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# On submanifolds immersed in a manifold with quarter symmetric connection (**)

## 1 - Preliminaries

Let $M^{n+1}$ be an $(n+1)$-dimensional differentiable manifold of class $C^{\infty}$ and $M^n$ be the hypersurface immersed in $M^{n+1}$ by a differentiable immersion $i : M^n \to M^{n+1}$. The differential $di$ of the immersion $i$ will be denoted by $B$ so that the vector-field $X$ in the tangent space of $M^n$ corresponds to a vector field $BX$ in that of $M^{n+1}$. Suppose that the enveloping manifold $M^{n+1}$ is a Riemannian manifold with metric tensor $\tilde{g}$. Then the hypersurface $M^n$ is also a Riemannian manifold with induced metric tensor $g$ defined by

$$g(FX, Y) = \tilde{g}(BFX, BY),$$

$X$ and $Y$ being arbitrary vector fields in $M^n$ and $F$ is a tensor of type (1,1). If the Riemannian manifolds $M^{n+1}$ and $M^n$ are both orientable, we can choose a unique vector field $N$ defined along $M^n$ such that

$$\tilde{g}(BFX, N) = 0$$

and

$$\tilde{g}(N, N) \equiv 1,$$

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for arbitrary vector field $X$ in $M^n$. We call this vector field the unit normal vector field to the hypersurface $M^n$.

We now suppose that the Riemannian manifold $M^{n+1}$ admits a quarter symmetric metric connection given by [3]

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \tilde{\dot{\nabla}}_{\tilde{X}}\tilde{Y} + \tilde{\pi}(\tilde{Y})\,\tilde{F}(\tilde{X}) - \tilde{g}(\tilde{F}(\tilde{X}), \tilde{Y})\,\tilde{P}, \tag{1.1}$$

for arbitrary vector fields $\tilde{X}$ and $\tilde{Y}$ tangents to $M^{n+1}$, where $\tilde{\dot{\nabla}}$ denotes the Levi-Civita connection with respect to Riemannian metric $\tilde{g}$, $\tilde{\pi}$ is a 1-form $\tilde{F}$ is a tensor of type (1,1) and $\tilde{P}$, the vector field defined by

$$\tilde{g}(\tilde{P}, \tilde{X}) = \tilde{\pi}(\tilde{X})$$

for an arbitrary vector field $\tilde{X}$ of $M^{n+1}$. Also

$$\tilde{g}(\tilde{F}(\tilde{X}), \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{F}(\tilde{Y}))\,.$$

Let us put

$$\tilde{P} = BP + \lambda N\,, \tag{1.2}$$

where $P$ is a vector field and $\lambda$ a function on $M^n$.

We have the following theorem:

Theorem 1.1. *The connection induced on the hypersurface of a Riemannian manifold with a quarter symmetric metric connection with respect to the unit normal is also quarter symmetric.*

Proof. Let $\dot{\nabla}$, the connection induced on the hypersurface from $\tilde{\dot{\nabla}}$ with respect to the unit normal $N$. Then we have

$$\tilde{\dot{\nabla}}_{BX}BY = B(\dot{\nabla}_X Y) + h(X,\ Y)\,N \tag{1.3}$$

for arbitrary vector fields $X$ and $Y$ on $M^n$. Also $h$ is the second fundamental tensor of the hypersurface $M^n$. Similarly, let $\nabla$ be connection induced on the hypersurface from $\tilde{\nabla}$ with respect to the unit normal $N$. We have

$$\tilde{\nabla}_{BX}BY = B(\nabla_X Y) + m(X,\ Y)\,N \tag{1.4}$$

for arbitrary vector fields $X$ and $Y$ of $M^n$, $m$ being a tensor field of type (0,2) on the hypersurface $M^n$. From (1.1) we obtain

$$\tilde{\nabla}_{BX} BY = \tilde{\dot{\nabla}}_{BX} BY + \tilde{\pi}(BY)\, \tilde{F}(BX) - \tilde{g}(\tilde{F}(BX),\, BY)\, \tilde{P}\,.$$

