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Computation of Newton sum rules for polynomial solutions of O.D.E. with polynomial coefficients (**)

1 - Introduction

Consider polynomial eigenfunctions $P_N(x)$ of a linear differential operator of order m:

$$\sum_{i=0}^{m} g_i(x)\, f^{(i)}(x) = 0\,, \tag{1.1}$$

where the coefficients $g_i(x)$ are polynomials of degree c_i:

$$g_i(x) = \sum_{j=0}^{c_i} a_j^{(i)} x^j .$$

We will assume that $P_N(x) = \text{const.} \prod_{l=1}^{N} (x - x_l)$, where all x_l are different, so that zeros of $P_N(x)$ are all simple, and we will write in the following:

$$P_N(x) = x^N - u_{N,1} x^{N-1} + u_{N,2} x^{N-2} + \ldots + (-1)^N u_{N,N} \tag{1.2}$$

or

$$P_N(x) = x^N - u_1 x^{N-1} + u_2 x^{N-2} + \ldots + (-1)^N u_N . \tag{1.2'}$$

If $c_i \leq i$ $(i = 0, 1, \ldots, m)$ the differential operator (1.1) is called of hypergeometric type. When $m = 2$, and $c_i \leq i$ $(i = 0, 1, \ldots, m)$, polynomial solutions of (1.1)

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are very classical since they are connected with classical orthogonal polynomials, and have been deeply studied by A. F. Nikiforov - V. B. Uvarov in [**1**].

The case when $m=4$ was first considered by H. L. Krall [**2**], [**3**] in the thirties, but more recently in many papers by A. M. Krall [**4**], L. L. Littlejohn [**5**], [**6**] and others [**7**], [**8**], [**9**], [**10**]. New classes of orthogonal polynomials can be found in this way, such as the Heine polynomials (see T. S. Chihara [**11**]), and some generalizations of the classical polynomials obtained by adding Dirac measures in the support of the corresponding absolutely continuous Borel measure (see R. Alvarez Nodarse - F. Marcellan [**12**]).

To any polynomial $P_N(x)$ it is possible to associate a normalized discrete density distribution $\varrho_N(x)$ defined by

$$\varrho_N(x) = \frac{1}{N}\sum_{l=0}^{N} \delta(x-x_l) \qquad (\delta = \text{Dirac delta})$$

whose moments around the origin are given by

$$\mu_h = \frac{1}{N} y_h = \frac{1}{N}\sum_{i=1}^{N} x_i^h .$$

Computation of the y_h (Newton sum rules) has been considered by K. M. Case [**13**] for the hypergeometric case $c_i \leqslant i$ $(\forall i = 0, 1, \ldots, m)$, and by E. Buendia - J. S. Dehesa - F. J. Gálvez in [**14**] in the general case.

A computation of the Case method was given by P. E. Ricci [**15**] and P. Natalini [**16**] for the hypergeometric case. We used the generalized Lucas polynomials of the second kind in order to represent the Case sum rules.

In this paper, starting from the above mentioned paper [**14**], we first extend our method to this general case. Then, considering the recursive formula representing the coefficients of $P_N(x)$ in terms of the coefficients of the differential operator (1.1), introduced in [**14**], formula 13, we simply use the generalized Lucas polynomials of first kind in order to compute numerically the Newton sum rules.

2 - The generalized Case method

We recall first the definition of the generalized Lucas polynomials of second kind. They are defined as the solution of the bilateral linear homogeneous recurrence relation

$$\Phi_n = u_1 \Phi_{n-1} - u_2 \Phi_{n-2} + \ldots + (-1)^r u_r \Phi_{n-r}, \qquad (n \in \mathbf{Z})$$

corresponding to the initial conditions

$$\Phi_{-1} = 0,\ \Phi_0 = 0,\ \Phi_1 = 0,\ \ldots,\ \Phi_{r-2} = 1 .$$

This solution is called the *fundamental solution* of the above mentioned recurrence relation since all solutions of it can be expressed in terms of this particular solution (see e.g. [**17**], [**18**]).

E. Buendia, J. S. Dehesa, F. J. Gálvez [**14**] by generalizing the Case paper [**13**] proved the following recursive relation for the y_h Newton sum rules:

$$\sum_{i=2}^{m} i \sum_{l=-1}^{s+c_i-i-1} a_{i+l+1-s}^{(i)} J_{i+l}^{(i)} = -\sum_{j=0}^{c_1} a_j^{(1)} y_{s+j-1}, \qquad (s \geq 1) \tag{2.1}$$

assuming, by definition:

$$J_r^{(i)} = 0 \qquad \text{for } 0 \leq r \leq i-2,$$

and

$$J_r^{(i)} = \sum_{\neq (l_1, \ldots, l_i)}^{(1, \ldots, N)} \frac{x_{l_1}^r}{\prod_{k=1}^{i} (x_{l_1} - x_{l_k})}. \tag{2.2}$$

The $J_r^{(i)}$ are so called Case sum rules (see [**13**]), and in the last formula, $\sum_{\neq (l_1, \ldots, l_i)}^{(1, \ldots, N)}$ means that the sum runs over all l_s $(s = 1, \ldots, N)$ provided that $\forall i \neq j$, $l_i \neq l_j$.

