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Widened Fibonacci cubes (**)

1 - Introduction

Let C_n be the set of binary strings of length n without two consecutive ones.

If α and β denote two strings, then $\alpha\beta$ is the string obtained by concatenating α and β. More generally if S is a set of strings, then $\alpha S\beta$ denotes the set of strings $\alpha\gamma\beta$, where $\gamma \in S$.

Clearly the set C_{n+2} can be partitioned into two disjoint subsets as long as the first element is 0 or 1. If it is 1 the second has to be 0 and we obtain $C_{n+2} = 0C_{n+1} + 10C_n$.

It is well know that the cardinality of C_n is the Fibonacci number F_n. Recall that the Fibonacci numbers F_n form a sequence of positive integers, where $F_0 = 1$, $F_1 = 2$ and every other integer satisfies the recursion $F_{n+2} = F_{n+1} + F_n$.

The Fibonacci cube Γ_n is the graph whose set of vertices is C_n and whose edges are the couples of vertices with unitary Hamming distance.

This graph was introduced in [**6**] as a new interconnection topology, alternative to the classical one, namely the hypercube Q_n; indeed it has been shown that the Fibonacci cube can efficiently simulate many hypercube algorithms.

We define the *widened Fibonacci cube* WFC_{n+4}, $n \geq 0$, as the graph whose

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set of vertices is the set

(1) $$W_{n+4} = 00C_{n+2} + 10C_{n+2} + 0100C_n + 0101C_n$$

and whose edges are again the pairs of vertices with unitary Hamming distance.

Clearly this graph is embedded in the hypercube Q_{n+4}. Moreover it maintains all the desirable properties of the Fibonacci cube, having in addition the hamiltonicity, not satisfied by all the Fibonacci cubes.

We prove the Fibonacci cube is embedded in the widened Fibonacci cube and moreover this one is embedded in the hypercube; it can be introduced as a new type of interconnection network for multiprocessors systems.

In Section **2** we describe a particular decomposition of this graph and we prove it is Hamiltonian. In Section **3** we determine the values of its diameter, radius, center and independence number of vertices. In Section **4** we find its observability, which is the minimum number of colors assignable to the edges so that the coloring is proper and the vertices are distinguished by their color sets. Finally, in Section **5**, we prove it is isomorphic to the graph underlying the Hasse diagram of a particular semilattice.

2 - Hamiltonicity

In [**7**] it is proved that a Fibonacci cube Γ_n contains an Hamiltonian path; in addition, when n is even, Γ_n is Hamiltonian.

In this section we prove that the widened Fibonacci cube WFC_m, $m \geq 4$, is Hamiltonian for every m.

First we prove the following

Proposition 1. *For $n > 1$, Γ_{n-1} and Γ_n contain Hamiltonian paths in which an end-vertex is respectively v and $0v$, where $v \in C_{n-1}$.*

Proof. The result is true for $n = 2$. Indeed Γ_1 is reduced to one edge having vertices 0, 1, while Γ_2 has vertices 01, 00, 10.

We now proceed by induction on n.

Let us assume it is true for $n > 2$ and we prove it for $n + 1$.

Denote by P and Q Hamiltonian paths of Γ_{n-1} and Γ_n having end-vertices v, w and $0v$, z respectively, where v, $w \in C_{n-1}$, $z \in C_n$.

Then Γ_{n+1}, whose set of vertices is $0C_n + 10C_{n-1}$, contains the Hamiltonian path $(10w, \ldots, 10v, 00v, \ldots, 0z)$ and the result follows. ■

By repeating a suitable number of times the decomposition

$$C_{m+2} = 0C_{m+1} + 10C_m \tag{2}$$

we obtain from (1)

$$\begin{aligned} W_{n+4} = {} & 0010C_n + 0000C_n + 00010C_{n-1} + 1010C_n \\ & + 1000C_n + 10010C_{n-1} + 0100C_n + 0101C_n. \end{aligned} \tag{3}$$

Denote by A, B, C, D, E, F the graphs isomorphic to Γ_n, having as sets of vertices XC_n where X coincides with 1010, 0010, 1000, 0000, 0100, 0101 respectively. Moreover G and H denote graphs isomorphic to Γ_{n-1}, having as sets of vertices YC_{n-1}, where Y is 00010 and 10010 respectively. Thus WFC_{n+4} can be decomposed into the graph of Fig. 1

