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On an extension of the B-scroll surface in Lorentz 3-space R_1^3 (**)

1 - Elementary Lorentz geometry

Definition 1. Let R_1^3 be a 3-dimensional Lorentzian space with the pseudo metric $ds^2 = dx^2 + dy^2 - dz^2$. If $\langle \vec{X}, \vec{Y} \rangle = 0$ for all $\vec{X}$ and $\vec{Y}$, the vectors $\vec{X}$ and $\vec{Y}$ are called *perpendicular in the sense of Lorentz,* where $\langle . \rangle$ is the induced inner product in R_1^3.

Definition 2. The norm of $\vec{X} \in R_1^3$ is denoted by $\| \vec{X} \|$ and defined as

$$\| \vec{X} \| = \sqrt{| \langle \vec{X}, \vec{Y} \rangle |}\,.$$

The vector $\vec{X} \in R_1^3$ is called *a spacelike, timelike and null* (*lightlike*) *vector* if $\langle \vec{X}, \vec{X} \rangle > 0$ or $\vec{X} = 0$, $\langle \vec{X}, \vec{X} \rangle < 0$, and $\langle \vec{X}, \vec{X} \rangle = 0$ for $\vec{X} \neq 0$ respectively.

Definition 3. A regular curve $\alpha(s) : I \to R_1^3$, $I \in R$ in R_1^3 is said to be *a spacelike timelike and null curve* if the velocity vector $\alpha'(s) = \dfrac{d\alpha}{ds}$ is a spacelike, timelike or null vector respectively [**6**].

Definition 4. A surface in a 3-dimensional Lorentz space is called *a timelike surface* if the induced metric on the surface ia a Lorentz metric, i.e., the normal on the surface is a spacelike vector [**1**].

As revealed from the foregoing definitions, one can prove the following

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Lemma 1. *In the Lorentz space R_1^3, the following properties are satisfied*

(i) *Two timelike vectors are never orthogonal.*

(ii) *Two null vectors are orthogonal if and only if they are linearly dependent.*

(iii) *A timelike vector is never orthogonal to a null (lightlike) vector* [**5**].

2 - Null curve and frames in R_1^3

Definition 5. *The null frame of a null curve* $\alpha = \vec{\alpha}(s) \in R_1^3$ parametrized by the natural parametrization, is a frame field $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, having properties

$$
\begin{gathered}
\langle \vec{e}_1, \vec{e}_1 \rangle = \langle \vec{e}_2, \vec{e}_2 \rangle = 0\,, \qquad \langle \vec{e}_3, \vec{e}_3 \rangle = 1 \\
\langle \vec{e}_1, \vec{e}_3 \rangle = \langle \vec{e}_2, \vec{e}_3 \rangle = 0\,, \qquad \langle \vec{e}_1, \vec{e}_2 \rangle = -1 \\
\det(\vec{e}_1, \vec{e}_2, \vec{e}_3) = 1\,.
\end{gathered}
\tag{1.1}
$$

The infinitesimal displacements of null frame are given as

$$
\begin{gathered}
\vec{\alpha}'(s) = \vec{e}_1, \qquad \vec{e}_1'(s) = k(s)\,\vec{e}_3 \\
\vec{e}_2'(s) = \tau(s)\,\vec{e}_3, \qquad \vec{e}_3'(s) = \tau(s)\,\vec{e}_1 + k(s)\,\vec{e}_2, \qquad {}' = \frac{d}{ds}
\end{gathered}
\tag{1.2}
$$

where $\vec{e}_1(s)$ is the unit tangent null vector of curve $\alpha(s)$, $\vec{e}_2(s)$ is the principal normal vector field of type null and $\vec{e}_3(s)$ is the binormal vector field of type spacelike. The functions $k(s)$ and $\tau(s)$ are curvature and torsion of the curve $\alpha(s)$ respectively [**6**].

Thus, one can prove

Lemma 2. *The null frame of a null curve $\vec{\alpha} = \alpha(s)$ in R_1^3 has the following properties*

$$
\vec{e}_1 \wedge \vec{e}_2 = \vec{e}_3,\ \vec{e}_1 \wedge \vec{e}_3 = \vec{e}_1,\ \vec{e}_3 \wedge \vec{e}_2 = \vec{e}_2. \tag{1.3}
$$

Definition 6. A null curve, in Lorentz space R_1^3, having zero torsion is called *a generalized null cubic* [**6**].