The Case sum rules $J_r^{(i)}$ can be expressed in terms of the Newton sum rules y_t with $t \leq r - i + 1$ by means of the following representation theorem:

Proposition. *For any* $N \in \boldsymbol{N}$ $(N \geq 2)$, $r \in \boldsymbol{N}_0 := \boldsymbol{N} \cup \{0\}$, $i \in \boldsymbol{N}$, *and s.t.* $2 \leq i \leq N$, *then*

$$J_r^{(i)} = (i-1)! \sum_{k=0}^{N-i} (-1)^k \binom{N-k}{i} u_k \Phi_{N+r-i-k-1}(u_1, u_2, \ldots, u_N), \tag{2.3}$$

where $\Phi_h(u_1, u_2, \ldots, u_N)$ *denote the generalized Lucas polynomials of second kind in* N *variables.*

The proof is exactly the same as in the above mentioned paper [**15**], [**16**], since formula (2.3) gives a representation of the function $J_r^{(i)}$, which is a symmetric function of the zeros of $P_N(x)$, in terms of the coefficients of $P_N(x)$. *The possibility to obtain such a formula is a consequence of a well known Gauss' theorem on symmetric functions* (see e.g. [**19**], [**20**], p. 210), *and obviously, this formula is independent of the differential equation satisfied by* $P_N(x)$. Note that if the polynomials $P_N(x)$ satisfies an hypergeometric type differential equation (i.e. if $c_i \leq i$, $\forall i = 0, 1, \ldots, m$), then the representation formula (2.1) simplifies into:

$$\sum_{i=2}^{m} i \sum_{j=0}^{i} a_j^{(i)} J_{s+j}^{(i)} = -a_0^{(1)} y_s - a_1^{(1)} y_{s+1}, \qquad (s \geq 0) \tag{2.4}$$

and every $J_{s+j}^{(i)}$ can be computed in terms of the y_t $(t \leq s)$ so that starting from

$$y_0 = \sum_{i=1}^{N} x_i^0 = N ,$$

the recurrence relation (2.4) permits the computation of all Newton sum rules.

In the general case, since in the right hand side of (2.1) the more general combination

$$-a_0^{(1)} y_{s-1} - a_1^{(1)} y_s - \ldots - a_{c_1}^{(1)} y_{s+c_1-1}$$

occurs, then for computing all Newton sum rules it is necessary to construct separately the first values

$$y_0 = N, y_1, \ldots, y_{c_1-1}.$$

But this is not sufficient, since a similar indeterminacy problem arises in the left hand side, in which quantities $J_r^{(i)}$ appear, involving y_t with $t \leq r - i + 1$, so that $J_{i+l}^{(i)}$ is expressed in terms of the y_t, where $t \leq i + l - i + 1 = l + 1 \leq s + c_i - i - 1 + 1 = s + c_i - i$.

Then, in order that recurrence (2.1) works, it is sufficient to know y_t for $t \leq s + q$ $(s \geq 0)$, where

$$q := \max \{c_i - i; \quad i = 0, 1, 2, \ldots, m\} , \tag{2.5}$$

i.e. to know

$$y_0 = N, y_1, y_2, \ldots, y_q. \tag{2.6}$$

In the above mentioned paper [**14**] the Authors give expressions for the initial conditions (2.6) of the recurrence relation (2.1) in terms of the coefficients of the polynomial $P_N(x)$, by using the Newton-Girard formulas:

$$\begin{cases} u_1 = y_1 \\ u_2 = \dfrac{1}{2}(u_1 y_1 - y_2) = \dfrac{1}{2}(y_1^2 - y_2) \\ u_3 = \dfrac{1}{3}(-u_1 y_2 + u_2 y_1 + y_3) = \dfrac{1}{3}(y_1^3 - 3y_1 y_2 + 2y_3) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_N = \dfrac{1}{N}[(-1)^N u_1 y_{N-1} + (-1)^{N-1} u_2 y_{N-2} + \ldots + u_{N-1} y_1 + (-1)^{N-1} y_N]. \end{cases} \tag{2.7}$$

More precisely, initial conditions (2.6) are found by using (2.7) and the following explicit recurrent expressions for the coefficients of $P_N(x)$ in terms of the coefficients of the differential equation (1.1):

$$u_s = -\frac{\sum_{k=1}^{s}(-1)^k u_{s-k}\sum_{i=0}^{m}\frac{(N-s+k)!}{(N-s+k-i)!}a_{i+q-k}^{(i)}}{\sum_{i=0}^{m}\frac{(N-s)!}{(N-s-i)!}a_{i+q}^{(i)}}, \tag{2.8}$$

where $u_0 := 1$. The Authors also note that eqs. (2.7), (2.8) give also possibility to compute recursively the Newton sum rules y_t, but due to high non linearity of (2.7), they use only this method in order to compute initial condition (2.6), and subsequently, they use recurrence relation (2.1).