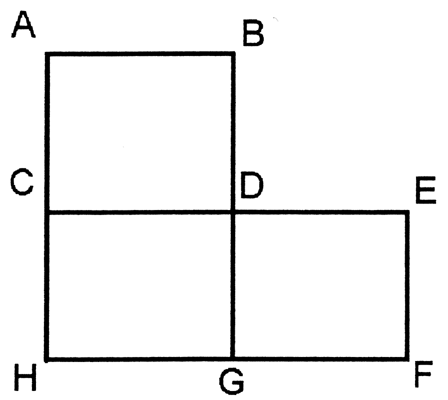

Fig. 1.

where an edge represents all the edges which connect corresponding subgraphs.

Theorem 1. *For $n \geqslant 4$, WFC_n is Hamiltonian.*

Proof. Consider the decomposition of Fig. 1. By Proposition 1, Γ_{n-1} and Γ_n contain Hamiltonian paths, denoted by P and Q, having as end-vertices v, w and $0v$, z respectively, where $v, w \in C_{n-1}$ and $z \in C_n$.

Thus the subgraph H of Fig. 1, isomorphic to Γ_{n-1}, contains the Hamiltonian path whose vertices are obtained by concatenating 10010 to all the vertices of P. We denote by H $(10010v, 10010w)$ the same path when we fix $10010v$ as first vertex and the other one as last vertex of the path.

In a similar way we obtain and denote Hamiltonian paths of the subgraphs corresponding to each vertex of Fig. 1; in the case of G and H we use the path P and, in all other cases, we use the path Q.

Now consider the sequence of paths:

$H\,(10010v,\, 10010w)$, $G\,(00010w,\, 00010v)$, $F\,(01010v,\, 0101z)$, $E\,(0100z,\, 01000v)$,

$D\,(00000v,\, 0000z)$, $B\,(0010z,\, 00100v)$, $A\,(10100v,\, 1010z)$, $C\,(1000z,\, 10000v)$.

We note that the last vertex of every path is adjacent to the first one of the consecutive path. In particular the last vertex of the last path is adjacent to the first vertex of the first path and the graph has an Hamiltonian cycle. ■

3 - Some structural properties

Let $G(V, E)$ be a connected graph.

The *eccentricity* of a vertex v of G is the maximum distance between v and the other vertices, i.e. the number

$$e(v) := \max_{u \in V(G)} d(u, v);$$

the *diameter* of G, denoted diam(G), is the maximal eccentricity when v runs through G; the *radius* of G is the minimum eccentricity of the vertices of G, i.e. the number

$$\mathrm{rad}(G) := \min_{v \in V(G)} e(v).$$

A vertex v is central if $e(v) = \mathrm{rad}(G)$; the center $Z(G)$ is the set of all central vertices.

First we note the diameter of WFC_m is m; indeed the strings having 1 in all the odd or even positions, and 0 otherwise, are at distance m.

Theorem 2. *For* $n \geqslant 0$, WFC_{n+4} *satisfies the following property*

$$\mathrm{rad}(WFC_{n+4}) = \lceil (n+4)/2 \rceil .$$

Proof. First we prove $e(\hat{o}) = \lceil (n+4)/2 \rceil$, where $\hat{o}$ is the string $00\ldots0$ of length $n+4$. Indeed a vertex at maximal distance from $\hat{o}$ is one having a maximal number of ones obtained by concatenating 10 the maximum number of times and ending with 1 when n is odd.

A vertex v of WFC_{n+4} can be decomposed into $v = wz$, where w and z are two Fibonacci strings of length 4 and n respectively.

We prove $e(v) \geqslant \lceil (n+4)/2 \rceil$.

