Riv. Mat. Univ. Parma (6) **3** (2000), 87-100

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Existence of nontrivial solutions to a nonlinear Dirichlet problem for the Q-Laplacian relative to Hörmander vector fields (**)

1 - Introduction

This paper deals with the existence of solutions of the problem

$$\begin{aligned} &u \in W_0^{1,Q}(\Omega, X) \\ &-\Delta_Q u \equiv -div(|Xu|^{Q-2} Xu) = f(x, u) \quad \text{in } \Omega \end{aligned} \tag{1}$$

where Ω is an open, bounded, connected subset of $\boldsymbol{R}^N$; X_j, for $j = 1, \dots, m$, are vector fields satisfying Hörmander's condition [**11**], [**12**]. Xu denotes the vector function $Xu := (X_1 u, \dots, X_m u)$ whereas div $\vec{w} := \sum_{j=1}^{m} X_j w_j$ for any vector function $\vec{w}: \boldsymbol{R}^N \to \boldsymbol{R}^m$.

For any $p \geq 1$, $W_0^{1,p}(\Omega, X)$ denotes the closure of $C_0^\infty(\Omega)$ under the norm $\|u\|_{1,p} = \left(\|u\|_p^p + \sum_{j=1}^{m} \|Xu\|_p^p\right)^{1/p}$, where $\|.\|_p$ denotes the $L^p(\Omega)$ norm. $Q \geq N$ is the homogeneous dimension associated to Ω and the vector fields [**13**].

We suppose that the nonlinearity $f(x, u)$ has a *subcritical growth on* Ω, i.e.

$$\lim_{|u| \to \infty} \frac{|f(x, u)|}{\exp(\alpha |u|^{Q'})} = 0 \quad \text{uniformly on } x \in \Omega, \ \forall \alpha > 0 \tag{2}$$

where $Q' = Q/(Q-1)$.

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(**) Received March 15, 2000. AMS classification 35 D 05, 35 J 20, 35 J 60,35 J 65, 35 J 70.

Other conditions we impose to f are

(F_1) $f : \overline{\Omega} \times \boldsymbol{R} \to \boldsymbol{R}$ is continuous , $f(x, 0) = 0$

(F_2) $\exists R, M > 0$ such that $\forall |u| \in \boldsymbol{R}, \forall x \in \Omega$

$$0 < F(x, u) = \int_0^u f(x, t)\, dt \leq M |f(x, t)|$$

(F_3) $0 < F(x, u) \leq \dfrac{1}{Q} f(x, u)\, u, \quad \forall |u| \in \boldsymbol{R} \backslash \{0\}, \quad \forall x \in \Omega$

(F_4) $\limsup\limits_{|u| \to 0} \dfrac{QF(x, u)}{|u|^Q} < \lambda_1$ uniformly on Ω

where $\lambda_1 > 0$ denotes the smallest eigenvalue of the problem $-\Delta_Q u = \lambda |u|^{Q-2} u$, $u \in W_0^{1, Q}(\Omega, X)$, which is variationally characterized as

$$\lambda_1 = \inf \left\{ \int |Xu|^Q dx \,|\, u \in W_0^{1, Q}(\Omega, X), \|u\|_Q = 1 \right\}. \tag{3}$$

We will prove the following result

Theorem 1. *Assume that f satisfies* (2), $(F_1), \ldots, (F_4)$. *Then the problem* (1) *has a nontrivial solution.*

The interest of Theorem 1 rests in the fact that f growth faster than any polynomial as $|u| \to +\infty$. In this case the «standard» methods for analyzing critical growth problems don't work. Recently Theorem 1 has been proved in the euclidean setting in [**6**] and [**7**] assuming f in the *critical or subcritical growth range* and using the Mountain Pass Lemma whithout the Palais-Smale condition.

In this paper we extend these new methods to the Hörmander vector field's setting for f in the subcritical growth range. On this object in Section **2** we state the existence of positive structural constants C and α_Q such that

$$\int_\Omega \exp(\alpha |u|^{Q'}) \leq C \tag{4}$$

for all $u \in W_0^{1, Q}(\Omega, X)$, $\|u\|_{1, Q}$ for every $\alpha \leq \alpha_Q$.

This result is well known in the euclidean setting, where Q=N, as the Trudinger-Moser inequality [**14**], [**18**]. A sharp version for higher order derivatives is due to D. R. Adams [**2**].

In these papers the largest positive real number α for which (4) holds, let it be α_N, is precisely calculated. Here we are able to determine the constant α_Q in terms of the other structural constants. Moreover this value is the best possible being equal to α_N in the euclidean case. We underline the interest of (4) not only in order to prove Theorem 1 but also as it permits us to extends to a larger setting many other important applications of the Trudinger-Moser inequality. In Section **2** we prove also a compact imbedding result.