Theorem 1 (without proof). *Let $\alpha = \vec{\alpha}(s)$ be a null curve of a Lorentz surface $M \subset R_1^3$. Then α is a null geodesic on M if and only if the curvature vanishes identically on α* [**6**].

3 - An extension of the B-scroll surface

Let $\alpha = \vec{\alpha}(s)$ be a null curve in R_1^3 space with null frame (1.1) and the infinitesimal displacements (1.2). The immersion

$$\Phi : U \to R_1^3, \qquad U \subset R^2,$$
$$\Phi(s, v) = \alpha(s) + v\,\vec{e}_2(s), \qquad \forall (s, v) \in U \tag{1.4}$$

define a ruled surface generated by the principal normal (null) of the null curve $\alpha = \vec{\alpha}(s)$. This ruled surface is called a *B-scroll surface*, which has introduced by Graves [4].

Here, consider a ruled surface in R_1^3 generated by a null generator $\vec{L}(s)$ moving with Cartan's frame (null frame) of a null curve $\alpha = \alpha(s)$, i.e.,

$$\vec{L}(s) = \sum_{i=1}^{3} l_i(s)\,\vec{e}_i(s) \tag{1.5}$$

where the components $l_i = l_i(s)$, $(i = 1, 2, 3)$ are scalar functions of the parameter of arc length of the null curve $\alpha = \vec{\alpha}(s)$. Thus if $\vec{L}$ moves with Cartan's frame, the constructed ruled surface is given by the following parametrization

$$M : \Phi(s, v) = \vec{\alpha}(s) + v\vec{L}(s), \qquad (s, v) \in U \subset R^2,$$
$$\langle \vec{L}(s), \vec{L}(s) \rangle = l_3^2 - 2 l_1 l_2 = 0\,. \tag{1.6}$$

This ruled surface is called an *extension B-scroll surface*, and is denoted by EB.

From (1.5) and using (1.2), we obtain

$$\vec{L}'(s) = (l_1' + l_3\tau)\,\vec{e}_1 + (l_2' + l_3 k)\,\vec{e}_2 + (l_1 k + l_3' + l_2\tau)\,\vec{e}_3. \tag{1.7}$$

From Lemma 1, it follows that the vector $\vec{L}'(s)$ is a spacelike vector or null vector and in the second case it is linearly dependent with the generator.

The assumption $\vec{L}'(s) \neq 0$, is usually expressed by saying that the ruled surface M is a non cylindrical.

From (1.6), one can obtain the first fundamental quantities of the extension

$$g_{11} = -2v(l_2' + l_3 k) + v^2 \| \vec{L}' \|^2,$$
$$g_{12} = \langle \vec{\alpha}'(s), \vec{L}'(s) \rangle = -l_2, \tag{1.8}$$
$$g_{22} = \langle \vec{L}(s), \vec{L}(s) \rangle = 0\,.$$

Thus, the induced metric $g = g_{11} g_{22} - g_{12}^2 = -l_2^2 < 0$ on the extension B-scrool surface is a Lorentz metric. Therefore, we have:

Lemma 3. *The extension B-scrool surface is a timelike ruled surface.*

Then the unit normal vector field $\vec{n} = \vec{n}(s, v)$ on the extension B-scroll surface in R_1^3 is

$$\vec{n} = \frac{\vec{\alpha}'(s) \wedge \vec{L}(s) + v\vec{L}'(s) \wedge \vec{L}(s)}{\sqrt{|g|}}, \quad \text{or equivalently} \tag{1.9}$$
$$\vec{n} = \frac{\vec{\alpha}'(s) \wedge \vec{L}(s) + v\vec{L}'(s) \wedge \vec{L}(s)}{l_2}.$$

Striction curve

If there exists a common perpendicular to two consecutive rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central point is called the striction curve [**11**].