Consider the vertex $v' = w'z'$ at maximal distance from v, where w' and z' are Fibonacci strings of length 4 and n respectively. Clearly $\operatorname{dist}(w, w') \geq 2$. Moreover we can assume z' is at maximal distance from z, that is $\operatorname{dist}(z, z') \geq \lceil n/2 \rceil$. Thus $\operatorname{dist}(v, v') = \operatorname{dist}(w, w') + \operatorname{dist}(z, z') \geq 2 + \lceil n/2 \rceil = \lceil (n+4)/2 \rceil$. ■

Denote by T_n the set of strings $\omega\omega'$, where $\omega = 0000$ and ω' is a string of odd length n having only one 1 and maximal subsequences of zeros of even length. For instance $00100 \in T_5$, while $01000 \notin T_5$.

Theorem 3. *For every $n \in N$ we have*

$$Z(WFC_{n+4}) = \begin{cases} \{\hat{o}\} & \text{for } n \text{ even} \\ \{\hat{o}\} + T_n & \text{for } n \text{ odd}. \end{cases}$$

Proof. By the proof of Theorem 2, $\hat{o} \in Z(WFC_{n+4})$.

Let n be even. Consider a vertex $v = wz$ of the graph, where w and z are Fibonacci strings of length 4 and n respectively. Denote by $v' = w'z'$ a vertex of WFC_{n+4} at maximal distance from v. By the properties of the Fibonacci strings of even length, if w or z contains at least one 1, then either $\operatorname{dist}(w, w') > 2$ or $\operatorname{dist}(z, z') > \lceil n/2 \rceil$. Thus if v contains at least one 1, then $\operatorname{dist}(v, v') > \lceil (n+4)/2 \rceil = e(\hat{o})$.

Now, let n be odd and $u = \omega\omega'$ an element of T_n. If $u' = w'z'$ is a string at maximal distance from u, then $\operatorname{dist}(\omega, w') = 2$ and $\operatorname{dist}(\omega', z') = \lceil n/2 \rceil$. Thus $\operatorname{dist}(u, u') = \lceil (n+4)/2 \rceil$. In each other case for a vertex $v = wz$ of the graph and $v' = w'z'$ at maximal distance, either $\operatorname{dist}(w, w') > 2$ or $\operatorname{dist}(z, z') > \lceil n/2 \rceil$ and v does not belong to the center. ■

Proposition 2. *The number of pairs of vertices of WFC_{n+4} at distance equal to the diameter is* 2.

Proof. Denote by α and β the strings 0101 and 1010 respectively, while ε and δ are strings of length n having 1 in all the even or odd positions respectively. Clearly ε and δ are at distance n and they are the only possible strings at distance n. Thus the pairs $\alpha\varepsilon$, $\beta\delta$ and the pairs $\alpha\delta$, $\beta\varepsilon$ are the only pairs of vertices at distance equal to $n+4$, the diameter of WFC_{n+4}. ■

Let E_{n+4}, O_{n+4} be the sets of vertices having an even, odd number of ones respectively; we call such vertices even or odd. They form the partite sets of WFC_{n+4}. From the condition of the existence of an Hamiltonian cycle it follows that $|E_{n+4}| = |O_{n+4}| = F_{n+2} + F_n = L_{n+3}$.

Recall that the *independence number* $\beta(G)$ is the maximum cardinality among those of independent sets of vertices of G.

Theorem 4. *For every* $n \geq 0$, $\beta(WFC_{n+4}) = L_{n+3}$.

Proof. The subset E_{n+4} of W_{n+4} is an independent set having cardinality L_{n+3}. Let us assume there exists a subset H of independent vertices of cardinality greater then L_{n+3}. We can decompose H into the subsets A and B of even and odd vertices respectively. The vertices of B are adjacent only to vertices of $A' = E_{n+4} - A$, while on the assumption on the cardinality of H, $|B| > |A'|$. By Theorem 1, WFC_{n+4} contains an Hamiltonian cycle C. If we assign an order to the vertices of C, we obtain that every vertex of B is followed by a vertex of A'. This implies $|B| \leq |A'|$, a contradiction. ■

4 - Observability

In [3] it has been proved that $\text{obs}(\Gamma_n) = n$ for $n \geq 4$. Now we prove a similar result for the widened Fibonacci cubes using the decomposition of Fig. 2.

Let us denote by A_1, A_2 the subsets of the set $1010C_n$, $n > 1$, given by the decomposition (2), that is A_1 is the set $10100C_{n-1}$ and A_2 is $101010C_{n-2}$.