Using (1.3) and (1.4), the above equation takes the form

$$B(\nabla_X Y) + m(X, Y)\, N = B(\dot{\nabla}_X Y) + h(X,\, Y)\, N + \pi(Y)\, BFX - \tilde{g}(BFX, BY)\, \tilde{P}\,, \tag{1.5}$$

where

$$\tilde{\pi}(BX) = \pi(X)$$

and

$$\tilde{F}(BX) = BFX\,.$$

Substituting (1.2) into (1.5), and using $\tilde{g}(BFX,\, BY) = g(FX,\, Y)$, we get

$$B(\nabla_X Y) + m(X,\, Y)\, N = B(\dot{\nabla}_X Y) + h(X,\, Y) N + \pi(Y)\, BFX$$
$$- g(FX,\, Y)(BP + \lambda N)\,.$$

Comparison of tangential and normal vector fields yields

$$\nabla_X Y = \dot{\nabla}_X Y + \pi(Y)\, F(X) - g(FX,\, Y)\, P \tag{1.6}$$

and

$$m(X,\, Y) = h(X,\, Y) - \lambda g(FX,\, Y)\,. \tag{1.7}$$

Thus,

$$\nabla_X Y - \nabla_Y X - [X,\, Y] = \pi(Y)\, F(X) - \pi(X)\, F(Y)\,. \tag{1.8}$$

Hence the connection $\nabla$ induced on $M^n$ is quarter symmetric one [3].

## 2 - Totally geodesic and totally umbilical hypersurfaces

We define $\dot{\nabla} B$ and $\nabla B$ respectively by

$$(\dot{\nabla} B)(X,\, Y) = (\dot{\nabla}_X B)(Y) = \tilde{\dot{\nabla}}_{BX} BY - B(\dot{\nabla}_X Y)$$

and

$$(\nabla B)(X,\, Y) = (\nabla_X B)(Y) = \tilde{\nabla}_{BX} BY - B(\nabla_X Y)\,,$$

$X$ and $Y$ being arbitrary vector fields on $M^n$. Then (1.3) and (1.4) take the form

$$(\dot{\nabla}_X B)\, Y = h(X,\, Y)\, N$$

and

$$(\nabla_X B)\, Y = m(X,\, Y)\, N\,.$$

These are the equations of Gauss with respect to the induced connection $\dot{\nabla}$ and $\nabla$ respectively.

Let $X_1, X_2, \ldots X_n$ be $n$ orthonormal vector fields in $M^n$. Then the function

$$\frac{1}{n} \sum_{i=1}^{n} h(X_i,\, X_i)$$

is the mean curvature of $M^n$ with respect to Riemannian connection $\dot{\nabla}$ and

$$\frac{1}{n} \sum_{i=1}^{n} m(X_i,\, X_i)$$

is called the mean curvature of $M^n$ with respect to the quarter symmetric connection $\nabla$.

From this we have the following definitions:

Definition 2.1. If $h$ vanishes, we call $M^n$ as *totally geodesic hypersurface of* $M^{n+1}$ *with respect to the Riemannian connection* $\dot{\nabla}$.

Definition 2.2. The hypersurface $M^n$ is called *totally umbilical with respect to connection* $\dot{\nabla}$ if $h$ is proportional to the metric tensor $g$.

We call $M^n$ is totally geodesic and totally umbilical with respect to quarter symmetric connection $\nabla$ according as the function $m$ vanishes and proportional to the metric tensor $g$ respectively.

Now we have following theorems:

Theorem 2.1. *In order that the mean curvature of* $M^n$ *with respect to* $\dot{\nabla}$ *coincides with that of* $M^n$ *with respect to* $\nabla$, *it is necessary and sufficient that the vector field* $\tilde{P}$ *is tangent to* $M^{n+1}$.

