One can show that this modulus can be defined equivalently as

$$\delta_X(\varepsilon) := \inf\left\{1 - \left\| \frac{x+y}{2} \right\| \,\middle|\, x, y \in S(X),\ \|x-y\| = \varepsilon\right\},$$

(see [**2**], for example).

Definition 4 [**1**]. *The modulus of smoothness of X* is the function $\varrho_X:[0,2]\to[0,1]$ defined by

$$\varrho_X(\varepsilon) := \sup\left\{1 - \left\| \frac{x+y}{2} \right\| \,\middle|\, x, y \in S(X),\ \|x-y\| \leqslant \varepsilon\right\}.$$

Definition 5 [**1**]. *The modulus of deformation of X* is the function $d_X:[0,2]\to[0,1]$ defined by

$$d_X(\varepsilon) := \varrho_X(\varepsilon) - \delta_X(\varepsilon).$$

For any Banach space X the following estimate is true:

$$\delta_X(\varepsilon) \leqslant \varrho_X(\varepsilon),$$

(7) $$\delta_X(\varepsilon) \leqslant 1 - \sqrt{1 - \frac{\varepsilon^2}{4}},$$

(8) $$1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \leqslant \varrho_X(\varepsilon) \leqslant \frac{\varepsilon}{2},$$

(9) $$0 \leqslant d_X(\varepsilon) \leqslant \frac{\varepsilon}{2},$$

(see [**1**]).

Some additional properties of functional g are quoted below.

Lemma 1. *For $x, y \in S(X)$ we have*

(10) $$1 - \|x - y\| \leqslant g(x, y) \leqslant \|x + y\| - 1,$$

(11) $$1 - \|x - y\| \leqslant \cos(x, y) \leqslant \|x + y\| - 1,$$

(12) $$1 - \left\| \frac{x + y}{2} \right\| \leqslant \frac{1 - \cos(x, y)}{2} \leqslant \left\| \frac{x - y}{2} \right\|,$$

(13) $$g(x + y, x + y) = g(x + y, x) + g(x + y, y).$$

Proof. Using (3) and (4) we deduce that

$$g(x, x \pm y) = 1 \pm g(x, y) \leqslant \|x \pm y\|.$$

Hence (10) is true. Inequality (10) implies (11) and (12). On the other hand, by (3) we obtain $g(x + y, x) = g(x + y, x + y - y) = g(x + y, x + y) - g(x + y, y)$. Hence (13) is true.

It is easily seen that in an i.p. space, for $x, y \in S(X)$ we have

(14) $$\frac{1 - \cos(x, y)}{2} = \left\| \frac{x - y}{2} \right\|^2.$$

In accordance with (12) and (14) we define new moduli of convexity, smoothness and deformation of X.

Definition 6. The function $\delta'_X:[0,2]\to[0,1]$ defined by

$$\delta'_X(\varepsilon) := \inf\left\{ \frac{1-\cos(x,y)}{2} \,\middle|\, x,y\in S(X),\ \|x-y\|\geqslant\varepsilon \right\},$$

will be called *the angle modulus of convexity of space X.*

Definition 7. The function $\varrho'_X:[0,2]\to[0,1]$ defined by

$$\varrho'_X(\varepsilon) := \sup\left\{ \frac{1-\cos(x,y)}{2} \,\middle|\, x,y\in S(X),\ \|x-y\|\leqslant\varepsilon \right\},$$

is called *the angle modulus of smoothness of space X.*

Definition 8. The function $d'_X:[0,2]\to[0,1]$ defined by

$$d'_X(\varepsilon) := \varrho'_X(\varepsilon) - \delta'_X(\varepsilon),$$

is called *the angle modulus of deformation of space X.*

Now, we note some elementary properties of the moduli δ'_X and ϱ'_X.

Theorem 1. (a) *If X is arbitrary, then* $\delta_X(\varepsilon)\leqslant\delta'_X(\varepsilon)$ *and* δ'_X *is nondecreasing on* [0,2].

(b) *If X is an i.p. space, then* $\delta'_X(\varepsilon)=\varepsilon^2/4$.

(c) *If X is a complete q.i.p. space then* $\delta'_X(\varepsilon)\leqslant\varepsilon^2/4$.