Moreover, denote by B_i, C_i, D_i, E_i, F_i, where $i = 1, 2$, similar subsets of $0010C_n$, $1000C_n$, $0000C_n$, $0100C_n$, $0101C_n$, by G_1 the set $00010C_{n-1}$ and by H_1 the set $10010C_{n-1}$.

Such a decomposition can be represented by the graph in Fig. 2, to which it has been assigned a partial edge-coloring α, using the four colors a, b, c, d.

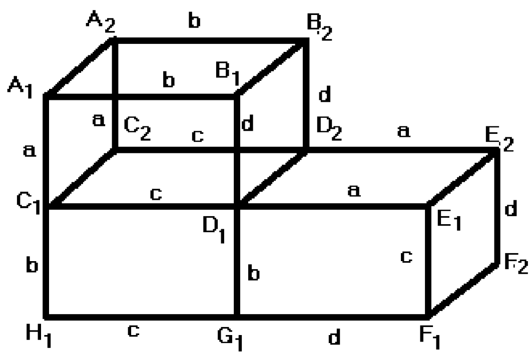

Fig. 2.

The subgraphs induced by the sets of vertices X_1, X_2, $X_1 \cup X_2$, where $X \in \{A, B, C, D, E, F\}$, are isomorphic to Γ_{n-1}, Γ_{n-2}, Γ_n respectively.

The edge which connects X_1 and X_2 represents the set of edges $(y10s, y00s)$ where $s \in C_{n-2}$ and y is a sequence of length 4 which coincides with 1010, 0010, 1000, 0000, 0100, 0101 for A_2, B_2, C_2, D_2, E_2, F_2 respectively. The edge which connects A_1 and B_1 represents all the edges which connect a vertex $10100v$ of A_1 with its corresponding one $00100v$ of B_1, for $v \in C_{n-2}$. It is clear that by deleting F_2 and all the incident edges, the subgraph induced is isomorphic to the Fibonacci cube. Indeed the vertices of F_2 are the unique strings with two consecutive ones, while other vertices are all the Fibonacci strings of length $n+4$.

The partial edge-coloring α assigns to all the vertices of A_1, A_2 the set $\{a, b\}$, to all the vertices of B_1, B_2 the set $\{b, d\}$, to the vertices of C_1, C_2, D_1, D_2, E_1, E_2, F_1, F_2, G_1, H_1 the sets $\{a, b, c\}$, $\{a, c\}$, $\{a, b, c, d\}$, $\{a, c, d\}$, $\{a, c\}$, $\{a, d\}$, $\{c, d\}$, $\{d\}$, $\{b, c, d\}$, $\{b, c\}$ respectively.

Proposition 3. *For* $n = 5, 6, 7$, $\mathrm{obs}(WFC_n) = n$.

Proof. An obs-coloring of WFC_5 is given in Fig. 3

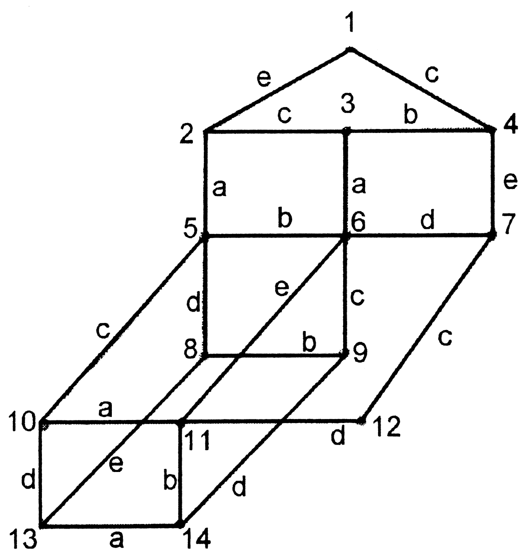

Fig. 3.

where the vertices $1, 2, \ldots, 14$ represent the strings 01011, 01001, 01000, 01010, 00001, 00000, 00010, 00101, 00100, 10001, 10000, 10010, 10101, 10100 respectively.

Consider WFC_6.