In Section **3** we give a variational formulation of the problem and in Section **4** we prove Theorem 1.

2 - Preliminary results

Let Σ be an open connected subset of $\boldsymbol{R}^N$. Let us suppose that the rank of the Lie algebra generated by the vector fields X_j, $j = 1, \ldots, m$, equals N at each point of a neighbourhood Σ_0 of $\overline{\Sigma}$.

Let $\varrho(x, y)$, $x, y \in \Sigma$, be the metric associated to the vector fields X_j and let $B(x, r) := \{y \in \Sigma | \varrho(x, y) < r\}$, $x \in \Sigma$, $r > 0$ be the corresponding balls [**15**]. Let K_0 be an arbitrary compact subset of Σ. By the results of [**15**] there exist positive constants r_0, $c_0 = r_0$, $c_0(K_0)$ such that

$$|B(x, 2r)| \leq c_0 |B(x, r)| , \qquad \text{for any } 0 < r < r_0,\ x \in K_0 \tag{5}$$

where $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \boldsymbol{R}^N$. Moreover (see also [**9**]) there exist positive constants c_1, $c_2 = c_1$, $c_2(K_0)$ such that

$$c_1 \left(\frac{r}{s} \right)^{\alpha} \leq \frac{|B(x, r)|}{|B(x, s)|} \leq c_2 \left(\frac{r}{s} \right)^{\beta} \tag{6}$$

for any $0 < s < r < r_0$, $x \in K_0$, for suitable $\alpha = \alpha(x)$ and $\beta = \beta(x)$ with $N \leq \alpha \leq \beta \leq Q$. By the results of [**13**] a Sobolev-Poincaré inequality holds: there exist a positive contant $c_3 = c_3(K_0)$ such that, for any $0 < r < r_0$, $x \in K_0$ and for any $f \in C_0^\infty(B(x, r))$

$$\left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_B|^q dy \right)^{1/q} \leq c_3\, r \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} \left| \sum_{j=1}^{m} X_j f(y) \right|^p dy \right)^{1/p} \tag{7}$$

provided $1 \leq q < \dfrac{pQ}{Q-p}$, where $f_B = \dfrac{1}{|B(x, r)|} \displaystyle\int_{B(x, r)} f(y)\, dy$. By the results of [**17**], [**15**] (see also [**13**], [**8**]) the following estimate involving the Riesz potentials related to the vector fields holds:

$$|f(z)| \leq c_4 \int |Xf(y)| \frac{\varrho(z, y)}{|B(z, \varrho(z, y))|} dy \tag{8}$$

for any $0 < r < r_0$, $x \in K_0$ and for any $f \in C_0^\infty(B(x, r))$, for a suitable positive constant $c_4 = c_4(K_0)$.

Let x_0 be an arbitrary point of K_0. We will suppose $x_0 = 0$ for sake of simplicity and we will denote $\alpha_0 = \alpha(0)$, $\beta_0 = \beta(0)$ the constants appearing in (6). Let $\Omega \subset B(0, r_1/2)$ for $0 < r_1 < \dfrac{r_0(K_0)}{2}$, where $r_0(K_0)$ is the constant appearing in (5). In the following C will denote a positive structural constant not necessarily the same at each occurrence.

Lemma 2.1 ([2], Lemma 1). *Let $1 < p < +\infty$. Let $a(s, t)$ be a nonnegative measurable function on $(-\infty, +\infty) \times [0, +\infty)$ such that a.e. $a(s, t) \leq 1$ when $0 < s < t$ and $\sup_{t>0} \left(\int_{-\infty}^{0} + \int_{t}^{+\infty} a(s, t)^{p'} ds \right)^{1/p'} = b < +\infty$. Then there is a constant $\overline{C} = \overline{C}(p, b)$ such that, for any $\varphi \geq 0$ satisfying $\int_{-\infty}^{+\infty} \varphi(s)^p ds \leq 1$, we have $\int_0^{+\infty} e^{-F(t)} dt \leq \overline{C}$ if $F(t) = t - \left(\int_{-\infty}^{+\infty} a(s, t) \varphi(s) ds \right)^{p'}$.*

Proposition 2.2. *There is a constant $C_5 = C_5(Q, K_0, \Omega)$ such that, for all $f \in L^Q(\boldsymbol{R}^N)$ with support contained in Ω, $f \geq 0$,*

$$\int_\Omega \exp\left(\Gamma \left| \frac{I_1 \star f(x)}{\|f\|_Q} \right|^{Q'} \right) dx \leq C_5 \tag{9}$$

where $I_1 \star f(x) = \int \dfrac{\varrho(x, y) f(y)}{|B(x, \varrho(x, y))|} dy$ *and* $\Gamma = \dfrac{c_1 r_1^{\alpha_0}}{|B_{r_1}|} c_2^{-\frac{\alpha_0^2}{\beta_0(\alpha_0 - 1)}}$ *if* $B_{r_1} = B(0, r_1)$.