Concluding this section we can say that *even in this more general case (with respect to the hypergeometric case considered in* [**15**], [**16**]), *representation formula* (2.3), *Newton-Girard formulas* (2.7), *and initial condition obtained by using* (2.8) *completely solve the problem of computing by recursion Newton sum rules*, where as in [**14**], the problem is solved only in particular (but relevant) cases.

3 - Computation of Newton sum rules by using generalized Lucas polynomials of the first kind

We recall here the definition of the generalized Lucas polynomials of the first kind:

$$\begin{cases} \Psi_{N-1}(u_1,u_2,\dots,u_N)=u_1 \\ \Psi_N(u_1,u_2,\dots,u_N)=u_1^2-2u_2 \\ \Psi_{N+1}(u_1,u_2,\dots,u_N)=u_1^3-3u_1u_2+3u_3 \\ \dots\dots\dots\dots\dots\dots \\ \Psi_{2N-2}(u_1,u_2,\dots,u_N)=u_1\Psi_{2N-3}-u_2\Psi_{2N-4}+\dots+(-1)^{N-2}u_{N-1}\Psi_{N-1} \\ \qquad\qquad +(-1)^{N-1}Nu_N \end{cases} \tag{3.1}$$

and for $h > 2N-2$:

$$\begin{aligned}&\Psi_h(u_1, u_2, \dots, u_N)\\ &= u_1\Psi_{h-1} - u_2\Psi_{h-2} + \dots + (-1)^{N-2}u_{N-1}\Psi_{h-N+1} + (-1)^{N-1}u_N\Psi_{h-N}.\end{aligned} \tag{3.2}$$

Then, according to the above mentioned definition, the generalized Lucas polynomial of the first kind $\Psi_h(u_1, u_2, \ldots, u_N)$ gives the sum of the $(h - N + 2)$-th powers of the roots of $P_N(x)$, i.e. the Newton sum rule y_{h-N+2}.

Then it is possible to formalize connection between coefficients of differential equation (1.1) and Newton sum rules of zeros of $P_N(x)$, *via the Newton-Girard formulas* (2.7), *and avoiding the generalized Case method*, by using the following

Proposition. *Consider a polynomial* $P_N(x)$, *given by* (1.2), *which satisfies differential equation with polynomial coefficients* (1.1). *Then, coefficients of* $P_N(x)$ *are recursively linked to the coefficients of* (1.1) *by formula* (2.8), *and for the Newton sum rules the following representation formula holds true*:

$$y_h = \sum_{k=1}^{N} x_k^h = \Psi_{h+N-2}(u_1, u_2, \ldots, u_N). \tag{3.3}$$

This formula, provided that initial conditions (3.1) are computed, permits recursive computation of moments via (3.2).

Remark. Note that the starting set of the Lucas polynomials of first kind is obtained by inverting by the Newton-Girard formulas (2.7). In formulas (3.1) the coefficients $u_1, u_2, \ldots, u_N$ are considered as independent variables. This assumption is important since by the physical point of view it is interesting to test the variation of moments in terms of the variation of coefficients.

4 - Numerical examples

We present here a numerical example in which computation of moments for some generalized classical polynomials (obtained by adding Dirac measures to the classical measures) considered in [**12**] is given by using the last formula which uses representation formula (3.3) i.e. the generalized Lucas polynomials of the first kind.

Generalized Hermite polynomials $H_{2N}^A(x)$

$\mu_{2i+1} = 0 \ \forall N, A$

$A = 1$	$N = 9$	$N = 12$	$N = 15$	$N = 18$
μ_2	8.4637109	11.4714258	14.4763419	17.4797658
μ_4	139.6449379	258.1025935	412.5723070	603.0491760
μ_6	2'809.1547280	7'128.2790331	14'489.1565804	25'701.8602029
μ_8	62'006.3541057	217'187.9087867	563'137.3668598	1'214'790.5356837

$A = 2$	$N = 9$	$N = 12$	$N = 15$	$N = 18$
μ_2	8.4560984	11.4660998	14.4723257	17.4765866
μ_4	139.5210206	257.9840863	412.4589377	602.9403909
μ_6	2'806.0889184	7'124.2801457	14'484.3112674	25'696.2299883
μ_8	61'924.1366998	217'040.6653181	562'910.5387821	1'214'470.5089642

$A = 3$	$N = 9$	$N = 12$	$N = 15$	$N = 18$
μ_2	8.4527979	11.4638539	14.4706655	17.4752925
μ_4	139.4676167	257.9343205	412.4122004	602.8962088
μ_6	2'804.7675299	7'122.6007871	14'482.3124078	25'693.9425318
μ_8	61'888.7046914	216'978.8340725	562'816.8837283	1'214'340.4213211

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Abstract

Generalized Lucas polynomials of second and first kind are used in order to compute the Newton sum rules of polynomial solutions of all ordinary differential equations with polynomial coefficients.

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