The subgraphs induced by the sets of vertices $X_1 \cup X_2$, for $X \in \{A, B, \ldots, F\}$, are isomorphic to Γ_2, while the subgraphs induced by X_1, for $X \in \{A, B, \ldots, H\}$, are isomorphic to Γ_1, the subgraphs induced by X_2, for $X \in \{A, B, \ldots, F\}$, are isomorphic to Γ_0.

In Fig. 4 we represent a graph isomorphic to Γ_2

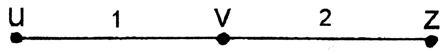

Fig. 4.

to which it has been assigned the coloring β using the colors 1, 2.

The vertices u, v, z represent the strings 01, 00, 10 respectively. Then the vertices of X_1 for $X \in \{A, B, \ldots, H\}$ are u, v, while the vertex of X_2 for $X \in \{A, B, \ldots, F\}$ is z.

Let γ be the coloring which assigns the color 1 to the unique edge $u'v'$ of the subgraphs induced by G_1 and H_1.

If we consider the partial edge-coloring α represented in Fig. 2 and the new colorings β and γ, we obtain that all the vertices of WFC_6 are distinguished, except the vertices of G_1 and H_1. If we change into 2 the color c assigned to the edges $u'(G_1)$, $u'(H_1)$, we have an appropriate edge-coloring, given in Fig. 5.

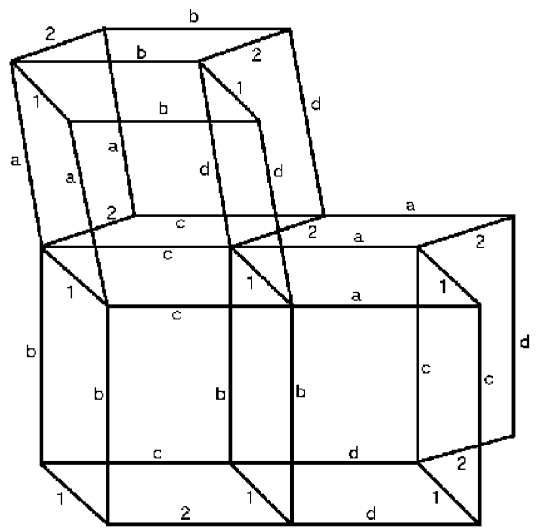

Fig. 5.

Finally consider WFC_7.

In this case X_1, X_2 are isomorphic to Γ_2 and Γ_1 respectively.

Assign to the edges of the subgraphs induced by $X_1 \cup X_2$, isomorphic to Γ_3, the following 3-edge-coloring δ

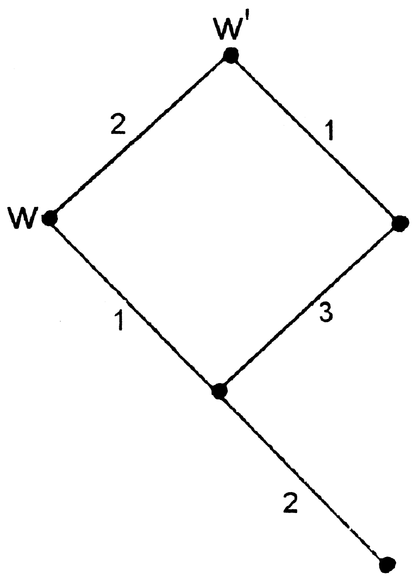

Fig. 6.

which distinguishes the vertices of Γ_3 but w and w'.

Let us assign the partial coloring α, given in Fig. 2, to the edges of WFC_7, δ to the edges of $X_1 \cup X_2$ and β of Fig. 4 to the edges of G_1, H_1. Then change into 3 the color b assigned to the edge which connects w' of A_2 with the corresponding vertex w' of B_2 and the color a assigned to the edge which connects the vertices w of D_1 and E_1.