Let's observe that, in the euclidean setting where $c_1 = c_2 = 1$, $\alpha_0 = \beta_0 = N$, $|B_{r_1}| = r_1^N \dfrac{\omega_{N-1}}{N}$, it results $\Gamma = \dfrac{N}{\omega_{N-1}}$, which is proved to be an upper bound for (9), [2].

Proof. Let $f \in L^Q(\boldsymbol{R}^N)$ with support contained in Ω, $f \geq 0$, $\|f\|_{\alpha_0} \leq 1$. For any $s > 0$, let $\lambda_f(s) = |\{x \in \Omega \,|\, f(x) > s\}|$, and for any $t > 0$ let $f^*(t) = \inf\{s > 0 \,|\, \lambda_f(s) \leq t\}$ and $f^{**}(t) = t^{-1} \int_0^1 f^*(s) ds$.

Let $g(x) = \dfrac{\varrho(0, x)}{r_1^{\alpha_0}} \left(\dfrac{|B_{r_1}|}{|B(0, \varrho(0, x))|} \right)^{\frac{\alpha_0}{\beta_0}}$, $x \in \Omega$. By (6) we have $g(x) \leqslant \dfrac{c_2^{\frac{\alpha_0}{\beta_0}}}{\varrho(0, x)^{\alpha_0 - 1}}$. Then $\lambda_g(s) \leqslant \left| \left\{ x \in \Omega \,|\, \varrho(0, x) \leqslant \left(\dfrac{c_2^{\frac{\alpha_0}{\beta_0}}}{s} \right)^{\frac{1}{\alpha_0 - 1}} \right\} \right|$. If

$$\left(\frac{c_2^{\frac{\alpha_0}{\beta_0}}}{s} \right)^{\frac{1}{\alpha_0 - 1}} \leqslant r_1 \tag{10}$$

then, from (10) and (6) we obtain

$$\lambda_g(s) \leqslant \frac{|B_{r_1}|}{c_1 r_1^{\alpha_0}} c_2^{\frac{\alpha_0^2}{\beta_0(\alpha_0 - 1)}} s^{-\frac{\alpha_0}{\alpha_0 - 1}} = \Gamma^{-1} s^{-\frac{\alpha_0}{\alpha_0 - 1}}. \tag{11}$$

If (10) is not satisfied then, by (6),

$$\Gamma^{-1} s^{-\frac{\alpha_0}{\alpha_0 - 1}} \geqslant \frac{|B_{r_1}|}{c_1} \geqslant \frac{|B_{r_1}|}{c_1 2^{\alpha_0}} \geqslant |B_{r_1/2}| \geqslant |\Omega| \lambda_g(s).$$

So (11) holds in any case. It follows from (11) that $t \leqslant \dfrac{\Gamma^{-1}}{g^*(t)^{\alpha_0'}}$, where $\alpha_0' = \dfrac{\alpha_0}{\alpha_0 - 1}$ and then

$$g^*(t) \leqslant (t\Gamma)^{-\frac{1}{\alpha_0'}}. \tag{12}$$

If $v(x) = (I_1 \star f)(x)$, then by (12) and O'Neil's Lemma [**16**], Lemma 1.5, we have

$$\begin{aligned} v^*(t) &\leqslant v^{**}(t) \leqslant tf^{**} g^{**}(t) + \int_t^{+\infty} f^*(s)\, g^*(s)\, ds \\ &\leqslant \Gamma^{-\frac{1}{\alpha_0'}} \left(\frac{\alpha_0}{t^{\frac{1}{\alpha_0'}}} \int_0^t f^*(s)\, ds + \int_t^{r_1} \frac{f^*(s)}{s^{\frac{1}{\alpha_0'}}}\, ds \right) \equiv \Gamma^{-\frac{1}{\alpha_0'}} \gamma(t). \end{aligned} \tag{13}$$