(d) *X is uniformly convex* (UC) *if and only if* $\delta'_X(\varepsilon)>0$.

Proof. (a) follows from (12) and from the implication

(15) $$\varepsilon_1<\varepsilon_2 \Rightarrow \{(x,y)\,|\,\|x-y\|\geqslant\varepsilon_1\}\supset\{(x,y)\,|\,\|x-y\|\geqslant\varepsilon_2\} \qquad (x,y\in B(X)).$$

(b) follows from (14).

(c) Assume that there is $\varepsilon>0$ such that $\delta'_X(\varepsilon)>\varepsilon^2/4$.

Then, for $x,y\in S(X)$

(16) $$\sup_{\|x-y\|\geqslant\varepsilon}\cos(x,y) < 1-\frac{\varepsilon^2}{2}.$$

By Definition 1 and Definition 2, for $x,y\in S(X)$ we have

(17) $$\left\|\frac{x+y}{2}\right\|^4 = \cos(x,y) + \left\|\frac{x-y}{2}\right\|^4.$$

Therefore, from (15) and (16) we have

$$\delta_X(\varepsilon) = 1 - \sup_{\|x-y\|=\varepsilon} \sqrt[4]{\cos(x, y) + \frac{\varepsilon^4}{16}} > 1 - \sqrt[4]{1 - \frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{16}} = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}},$$

which is impossible (see (7)).

(d) It is known that X is UC if and only if $\delta_X(\varepsilon) > 0$ for each $\varepsilon > 0$. So if X is UC, then $\delta'_X(\varepsilon) > 0$ for each $\varepsilon > 0$ (see (a)).

Suppose now that $\delta'_X(\varepsilon) > 0$ for each $\varepsilon > 0$, i.e.

$$\inf\left\{ \frac{1 - \cos(x, y)}{2} \;\middle|\; x, y \in S(X);\ \|x - y\| \geqslant \varepsilon \right\} > 0,$$

for each $\varepsilon > 0$. Therefore

$$\sup\{\cos(x, y) \mid \|x - y\| \geqslant \varepsilon\} < 1. \tag{18}$$

Since $\cos(x, y) = 1/2(g(x, y) + g(y, x))$ and $\sup_{\|x-y\| \geqslant \varepsilon} g(x, y) = \sup_{\|x-y\| \geqslant \varepsilon} g(y, x)$, the inequality

$$\sup_{\|x-y\| \geqslant \varepsilon} g(x, y) < 1 \qquad (x, y \in S(X)), \tag{19}$$

follows from (18).

Let $u = (x+y)/\|x+y\|$ and $\|u - x\| = \max\{\|u-x\|, \|u-y\|\}$. Then $\|x-y\| \geqslant \varepsilon$ implies $\|u - x\| \geqslant \varepsilon/2$. On the other hand we have

$$1 - \left\| \frac{x+y}{2} \right\| = \frac{2 - g(u, x+y)}{2} = \frac{1 - g(u, x)}{2} + \frac{1 - g(u, y)}{2} \geqslant \frac{1 - g(u, x)}{2}.$$

So, for $x, y \in S(X)$, we have

$$1 - \left\| \frac{x+y}{2} \right\| \geqslant \frac{1}{2}\Big(1 - \sup_{\|x-y\| \geqslant \varepsilon} g(u, x)\Big).$$

Hence, from (19), for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x - y\| \geqslant \varepsilon$ implies $1 - \|(x + y)/2\| \geqslant \delta$ i.e. X is UC.

Theorem 2. (a) *If X is arbitrary, then $\varrho_X(\varepsilon) \leqslant \varrho'_X(\varepsilon) \leqslant \varepsilon/2$ and ϱ'_X* is increasing on [0,2].