We obtain that the color sets of the vertices of $A_1 \cup A_2$ are: $\{1, 2, 3, a\}$, $\{1, 3, a, b\}$, $\{1, 2, a, b\}$, $\{1, 2, 3, a, b\}$, $\{2, a, b\}$, the color sets of the vertices of $B_1 \cup B_2$: $\{1, 2, 3, d\}$, $\{1, 3, b, d\}$, $\{1, 2, b, d\}$, $\{1, 2, 3, b, d\}$, $\{2, b, d\}$, the color sets of the vertices of $C_1 \cup C_2$: $\{1, 2, a, c\}$, $\{1, 3, a, c\}$, $\{1, 2, a, b, c\}$, $\{1, 2, 3, a, b, c\}$, $\{2, a, b, c\}$, the color sets of the vertices of $D_1 \cup D_2$: $\{1, 2, a, c, d\}$, $\{1, 3, a, c, d\}$, $\{1, 2, 3, b, c, d\}$, $\{1, 2, 3, a, b, c, d\}$, $\{2, a, b, c, d\}$, the color sets of the vertices of $E_1 \cup E_2$: $\{1, 2, a, d\}$, $\{1, 3, a, d\}$, $\{1, 2, 3, c,\}$, $\{1, 2, 3, a, c\}$, $\{2, a, c\}$, the color sets of the vertices of $F_1 \cup F_2$: $\{1, 2, d\}$, $\{1, 3, d\}$, $\{1, 2, c, d\}$, $\{1, 2, 3, c, d\}$, $\{2, c, d\}$, the color sets of the vertices of G_1: $\{1, b, c, d\}$, $\{1, 2, b, c, d\}$, $\{2, b, c, d\}$, the color sets of the vertices of H_1: $\{1, b, c\}$, $\{1, 2, b, c\}$, $\{2, b, c\}$.

Thus obs $(WFC_7) = 7$. ■

Theorem 5. *For* $n \geq 1$, obs $(WFC_{n+4}) = n + 4$.

Proof. By Proposition 3 the result is satisfied for positive $n \leq 3$.

Consider the decomposition of WFC_{n+4} of Fig. 2.

In [3] it was proved that obs $(\Gamma_n) = n$ for $n \geq 4$. Thus let β and γ be obs-colorings of Γ_n and Γ_{n-1} using n colors and α the partial coloring of Fig. 2 using 4 news colors.

The vertices of A_1 and A_2 have the same color set $\{a, b\}$ by α, but they are distinguished by β. A similar situation holds for B_1 and B_2. Moreover the vertices of C_2 and E_1 have assigned the same color set $\{a, c\}$ by α, but they are also distinguished because they correspond to disjoint induced subgraphs of Γ_n and they are distinguished by β. This completes the proof of the theorem. ■

5 - Semilattices

Consider the set W_{n+4} and two vertices $\alpha = a_1 a_2 \ldots a_{n+4}$, $\beta = b_1 b_2 \ldots b_{n+4}$ of such a set. On W_{n+4} we define an order relation by setting

$$\alpha \leq \beta \qquad \text{if and only if} \qquad a_i \leq b_i \qquad \text{for } i = 1, 2, \ldots, n+4 .$$

Then $\alpha \wedge \beta = [\min(a_1, b_1) \min(a_2, b_2) \ldots \min(a_{n+4}, b_{n+4})]$ and this sequence is a vertex of W_{n+4}. Note that $\alpha \vee \beta = [\max(a_1, b_1) \max(a_2, b_2) \ldots \max(a_{n+4}, b_{n+4})]$ could not belong to W_{n+4}. Then the order set $(W_{n+4}, \leq)$ is closed under inf and it has the minimal element $\hat{o} = [0\ 0 \ldots 0]$. Thus it is a meet-semilattice.

Let Atom(S) be the set of atoms of the semilattice S. S is atomic if for each $x \in S$ there exist a subset $A \subseteq$ Atom(S) such that $x = \vee A$.

In W_{n+4} for a string β having k ones there are exactly k atoms $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\beta = \alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_k$. Thus the order set $(W_{n+4}, \leq)$ is an atomic semilattice.

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Abstract

We introduce the widened Fibonacci cube, a graph, embedded in the hypercube, which contains the Fibonacci cube as induced subgraph and provides many of the properties of the Fibonacci cube with in addition the hamiltonicity for every number of vertices. The values of the diameter, radius, center and independence number are determined, together with its observability, which is the minimum number of colors assignable to the edges so that the coloring is proper and the vertices are distinguished by their color sets. Finally we prove that it is isomorphic to the Hasse diagram of a particular semilattice.

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