Let

$$(14)\qquad a(s,t)=\begin{cases}1 & \text{for } 0<s<\log\dfrac{|\Omega|}{t}\\ \alpha_0 t^{-\frac{1}{\alpha_0'}}e^{-\frac{s}{\alpha_0}}|\Omega|^{\frac{1}{\alpha_0'}} & \text{for } \log\dfrac{|\Omega|}{t}<s<+\infty\\ 0 & \text{for } -\infty<s\leqslant 0\end{cases}$$

$$(15)\qquad \varphi(s)=|\Omega|^{\frac{1}{\alpha_0}}f^*(|\Omega|e^{-s})\,e^{-\frac{s}{\alpha_0}}$$

$$(16)\qquad F(t)=t-\left(\int_{-\infty}^{+\infty}a(s,t)\,\varphi(s)\,ds\right)^{\alpha_0'}$$

for $(s,t)\in(-\infty,+\infty)\times[0,+\infty)$. Then

$$(17)\qquad \int_\Omega f(x)^{\alpha_0}dx=\int_0^{|\Omega|}f^*(t)^{\alpha_0}dt=\int_0^{+\infty}\varphi(s)^{\alpha_0}ds\,.$$

Moreover $F(t)=t-\gamma(t)^{\alpha_0'}$, $t\in[0,+\infty)$, and then, for any $\delta>0$,

$$(18)\qquad \int_\Omega e^{\delta v(x)^{\alpha_0'}}dx=\int_0^{|\Omega|}e^{\delta v^*(t)^{\alpha_0'}}dt\leqslant\int_0^{|\Omega|}e^{\delta\Gamma^{-1}\gamma(t)^{\alpha_0'}}dt=\int_0^{|\Omega|}e^{\delta\Gamma^{-1}(t-F(t))}dt\,.$$

From (17) we have $\int_0^{+\infty}\varphi(s)^{\alpha_0}ds\leqslant 1$. By Lemma 2.1, $\int_0^{+\infty}e^{-F(t)}dt\leqslant\bar{c}$, and then, by (18), for $\delta\leqslant\Gamma$, $\int_\Omega e^{\delta v(x)^{\alpha_0'}}\leqslant C$, where C is a positive constant independent of f. Hence (9) follows.

Proposition 2.3. *There exists a positive constant* $c_6=c_6(Q,K_0,\Omega)$ *such that, for all* $u\in C_0^\infty(\Omega)$, $\|Xu\|_Q\leqslant 1$,

$$(19)\qquad \int_\Omega e^{\alpha|u(x)|^{Q'}}dx\leqslant c_6,\qquad \text{for all } \alpha\leqslant\alpha_Q=\frac{\Gamma}{c_4^{Q'}}\,.$$

Proof. By (8) we have $|u(x)|^{Q'}\leqslant c_4^{Q'}\,|I_1\star|Xu|(x)|^{Q'}$. Now just apply Proposition 2.2.

Proposition 2.4. *Let $q < Q$ and let $B_{r_1/2} = B(0, r_1/2)$. Every sequence (f_n) bounded in $W^{1,q}(B_{r_1/2}, X)$ is relatively compact in $L^q(B_{r_1/2})$.*

Proof. By a method of M. Biroli and S. Tersian [4] we will prove that (f_n) admits a subsequence convergent in $L^q(B_{r_1/2})$. On account of [5], the ball $B_{r_1/2}$ can be covered by a finite number of balls $B(x_j, r)$, $r \leq r_1/8$, $j = 1, \ldots, \nu$, such that $d(x_i, x_j) \geq r$, $\forall i, j = 1, \ldots, \nu$, where ν depends on r, r_1. It follows from the doubling property that every point x in $B_{r_1/2}$ belongs at most to M balls, where M does not depend on r. In fact for every such j we have $B(x_j, r/2) \subseteq B(x, 2r) \subseteq B(x_j, 4r)$, and then $|B(x_j, r/2)| \geq 2^{-(3Q+1)} |B(x, 2r)|$. Therefore, taking into account that M is also the number of points x_j in $B(x, r)$ we have

$$M 2^{-(3Q+1)} |B(x, 2r)| \leq M \min_{x_j \in B(x,r)} |B(x_j, r/2)| \leq \Big| \bigcup_{x_j \in B(x,r)} B(x_j, r/2) \Big| \leq |B(x, 2r)| .$$

Let $w_{n,m} = f_n - f_m$ and $(w_{n,m})_j$ the average of $w_{n,m}$ on $B(x_j, r)$. We have

$$\int_{B_{r_1/2}} |w_{n,m}|^q \leq 2^q \sum_{j=1}^{\nu} \int_{B(x_j, r)} |w_{n,m} - (w_{n,m})_j|^q + \nu 2^q \sup_j \frac{1}{|B(x_j, r)|^{q-1}} \int_{B(x_j, r)} |w_{n,m}|^q$$