(b) *If X is an i.p. space, then $\varrho'_X(\varepsilon) = \varepsilon^2/4$ $(\varepsilon \in [0, 2])$.*

(c) *If X is a q.i.p. space, then $\varrho'_X(\varepsilon) \geqslant \varepsilon^2/4$.*

Proof. (a) Follows from (12) and from the implication (15). (b) Follows from (14). (c) Since $\varrho'_X(0) = 0$ and $\varrho'_X(2) = 1$, suppose that there exists $\varepsilon \in (0, 2)$ such that $\varrho'_X(\varepsilon) < \varepsilon^2/4$, i.e.

$$\inf\{\cos(x, y) \mid \|x - y\| \leqslant \varepsilon\} > 1 - \frac{\varepsilon^2}{2}. \tag{20}$$

Therefore, for $x, y \in S(X)$, from (20) and (17) we have

$$1 - \frac{\varepsilon^2}{2} < \inf_{\|x - y\| \leqslant \varepsilon} \cos(x, y) \leqslant \cos(x, y) = \left\| \frac{x + y}{2} \right\|^4 - \left\| \frac{x - y}{2} \right\|^4. \tag{21}$$

By inequalities $\varepsilon \geqslant \|x - y\| \geqslant 2 - \|x + y\|$ we have $1 - \varepsilon/2 \leqslant \|(x + y)/2\| \leqslant 1$ and $\left\| \frac{x + y}{2} \right\| \geqslant 1 - \left\| \frac{x + y}{2} \right\|$. Hence, from (21) we derive

$$1 - \frac{\varepsilon^2}{2} < \left\| \frac{x + y}{2} \right\|^4 - \left(1 - \left\| \frac{x + y}{2} \right\|\right)^4, \tag{22}$$

for $\|x - y\| \leqslant \varepsilon$.

Since the real function $t \mapsto f(t) = t^4 - (1 - t)^4$ is increasing on $[1 - \varepsilon/2, 1]$, it follows

$$\min_{t \in [1 - \varepsilon/2, 1]} f(t) = f(1 - \varepsilon/2) = \left(1 - \frac{\varepsilon}{2}\right)^4 - \frac{\varepsilon^4}{16}.$$

Because of that we have

$$\left(1 - \frac{\varepsilon}{2}\right)^4 - \frac{\varepsilon^4}{16} \geqslant 1 - \frac{\varepsilon^2}{2} \quad \text{i.e.} \quad \varepsilon(\varepsilon - 2)^2 \leqslant 0 \quad (\varepsilon \in (0, 2)),$$

which is impossible.

Lemma 2. *Let X is a q.i.p. space, $x, y \in S(X)$ and $\varepsilon \in [0, 2]$. The following implications hold*

$$\text{a)} \quad \|x - y\| \leqslant \varepsilon \Rightarrow \left(1 - \frac{\varepsilon}{2}\right)^4 - \frac{\varepsilon^4}{16} \leqslant \cos(x, y), \tag{23}$$

$$\text{b)} \quad \|x - y\| \geqslant \varepsilon \Rightarrow \cos(x, y) \leqslant 1 - \frac{\varepsilon^4}{16}. \tag{24}$$

Proof. (a) Let $x, y \in S(X)$ and $\|x-y\| \leqslant \varepsilon$. Then $2 = \|x+y+x-y\| \leqslant \|x+y\| + \varepsilon$, which implies $\|(x+y)/2\| \geqslant 1 - \varepsilon/2$. It follows from (17) that (23) holds. (b) Apply (17) to get (24).

Theorem 3. *Let X is a q.i.p. space. Then*

$$\delta_X(\varepsilon) = 1 - \sqrt[4]{1 - 2\delta'_X(\varepsilon) + \frac{\varepsilon^4}{16}}. \tag{25}$$

Proof. It follows from (17) that

$$\delta_X(\varepsilon) = \inf_{\|x-y\|=\varepsilon}\left(1 - \sqrt[4]{\cos(x, y) + \left\| \frac{x+y}{2} \right\|^4}\right) = 1 - \sqrt[4]{\sup_{\|x-y\|=\varepsilon} \cos(x, y) + \frac{\varepsilon^4}{16}}$$

$$\geqslant 1 - \sqrt[4]{\sup_{\|x-y\|\geqslant\varepsilon} \cos(x, y) + \frac{\varepsilon^4}{16}} = 1 - \sqrt[4]{1 - 2\delta'_X(\varepsilon) + \frac{\varepsilon^4}{16}}.$$

So,

$$\delta_X(\varepsilon) \geqslant 1 - \sqrt[4]{1 - 2\delta'_X(\varepsilon) + \frac{\varepsilon^4}{16}}. \tag{26}$$