Proof. In view of (1.7), we have

$$m(X_i,\, X_i) = h(X_i,\, X_i) - \lambda g(FX_i,\, X_i)\,.$$

Summing up for $i = 1, 2, \ldots, n$ and dividing by $n$, we obtain

$$\frac{1}{n} \sum_{i=1}^{n} m(X_i, X_i) = \frac{1}{n} \sum_{i=1}^{n} h(X_i, X_i)$$

if and only if, $\lambda = 0$. Hence from (1.2), we have

$$\tilde{P} = BP .$$

Thus the vector field $\tilde{P}$ is in tangent space of $M^{n+1}$.

Theorem 2.2. *The hypersurface $M^n$ will be totally umbilical with respect to the Riemannian connetion $\dot{\nabla}$, if and only if it is totally umbilical with respect to quarter symmetric connection $\nabla$.*

Proof. The proof follows easily from (1.7).

### 3 - Gauss, Weingarten and Codazzi equations

In this section we shall obtain Weingarten equations with respect to the quarter symmetric metric connection $\tilde{\nabla}$. For the Riemannian connection $\tilde{\dot{\nabla}}$, these equations are given by

$$\tilde{\dot{\nabla}}_{BX} N = -BHX \tag{3.1}$$

for any vector field $X$ in $M^n$, where $H$ is a tensor field of type (1,1) of of $M^n$ defined by

$$g(HX, Y) = h(X, Y) . \tag{3.2}$$

In view of the equation (1.1), we have

$$\tilde{\nabla}_{BX} N = \tilde{\dot{\nabla}}_{BX} N + \lambda BFX . \tag{3.3}$$

Since $\tilde{\pi}(N) = \tilde{g}(\tilde{P}, N) = \lambda$ and $\tilde{g}(BFX, N) = 0$.

Thus, from (3.1) and (3.3), we get

$$\tilde{\nabla}_{BX} N = -B(H - \lambda F) X \tag{3.4}$$

which is the equation of Weingarten with respect to the quarter symmetric metric connection.

Let $M = H - \lambda F$. Then from (3.4), we get

$$\tilde{\nabla}_{BX} N = -BMX, \tag{3.5}$$

for any vector field $X$ in $M^n$. Let us denote the curvature tensor of $M^{n+1}$ with respect to $\tilde{\dot{\nabla}}$ by $\tilde{K}$ and that of $M^n$ with respect to $\dot{\nabla}$ by $K$. Thus

$$\tilde{K}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\dot{\nabla}}_{\tilde{X}} \widetilde{\dot{\nabla}_{\tilde{Y}}} \tilde{Z} - \tilde{\dot{\nabla}}_{\tilde{Y}} \tilde{\dot{\nabla}}_{\tilde{X}} \tilde{Z} - \tilde{\dot{\nabla}}_{[\tilde{X}, \tilde{Y}]} \tilde{Z}$$

and

$$K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z .$$

Then the equation of Gauss is given by

$$K(X, Y, Z, U) = \tilde{K}(BX, BY, BZ, BU) + h(X, U)\, h(Y, Z) - h(Y, U)\, h(X, Z),$$

where

$$\tilde{K}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) = \tilde{g}(\tilde{K}(\tilde{X}, \tilde{Y})\, \tilde{Z}, \tilde{U})$$

and the similar expression for $K(X, Y, Z, U)$ for $M^n$.

The equation of Codazzi is given by

$$(\dot{\nabla}_X h)(Y, Z) - (\dot{\nabla}_Y h)(X, Z) = \tilde{K}(BX, BY, BZ, N).$$

We shall find the equations of Gauss and Codazzi with respect to the quarter symmetric connection. The curvature tensor with respect to the quarter symmetric metric connection $\tilde{\nabla}$ of $M^{n+1}$ is, by the definition

$$\tilde{R}(\tilde{X}, \tilde{Y})\, \tilde{Z} = \tilde{\nabla}_{\tilde{X}} \tilde{\nabla}_{\tilde{Y}} \tilde{Z} - \tilde{\nabla}_{\tilde{Y}} \tilde{\nabla}_{\tilde{X}} \tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{Z}. \tag{3.6}$$