$$\leq 2^q r^q c_3 \sum_{j=1}^{\nu} \int_{B(x_j, r)} |X w_{n,m}|^q + \nu 2^q \left[\frac{c_2}{|B_{r_1/2}|} \left(\frac{r_1}{r} \right)^{\beta} \right]^{q-1} \sup_j \int_{B(x_j, r)} |w_{n,m}|^q$$

on account of (7), $B(x_j, r_1) \supseteq B_{r_1/2} \supseteq B(x_j, r)$, and the doubling property. For small $\varepsilon > 0$ we choose $r = r_\varepsilon = \dfrac{1}{2} \left(\dfrac{\varepsilon}{2 c_3 C} \right)^{1/q}$, where $\int_{B_{r_1/2}} |X w_{n,m}| \leq C$ for any n, m. Taking into account that (f_n) is weakly convergent in $L^q(B_{r_1/2})$, we can choose n_ε such that, for $n, m > n_\varepsilon$,

$$\sup_j \int_{B(x_j, r)} |w_{n,m}|^q \leq \frac{\varepsilon}{\nu 2^{q+1}} \left[\frac{|B_{r_1/2}|}{c_2} \left(\frac{r}{r_1} \right)^{\beta} \right]^{q-1} .$$

Then for $n, m > n_\varepsilon$ we have $\int_{Br_1/2} |w_{n,m}|^q \leq \varepsilon$, i.e. (f_n) is a Cauchy sequence in $L^q(B_{r_1/2})$; then (f_n) converges strongly in $L^q(B_{r_1/2})$.

Corollary 2.5. *Let $1 \leq q < Q$. The imbedding of $W_0^{1,q}(\Omega, X)$ in $L^q(\Omega)$ is compact.*

Proposition 2.6. *Let* $1 \leqslant q < Q$ *and* $1 \leqslant p < \dfrac{qQ}{Q-q}$. *The imbedding of* $W_0^{1,q}(\Omega, X)$ *in* $L^p(\Omega)$ *is compact.*

Proof (see [**10**], Theorem 7.22). If $p \leqslant q$ the proof follows from Corollary 2.5. If $q < p < \dfrac{qQ}{Q-q}$, then let r be such that $q < p < r < \dfrac{qQ}{Q-q}$. From Hölder's inequality we have $\|u\|_p \leqslant \|u\|_q^{\lambda} \|u\|_r^{1-\lambda}$ where $\lambda \in (0, 1)$, $\dfrac{1}{p} = \dfrac{\lambda}{q} + \dfrac{1-\lambda}{r}$ and then, by (7), $\|u\|_p \leqslant \|u\|_q^{\lambda} \|Xu\|_q^{1-\lambda}$. Consequently a bounded set in $W_0^{1,q}(\Omega, X)$ must be precompact in $L^p(\Omega)$ for $p > 1$, and Proposition 2.6 is proved.

3 - The variational formulation

By (2) and (F_1) there exists a constant $C > 0$ such that

$$|f(x, u)| \leqslant C \exp(\alpha |u|^{Q'}), \qquad \forall (x, u) \in \Omega, \quad \forall \alpha > 0. \tag{20}$$

By (20) and (4), $f(x, u(x)) \in L^q(\Omega)$ for all $q > 1$ when $u \in W_0^{1,Q}(\Omega, X)$. In fact

$$\int_\Omega |f(x, u(x))|^q \leqslant C \int_\Omega \exp(\alpha q |u(x)|^{Q'}) \leqslant C \int_\Omega \exp\left(\alpha q \|u\|_{1,Q}^{Q'}\right) \left(\frac{|u(x)|}{\|u\|_{1,Q}} \right)^{Q'} \leqslant C$$

if $\alpha q \|u\|_{1,Q}^{Q'} \leqslant \alpha_Q$. The relation

$$I(u) = \frac{1}{Q} \int |Xu|^Q - \int F(x, u) \tag{21}$$

defines a C^1 functional $I : W_0^{1,Q}(\Omega, X) \to \boldsymbol{R}$ such that

$$\langle I'(u), v \rangle = \int |Xu|^{Q-2} Xu\, Xv - \int f(x, u)\, v, \qquad \forall v \in W_0^{1,Q}(\Omega, X) \tag{22}$$

where $< \cdot, \cdot >$ denotes the duality between $W_0^{1,Q}(\Omega, X)$ and its dual space. It follows from (F_1), (F_2), (F_3) that

$$\exists C > 0 \ \text{ such that } \ \forall |u| \geqslant R,\ \forall x \in \Omega,\ F(x, u) \geqslant C \exp\left(\frac{|u|}{M} \right) \tag{23}$$

$$\exists R_0 > 0,\ \theta > Q \ \text{ such that } \ \forall |u| \geqslant R_0,\ \forall x \in \Omega,\ \theta F(x, u) \leqslant u f(x, u). \tag{24}$$

Lemma 3.1. *Assume* (F_1), (F_2) *and* (F_3). *Then* $I(tu) \to -\infty$ *as* $t \to +\infty$, *for all* $u \in W_0^{1,Q}(\Omega, X) \setminus \{0\}$.