On the other hand

$$\delta_X(\varepsilon) = 1 - \sup_{\|x-y\|\geqslant\varepsilon} \sqrt[4]{\cos(x, y) + \left\| \frac{x-y}{2} \right\|^4}$$

$$\leqslant 1 - \sup_{\|x-y\|\geqslant\varepsilon} \sqrt[4]{\cos(x, y) + \frac{\varepsilon^4}{16}} = 1 - \sqrt[4]{1 - 2\delta'_X(\varepsilon) + \frac{\varepsilon^4}{16}}.$$

Hence,

$$\delta_X(\varepsilon) \leqslant 1 - \sqrt[4]{1 - 2\delta'_X(\varepsilon) + \frac{\varepsilon^4}{16}}. \tag{27}$$

Using (26) and (27) we obtain (25).

Corollary 1. *A q.i.p. space X is an i.p. if and only if*

$$\delta'_X(\varepsilon) = \frac{\varepsilon^4}{4}\,.$$

Proof. If $\delta'_X(\varepsilon) = \dfrac{\varepsilon^2}{4}$ then from (25) we get

$$\delta_X(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}\,,$$

i.e. X is an i.p. space ([**3**]). To complete the proof we use (b) of Theorem 1.

Corollary 2. *For a q.i.p. space it holds*

$$\frac{\varepsilon^4}{32} \leqslant \delta'_X(\varepsilon) \leqslant \frac{\varepsilon^2}{4} \qquad \varepsilon \in [0, 2]. \tag{28}$$

Proof. According to (24) we get

$$\sup_{\|x-y\| \geqslant \varepsilon} \cos(x, y) \leqslant 1 - \frac{\varepsilon^4}{16}\,.$$

This implies that

$$\delta'_X(\varepsilon) \geqslant \frac{\varepsilon^4}{32}\,.$$

To complete the proof we use (c) of Theorem 1.

Clearly, for $1 < \varepsilon < 2$ the inequality

$$\frac{1}{4}(\varepsilon^3 - 3\varepsilon^2 + 4\varepsilon) < \frac{\varepsilon}{2}\,,$$

holds. This inequality is important in the sequel.

Theorem 4. *For a q.i.p. space X one holds the estimate*

$$\varrho'_X(\varepsilon) \leqslant \begin{cases} \dfrac{\varepsilon}{2} & 0 \leqslant \varepsilon \leqslant 1 \\[2ex] \dfrac{1}{4}(\varepsilon^3 - 3\varepsilon^2 + 4\varepsilon) & 1 \leqslant \varepsilon \leqslant 2\,. \end{cases}$$

Proof. For $\varepsilon \in [1, 2]$ from (23) we get

$$\varrho'_X(\varepsilon) = \frac{1}{2}(1 - \inf_{\|x-y\| \leqslant \varepsilon} \cos(x, y)) \leqslant \frac{1}{4}(\varepsilon^3 - 3\varepsilon^2 + 4\varepsilon).$$

Theorem 5. *For a q.i.p. space X the following estimate is true*

$$\varrho_X(\varepsilon) \leqslant \begin{cases} 1 - \sqrt[4]{1 - \varepsilon + \dfrac{\varepsilon^4}{16}} & 0 \leqslant \varepsilon \leqslant 1 \\ \dfrac{1}{4}(\varepsilon^3 - 3\varepsilon^2 + 4\varepsilon) & 1 \leqslant \varepsilon \leqslant 2\,. \end{cases}$$

Proof. Using (12) we conclude that $\|x - y\| \leqslant \varepsilon$ implies $\cos(x, y) \geqslant 1 - \varepsilon$ and $\|x - y\| \geqslant 1 - \cos(x, y) \geqslant 0$. Then it follows that

$$\cos(x, y) + \left\| \frac{x-y}{2} \right\|^4 \geqslant \cos(x, y) + \left(\frac{1 - \cos(x, y)}{2} \right)^4.$$

The function $t \mapsto f(t) = t + \left(\frac{1-t}{2} \right)^4$ is increasing on $[1 - \varepsilon, 1]$. Then

$$\min_{t \in [0, 1]} f(t) = f(1 - \varepsilon) = 1 - \varepsilon + \frac{\varepsilon^4}{16}\,.$$