Using (1.2) and (1.7), it follows that, the striction curve $\beta = \vec{\beta}(s)$ for the extension B-scroll surface is given by

$$\vec{\beta}(s) = \vec{\alpha}(s) + \frac{l_2' + l_3 k}{\|\vec{L}'(s)\|^2}\vec{L}(s). \tag{1.10}$$

The base curve $\alpha = \alpha(s)$ is the striction curve $\beta = \vec{\beta}(s)$ if and only if

$$l_2' + l_3 k = 0. \tag{1.11}$$

Thus, from (1.10) and Theorem 1, it follows that Bonnet's theorem for EB surface given by

Theorem 2. *For a null geodesic curve $\alpha = \vec{\alpha}(s)$ on the extension B-scroll surface in Lorentz space, the following condions*:

(i) *The null curve cut the rulings at a constant angle* ($l_2 = \text{const}$),
(ii) The null curve is the striction curve ($l_2' + l_3 k = 0$), *are equivalent.*

Now, we study the extension B-scroll surface for which the striction curve is the base curve and we denot it by EB^s, i.e.,

$$EB^s\colon \Phi(s, v) = \vec{\alpha}(s) + v\vec{L}(s) \tag{1.12}$$
$$l_3^2 - 2l_1 l_2 = 0, \qquad l_2' + l_3 k = 0.$$

From (1.7), using (1.12), one can see that

Corollary 1. *The vector $\vec{L}'(s)$ is a spacelike vector.*

Since $\langle \vec{L}(s), \vec{L}'(s) \rangle = 0$ and $\langle \vec{\alpha}', \vec{L}'(s) \rangle = 0$, one can see that

$$\vec{\alpha}'(s) \wedge \vec{L}(s) = \lambda \vec{L}'(s),$$

where

$$\lambda = \lambda(s) = \frac{\det(\vec{\alpha}', \vec{L}(s), \vec{L}'(s))}{\| \vec{L}'(s) \|^2}. \tag{1.13}$$

The function $\lambda = \lambda(s)$ is called the distribution parameter of the extension B-scroll surface EB^s.

In more explicitly using (1.2), (1.5) and (1.7), we have

$$\lambda(s) = \frac{l_2}{l_1 k + l_2' + l_2 \tau}.$$

The normal vector field on EB^s takes the form

$$\vec{N} = \lambda \vec{L}'(s) + v \vec{L}'(s) \wedge \vec{L}(s). \tag{1.14}$$

Thus, from (1.8) and (1.14) we have

$$l_2^2 = \lambda^2 \| \vec{L}'(s) \|^2. \tag{1.15}$$

Thus, we have

Corollary 2. *The singular points on the extension B-scroll surface are the points for which* $\lambda = 0$.

Since $\| \vec{L}' \| \neq 0$, i.e., $l_1 k + l_3' + l_2 \tau \neq 0$. Then, the singular points are given by $l_2 = 0$.

4 - Intrinsic geometry

Now, we give a theorem similar to Chasles theorem for the extension B-scroll surface in R_1^3 space.

The unit normal vector to the extension B-scroll surface EB^s at (s, v) is given from (1.14) and (1.15) as

$$\vec{n}(s, v) = \frac{\lambda \vec{L}'(s) + v \vec{L}' \wedge \vec{L}}{\lambda \| \vec{L}' \|}$$

or in the form

$$\vec{n}(s, v) = \vec{l} + \frac{v}{\lambda}\,\vec{l} \wedge \vec{L}, \qquad \text{where } \vec{l} = \frac{\vec{L}'(s)}{\|\vec{L}'(s)\|}.$$

For a regular patch on $EB^s(\lambda \neq 0)$, it is easy to see that the normal along the striction curve on EB^s is given by

$$\vec{n}_0(s, 0) = \vec{l}.$$

Since $\vec{n}_0$ is a unit spacelike vector and $\vec{n}$ is unit spacelike vector. Thus if θ is the angle of rotation from the normal $\vec{n}_0$ to the normal $\vec{n}$ we get

$$\sin\theta = \|\vec{n}_0 \wedge \vec{n}\| = \left\|\left(\vec{l}(s) + \frac{v}{\lambda}\,\vec{l}(s) \wedge \vec{L}(s)\right) \wedge \vec{l}(s)\right\|.$$

Routine calculation, one can obtain $\theta = 0$. Thus, we have (without loos of generality)

Theorem 3. *For the extension B-scroll surface EB^s in R_1^3 space, the normal vector $\vec{n}$ at a point of a ruling and the normal vector $\vec{n}_0$ at the striction point of this ruling are parallel.*