Proof. Let $u \in W_0^{1,Q}(\Omega, X) \setminus \{0\}$, and let $p > Q$. By (20) and (23), there exists a positive constant C such that $\forall x \in \Omega$

$$F(x, u) \geq |u|^p - C \tag{25}$$

and then $I(tu) \leq \frac{t^Q}{Q} \int |Xu|^Q - Ct^Q \int |u|^p + C$. Since $p > Q$, we obtain $I(tu) \to -\infty$ as $t \to +\infty$.

Lemma 3.2. *Assume* (F_1), (F_2), (F_3). *Then there exist* δ, $\varrho > 0$ *such that*

$$I(u) \geq \delta\,, \qquad \text{if } \|u\|_{1,Q} = \varrho \tag{26}$$

Proof. Using (F_1), (F_2), (F_3) and (20) we can choose $\lambda < \lambda_1$ such that

$$F(x, u) \leq \frac{1}{Q} \lambda |u|^Q + C \exp(\alpha |u|^{Q'}) |u|^q, \qquad \forall (x, u) \in \Omega \times \boldsymbol{R}\,, \qquad \forall \alpha > 0 \tag{27}$$

if $q > Q$. By Hölder's inequality and (4) we obtain

$$\begin{aligned} \int \exp(\alpha |u|^{Q'}) |u|^q &\leq \left\{ \int \exp\left[\alpha r \|u\|_{1,Q}^{Q'} \left(\frac{|u|}{\|u\|_{1,Q}} \right)^{Q'} \right] \right\}^{1/r} \\ &\cdot \left\{ \int |u|^{sq} \right\}^{1/s} \leq C(Q) \left\{ \int |u|^{sq} \right\}^{1/s} \end{aligned} \tag{28}$$

if $\alpha r \|u\|_{1,Q}^{Q'} < \alpha_Q$, where $\frac{1}{r} + \frac{1}{s} = 1$. Then, by (27), (28), (3) and (7), we have $I(u) \geq \frac{1}{Q}\left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|_{1,Q}^Q - C(Q) \|u\|_{1,Q}^q$. As $\lambda < \lambda_1$ and $q > Q$, we can choose $\varrho > 0$ such that $I(u) \geq \delta$ if $\|u\|_{1,Q} = \varrho$.

Proposition 3.3. *Assume* (F_1), (F_2), (F_3). *Let* (u_n) *be a Palais-Smale sequence*, i.e.

$$I(u_n) \to c \text{ and } I'(u_n) \to 0 \text{ in } W^{-1,Q'}(\Omega, X) \text{ as } n \to +\infty\,.$$

Then (u_n) *has a subsequence, still denoted by* (u_n) *for sake of simplicity, and there exists* $u \in W_0^{1,Q}(\Omega, X)$ *such that*

(i) $f(x, u_n) \to f(x, u)$ in $L^1(\Omega)$ as $n \to +\infty$
(ii) $|Xu_n|^{Q-2} Xu_n \to |Xu|^{Q-2} Xu$ *weakly in* $(L^{Q'}(\Omega))^m$ *as* $n \to +\infty$
(iii) u *solves* (1).

Proof. The proof follows the outline of [7], Lemma 4. As (u_n) is a Palais-Smale sequence, then

$$\frac{1}{Q}\int |Xu_n|^Q - \int F(x, u_n) \to c \tag{29}$$

$$\left| \int |Xu_n|^{Q-2} Xu_n \cdot Xv - \int f(x, u_n)\, v \right| \leq \varepsilon_n \|v\|_{1, Q}, \qquad \forall v \in W_0^{1, Q}(\Omega, X) \tag{30}$$

where $\varepsilon_n \to 0$ as $n \to +\infty$. Multiplying (29) by the constant $\theta > Q$ of (24) and subtracting (30) with $v = u_n$, we obtain

$$\left(\frac{\theta}{Q} - 1\right)\int |Xu_n|^Q - \int [\theta F(x, u_n) - f(x, u_n)\, u_n] \leq C + \varepsilon_n \|u_n\|_{1, Q}. \tag{31}$$

From (31) and (F_3) we deduce that (u_n) is bounded in $W_0^{1, Q}(\Omega, X)$. Moreover, unless we extract a subsequence, still denoted by (u_n), we have as $n \to +\infty$

$$\begin{aligned} &u_n \to u \text{ weakly in } W_0^{1, Q}(\Omega, X) \\ &u_n \to u \text{ in } L^q(\Omega), \qquad \forall q \geq 1 \\ &u_n(x) \to u(x) \;\; a.\, e. \;\; x \in \Omega\, . \end{aligned} \tag{32}$$