So, for $0 \leqslant \varepsilon \leqslant 1$ and $\|x - y\| \leqslant \varepsilon$, we have

$$\cos(x, y) + \left\| \frac{x-y}{2} \right\|^4 \geqslant 1 - \varepsilon + \frac{\varepsilon^4}{16}\,.$$

Hence

$$(1 - \varrho_X(\varepsilon))^4 = \inf_{\|x-y\| \leqslant \varepsilon} \left\| \frac{x+y}{2} \right\|^4 = \inf_{\|x-y\| \leqslant \varepsilon} \left(\cos(x, y) + \left\| \frac{x-y}{2} \right\|^4 \right) \geqslant 1 - \varepsilon + \frac{\varepsilon^4}{16}\,,$$

i.e., for $\varepsilon \in [0, 1]$, we have

$$\varrho_X(\varepsilon) \leqslant 1 - \sqrt[4]{1 - \varepsilon + \frac{\varepsilon^4}{16}}\,.$$

Inequality $\varrho_X(\varepsilon) \leqslant \varrho'_X(\varepsilon)$ and Theorem 4 imply that Theorem 5 is true.

Theorem 6. (a) *For arbitrary X we have*

$$d'_X(\varepsilon) = \frac{1}{2}\Big[\sup_{\|x-y\|=\varepsilon} \cos(x, y) - \inf_{\|x-y\|\leqslant\varepsilon} \cos(x, y)\Big].$$

(b) *If X is an i.p. space, then $d'_X(\varepsilon) = 0$.*
(c) *If X is a q.i.p. space, then*

$$d'_X(\varepsilon) \leqslant \begin{cases} \dfrac{1}{2}\left(\varepsilon - \dfrac{\varepsilon^4}{16}\right), & 0 \leqslant \varepsilon \leqslant 1 \\ \dfrac{1}{2}\left[1 - \left(1 - \dfrac{\varepsilon}{2}\right)^4\right], & 1 \leqslant \varepsilon \leqslant 2 . \end{cases}$$

Proof. Using the Definition 6 and Definition 7 we conclude that (a) is true. (b) follows from Theorem 1 and Theorem 2. From Theorem 4 and (28), the statement (c) is true for $\varepsilon \in [0, 1]$. If $\varepsilon \in [1, 2]$, from Lemma 2 we have

$$\inf_{\|x-y\|\leqslant\varepsilon} \cos(x, y) \geqslant \left(1 - \frac{\varepsilon}{2}\right)^4 - \frac{\varepsilon^4}{16} \quad \text{and} \quad \sup_{\|x-y\|\geqslant\varepsilon} \cos(x, y) \leqslant 1 - \frac{\varepsilon^4}{16} .$$

Then by (a) we conclude that

$$d'_X(\varepsilon) \leqslant \frac{1}{2}\left[1 - \left(1 - \frac{\varepsilon}{2}\right)^4\right].$$

From inequality

$$\frac{1}{2}\left[1 - \left(1 - \frac{\varepsilon}{2}\right)^4\right] < \frac{\varepsilon}{2} \qquad (\varepsilon \in [0, 2]),$$

we have, for a q.i.p. space X, that

$$d'_X(\varepsilon) < \frac{\varepsilon}{2} .$$

References

[1] J. BANAS and K. FRACZEK, *On deformation of Banach spaces*, Comment. Math. Univ. Carolinae **34** (1993), 47-53.

[2] J. DANES, *On local and global moduli of convexity*, Comment. Math. Univ. Carolinae **17** (1976), 413-420.

[3] M. M. DAY, *Some characterizations of inner product spaces*, Trans. Amer. Math. Soc. **62** (1947), 320-337.

[4] P. M. MILIČIĆ, *Sur le produit scalaire généralisé*, Mat. Vesnik (25) **10** (1973), 325-329.

[5] P. M. MILIČIĆ, *Sur le g-angle dans un espace normé*, Mat. Vesnik **45** (1993), 43-48.

[6] P. M. MILIČIĆ, *A generalization of the paralellogram equality in normed spaces*, J. Math. Kyoto Univ. **38** (1998), 71-75.

Abstract

Using so called g-angle defined by (6) we introduce new notions of the modulus of a normed space X (the angle modulus of the convexity of X, the angle modulus of smoothness of X and the angle modulus of deformation of X). Some estimates of these moduli are described for so called a quasi-inner product spaces.

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