Theorem 4. *The Gaussian curvature K of the extension B-scroll surface in R_1^3 is positive.*

Proof. The coefficients of the second fundamental form II are given by $h_{\alpha,\beta} = \langle \vec{n}, \Phi_{\alpha,\beta}\rangle$. Explicitly, one can obtain

$$h_{11} = \frac{\langle \lambda\vec{L}'(s) + v\vec{L}'(s) \wedge \vec{L}(s),\, k\,\vec{e}_3 + v\vec{L}''(s)\rangle}{\lambda\|\vec{L}'(s)\|},\; h_{12}\|\vec{L}'(s)\|,\; h_{22} = 0. \tag{1.16}$$

The Gaussian curvature of a surface in Lorentz space is defined as

$$K = \varepsilon\,\frac{\det(h_{ij})}{\det(g_{ij})}$$

where $\varepsilon = 1$ or -1 according to the surface is timelike or spacelike respectively [**10**]. Then, from (1.8), (1.16) and (1.15) one can see that the Gaussian curvature of the extension B-scroll EB^s is

$$K(s) = \frac{1}{\lambda^2} \tag{1.17}$$

and this completes the proof of the theorem.

Theorem 5. *The mean curvature of the extension B-scroll surface is defined by*

$$H = \frac{1}{\lambda}.$$

Proof. The mean curvature of a surface in a Lorentz space is defined by [**10**]

$$H = \frac{\varepsilon}{2} g^{\alpha\beta} h_{\alpha\beta}$$

where $g^{\alpha\beta}$ are the controvariant metric quantities. Therefore, from (1.8) and (1.16) it follows that the mean curvature of the extension B-scroll surface EB^s is

$$H(s) = \frac{1}{\lambda} = \tau + \frac{l_1 k + l_3'}{l_2}. \tag{1.18}$$

Theorem 6. *The Gaussian and mean curvatures of the extension B-scroll surface are functions in the parameter of arc length of the base curve and not depend on the distance along the generator.*

For (1.17) and (1.18), we have

Lemma 4. *For the extension B-scroll surface* $K = H^2$.

Thus, from (1.15), (1.16) and (1.18), we have

Theorem 7. *The extension B-scroll* EB^s *is totally umbilical.*

The Laplacian operator for the mean curvature vector of the extension B-scoll surface EB^s is

$$\Delta H = \sum_{\alpha\beta} g^{\alpha\beta} \left(\frac{\partial^2 H}{\partial u^\alpha \partial u^\beta} - \sum_\gamma \Gamma^\gamma_{\alpha\beta} \frac{\partial H}{\partial u^\gamma} \right)$$

where

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} \sum_{\mu} g^{\gamma\mu}\left(\frac{\partial g_{\alpha\mu}}{\partial u^{\beta}} + \frac{\partial g_{\beta\gamma}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\mu}} \right), \qquad (u^{\alpha}) = (s, v).$$

Then from (1.18), we have

$$(1.19) \quad \Gamma^{1}_{11} = \frac{l_2' + v\|\vec{L}'\|^2}{l_2}, \qquad \Gamma^{2}_{12} = \frac{-v\|\vec{L}'(s)\|^2}{l_2}, \qquad \text{and } \Gamma^{1}_{12} = \Gamma^{1}_{22} = \Gamma^{2}_{22} = 0 .$$

Since $\langle \vec{L}'(s), \vec{L}(s)\rangle = 0, \langle \vec{L}(s), \vec{L}(s)\rangle = 0$ and using (1.13) and (1.5), on can see that

$$\vec{L}'(s) \wedge \vec{L}(s) = \|\vec{L}'(s)\| \, \vec{L}(s).$$

Then, the mean curvature vector for the extension B-scroll surface EB^s is

$$\vec{H} = \frac{\vec{l}(s)}{\lambda} + \frac{v}{\lambda^2}\vec{L}(s).$$

Therefore, one can see that

$$\Delta\vec{H} = \frac{-2}{\lambda^2}\vec{H} + \frac{4\lambda'}{\lambda^4\|L'\|}\vec{L}(s).$$

Thus we have the following theorem:

Theorem 8. *The mean curvature vector $\vec{H}$ of the extension B-scroll surface satisfies the differential equation*

$$(1.20) \qquad \Delta\vec{H} = A\vec{H} + \vec{C}$$

where

$$A = \frac{-2}{\lambda^2}, \qquad \textit{and} \qquad C = \frac{4\lambda'}{\lambda^4\|\vec{L}'(s)\|}\vec{L}(s).$$

Lemma 5. *For the extension B-scroll surface for which the distribution parameter is constant, we have $\Delta H = AH$, where A is constant.*

5 - Maximal extension of the B-scroll surface

Since the extension of the B-scoll surface is a timelike ruled surface, then, the necessary condition for it's maximality $(H = 0)$ that it's ruling coincides with the principal normal of the null base curve [**12**]. This means that $l_1 = l_3 = 0$, i.e., $\vec{L}(s) = \vec{e}_2$ and from (1.18), one can see that $H = \tau$. Then, we have the following theorem:

Theorem 9. *The extension B-scroll surface EB^s is maximal if it is a B-scroll surface EB with generalized null cubic base curve* $(\tau = 0)$.

Extension B-scroll surface with constant distribution parameter

If $\lambda = c$, where c is a non zero constant and from (1.17) and (1.18), we have the following theorem:

Theorem 10. *For the extension B-scroll surface EB^s in R_1^3, the following conditions*

(i) *The parameter of distribution is non zero constant.*
(ii) *The Gaussian curvature K of M is constant.*
(iii) *The mean curvature H of M is constant.*
(iv) *The B-scroll surface EB^s is isoparametric* [8] *are equivalent.*

As a continuation to Dillen [**2**], we introduce the function $\varrho = \dfrac{1}{\lambda^2}$.
Thus, we have

Theorem 11. *Every maximal extension B-scroll surface is a flat in the sense of Dillen* $(\varrho = 0)$.

6 - Surfaces with finite type Gauss map

Definition 7. *The Gauss map of the extension B-scroll surface EB^s* (*time-like*) *in R_1^3* is defined as

$$\vec{n} : M \to S_1^2 \subset R_1^3$$

where S_1^2 is the pseudosphere. The Gauss map is called finite of one type, if there exist a constant c and a constant vector $\vec{y}$ such that $\Delta \vec{n} = c(\vec{n} - \vec{y})$ [**5**].

For the extension B-scroll surface EB^s and from theorem (8) we have

$$\Delta \vec{H} = A\vec{H} + C\vec{L}(s).$$

After routine calculations, one can obtain

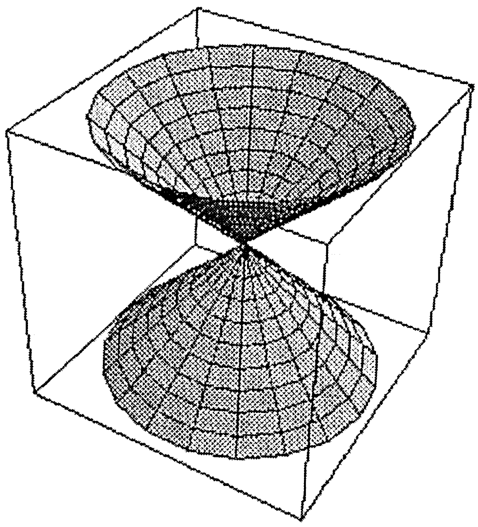
Theorem 12. *The extension B-scroll surface is a* 1-*type Gauss map if and only if, it has a constant parameter of distribution.*

The results in the study are confirmed by the following example.

Example. The ruled surface

$$\Psi(s, v) = (\cos s + v \sin s, \sin s + v \cos s, s + v)$$

is an extension B-scroll surface EB where $\vec{\alpha}(s) = (\cos s, \sin s, s)$, is a null base curve and $\vec{L}(s) = (\sin s, \cos s, 1)$ is a null generator. The striction curve is $\vec{\beta}(s) = \vec{\alpha}(s) - 2 \sin s \cos s \vec{L}(s)$. The distribution parameter is $\lambda = 2 \sin^2 s$. This ruled surface is translated to the figure



The extension B-scroll surface EB

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Summary

In this paper, an extension B-scroll surface is defined and studied as a continuation to Graves [4]. *Theorems due to Bonnet and Chasles are obtained. A theorem, for the maximal timelike ruled surface is proved. Finally the finite type Gauss map of the surface under investigation is defined and interesting result is given.*

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