Then $|Xu_n|^{Q-2} Xu_n$ is bounded in $(L^{Q'}(\Omega))^m$ and, from (30), we have

$$\int f(x, u_n)\, u_n \leq C\, . \tag{33}$$

From (32), (33) and [6], Lemma 2.1, we have

$$f(x, u_n) \to f(x, u) \quad \text{ in } L^1(\Omega) \text{ as } n \to +\infty \tag{34}$$

and the first assertion of Proposition 3.3 is so proved. It follows from (F_2) and (34), using the generalized Lebesgue dominated convergence theorem, that

$$F(x, u_n) \to F(x, u) \quad \text{ in } L^1(\Omega) \text{ as } n \to +\infty\, . \tag{35}$$

From (29), (30) we obtain

$$\begin{aligned} &\lim_{n \to +\infty} \|u_n\|_{1, Q}^Q = Q\left(c + \int F(x, u)\right) \\ &\lim_{n \to +\infty} \int f(x, u_n)\, u_n = Q(c + F(x, u))\, . \end{aligned} \tag{36}$$

By (F_3) and (36) we conclude $c \geq 0$. So any Palais-Smale sequence approaches a nonnegative level. To prove the second assertion we observe that, by simple calculation of vectorial algebra, we have

$$\begin{aligned}(37)\qquad &(|Xu_n|^{Q-2}Xu_n - |Xu|^{Q-2}Xu)(Xu_n - Xu)\\ &= \frac{1}{2}|Xu_n - Xu|^2(|Xu_n|^{Q-2} + |Xu|^{Q-2})\\ &+ \frac{1}{2}[|Xu_n|^2 - |Xu|^2][|Xu_n|^{Q-2} - |Xu|^{Q-2}] \geq 0 .\end{aligned}$$

By proving that

$$(38)\qquad \int_\Omega (|Xu_n|^{Q-2}Xu_n - |Xu|^{Q-2}Xu)(Xu_n - Xu)\,\psi \to 0$$

as $n \to +\infty$, for any test function $\psi \in C_0^\infty(B_r)$, $B_r \subset \Omega$, $\psi = 1$ on $\Omega \backslash B_r$, $0 \leq \psi \leq 1$, we obtain $Xu_n \to Xu$ as $n \to +\infty$ a.e. in Ω, and then, taking into account that $(|Xu_n|^{Q-2}Xu_n)$ is bounded in $(L^{Q'}(\Omega))^m$, unless we take a subsequence, we conclude the proof.

Let's observe at first that, by (2), (3) and the boundedness of (u_n) in $W_0^{1,Q}(\Omega, X)$, we have, for any $q > 1$

$$(39)\qquad \int_\Omega |f(x, u_n)|^q \leq C , \qquad \text{for every } n .$$

Notice that

$$\int_\Omega |f(x, u_n)\,u_n - f(x, u)\,u| \leq \int_\Omega |f(x, u_n) - f(x, u)|\,|u| + \int_\Omega |f(x, u_n)|\,|u_n - u| .$$

Since $f(x, u_n) \to f(x, u)$ in $L^1(\Omega)$ as $n \to +\infty$, then $f(x, u_n)v \to f(x, u)v$ in $L^1(\Omega)$ as $n \to +\infty$, $\forall v \in D(\Omega)$, and then

$$(40)\qquad \lim_{n \to +\infty} \int_\Omega |f(x, u_n) - f(x, u)|\,|u| = 0 .$$

On the other hand, by Hölder's inequality and (39) we have

$$(41)\qquad \int_\Omega |f(x, u_n)|\,|u_n - u| \leq C\left(\int_\Omega |u_n - u|^{q'}\right)^{1/q'} \to 0 \qquad \text{as } n \to +\infty .$$

Hence by (40) and (41)

$$\lim_{n\to+\infty}\int_\Omega |f(x,u_n)\,u_n - f(x,u)\,u| = 0\,. \tag{42}$$

To prove (38) we observe that, if we take $v=\psi u_n$ or $v=\psi u$ in (30), then we obtain respectively

$$\int_\Omega |Xu_n|^Q\psi + u_n\,|Xu_n|^{Q-2}Xu_nX\psi - \psi f(x,u_n)\,u_n \leqslant \varepsilon_n\|\psi u_n\|_{1,Q} \tag{43}$$

$$\int_\Omega -|Xu_n|^{Q-2}\psi Xu_nXu - |Xu_n|^{Q-2}uXu_nX\psi + \psi f(x,u_n)\,u \leqslant \varepsilon_n\|\psi u\|_{1,Q}. \tag{44}$$

Then

$$\begin{aligned}
0 &\leqslant (|Xu_n|^{Q-2}Xu_n - |Xu|^{Q-2}Xu)(Xu_n - Xu)\,\psi\\
&= \int_\Omega |Xu_n|^Q\psi - |Xu_n|^{Q-2}\psi Xu_nXu - |Xu|^{Q-2}\psi Xu\,Xu_n + |Xu|^Q\psi\\
&\leqslant -\int_\Omega u_n\,|Xu_n|^{Q-2}Xu_nX\psi + \int_\Omega \psi f(x,u_n)\,u_n + \varepsilon_n\|\psi u_n\|_{1,Q}\\
&\quad + \int_\Omega u|Xu_n|^{Q-2}Xu_nX\psi - \int_\Omega \psi f(x,u_n)\,u + \varepsilon_n\|\psi u\|_{1,Q}\\
&= \int_\Omega |Xu_n|^{Q-2}Xu_nX\psi(u-u_n) + \int_\Omega \psi f(x,u_n)(u_n-u) + \varepsilon_n(\|\psi u_n\|_{1,Q} + \|\psi u\|_{1,Q})\,.
\end{aligned} \tag{45}$$

Now it suffices to prove that each term in the last member of (45) tends to 0 as $n\to+\infty$. Using the interpolation inequality $ab\leqslant\delta a^{Q/(Q-1)}+C_\delta b^Q$ with $C_\delta=\delta^{1-Q}$, we have

$$\begin{aligned}
\int_\Omega |Xu_n|^{Q-2}Xu_nX\psi(u-u_n) &\leqslant \delta\int_\Omega |Xu_n|^Q + C_\delta\int_\Omega |X\psi|^Q\,|u-u_n|^Q\\
&\leqslant \delta C + C_\delta\Big(\int_\Omega |X\psi|^{rQ}\Big)^{1/r}\Big(\int_\Omega |u-u_n|^{sQ}\Big)^{1/s}
\end{aligned}$$

where $\dfrac{1}{r}+\dfrac{1}{s}=1$. Thus, since $u_n\to u$ in $L^{sQ}(\Omega)$ as $n\to+\infty$ and δ is arbitrarily small we obtain that

$$\limsup_{n\to+\infty}\int_\Omega |Xu_n|^{Q-2}Xu_nX\psi(u-u_n)\leqslant 0\,. \tag{46}$$

On the other hand, since $u_n \to u$ in $L^{q'}(\Omega)$ as $n \to +\infty$ and by (39), we have

$$(47) \qquad \int_\Omega \psi f(x, u_n)(u_n - u) \leq \Big(\int_\Omega f(x, u_n)^q\Big)^{1/q} \Big(\int_\Omega |u_n - u|^{q'}\Big)^{1/q'} \to 0 .$$

So (**ii**) is proved. (**iii**) follows from (**ii**) and (30).

4 - Proof of Theorem 1

It follows from Lemma 3.1, Lemma 3.2 and the Mountain-Pass Lemma [**1**] that there exists a positive level c and a Palais-Smale sequence (u_n) in $W_0^{1,Q}(\Omega, X)$ i.e. $I(u_n) \to c$, $I'(u_n) \to 0$ in $W^{-1,Q'}(\Omega, X)$ as $n \to +\infty$. In view of Proposition 3.3 there are a subsequence of (u_n), still denoted by (u_n), and $u \in W_0^{1,Q}(\Omega, X)$ such that (29), (30) hold and u solves (1). The proof is concluded if we prove that $u \neq 0$. If $u = 0$ we have from (36),

$$\lim_{n\to+\infty} \int_\Omega |Xu_n|^Q = \lim_{n\to+\infty} \int_\Omega f(x, u_n)\, u_n \leq \lim_{n\to+\infty} \Big(\int_\Omega |f(x, u_n)|^q\Big)^{1/q} \|u_n\|_{q'} = 0 .$$

But, from (F_3), $\lim_{n\to+\infty} \int_\Omega F(x, u_n) = 0$, and, by (36), $\lim_{n\to+\infty} \int_\Omega |Xu_n|^Q = Qc$. A contradiction.

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Abstract

In this paper we study the existence of solutions for the problem $-\Delta_Q u = f(x, u)$, $u \in W_0^{1,Q}(\Omega, X)$, *where* Δ_Q *is the Q-Laplacian in the Hörmander vector field setting, Q is the homogeneous dimension associated to* Ω *and the nonlinearity f has a subcritical growth on* Ω.

* * *