Putting $\tilde{X} = BX$, $\tilde{Y} = BY$ and $\tilde{Z} = BZ$, we have

$$\tilde{R}(BX, BY)\, BZ = \tilde{\nabla}_{BX} \tilde{\nabla}_{BY} BZ - \tilde{\nabla}_{BY} \tilde{\nabla}_{BX} \tilde{Z} - \tilde{\nabla}_{[BX, BY]} BZ .$$

By virtue of (1.4), (3.5) and (1.8), we get

$$\begin{aligned} \tilde{R}(BX, BY)\, BZ &= B\{R(X, Y)\, Z + m(X, Z)\, MY - m(Y, Z)\, MX\} \\ &+ \{(\nabla_X m)(Y, Z) - (\nabla_Y m)(X, Z) + m(\pi(Y)\, FX - \pi(X)\, FY, Z)\}\, N , \end{aligned} \tag{3.7}$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

is the curvature tensor of the quarter symmetric connection $\nabla$.

Putting

$$\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) = g(\tilde{R}(\tilde{X}, \tilde{Y})\, \tilde{Z}, \tilde{U})$$

and

$$R(X, Y, Z, U) = g(R(X, Y)\, Z, U).$$

Then from (3.7), we can easily show

$$\tilde{R}(BX, BY, BZ, BU) = R(X, Y, Z, U) - m(X, U)\, m(Y, Z) + m(Y, U)\, m(X, Z) \tag{3.8}$$

and

$$\begin{aligned} \tilde{R}(BX, BY, BZ, N) = {} & (\nabla_X m)(Y, Z) - (\nabla_Y m)(X, Z) \\ & + m(\pi(Y)\, F(X) - \pi(X)\, F(Y), Z). \end{aligned} \tag{3.9}$$

Equations (3.8) and (3.9) are the equations of Gauss and those of Codazzi with respect to the quarter symmetric connection.

## 4 - Submanifolds of codimensions 2

Let $M^{n+1}$ be an $(n+1)$-dimensional differentiable manifold of differentiability class $C^\infty$ and $M^{n-1}$, an $(n-1)$-dimensional manifold immersed differentiability in $M^{n+1}$ by the immersion $\tau : M^{n-1} \to M^{n+1}$. We denote the differential $d\tau$ of the immersion $\tau$ by $B$, so that the vector field $X$ in the tangent space of $M^{n-1}$ corresponds to a vector field $BX$ in that of $M^{n+1}$. Suppose that $M^{n+1}$ is a Riemannian manifold with metric tensor $\tilde{g}$. Then the submanifold $M^{n-1}$ is also Riemannian with metric tensor $g$ such that $\tilde{g}(BFX, BY) = g(FX, Y)$ for arbitrary vector fields $X$, $Y$ in $M^{n-1}$ [5].

If the Riemannian manifolds $M^{n-1}$ and $M^{n+1}$ are both orientable, we can choose mutually orthogonal unit normals $\underset{1}{N}$ and $\underset{2}{N}$ defined along $M^{n-1}$ such that

$$\tilde{g}(BFX, \underset{1}{N}) = \tilde{g}(BFX, \underset{2}{N}) = \tilde{g}(\underset{1}{N}, \underset{2}{N}) = 0$$

and

$$\tilde{g}(\underset{1}{N}, \underset{1}{N}) = \tilde{g}(\underset{2}{N}, \underset{2}{N}) = 1$$

for arbitrary vector field $X$ in $M^{n-1}$ [4].

We now suppose that the enveloping manifold $M^{n+1}$ admits a quarter symmetric metric connection given by [3]

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \tilde{\dot{\nabla}}_{\tilde{X}}\tilde{Y} + \tilde{\pi}(\tilde{Y})\,\tilde{F}\tilde{X} - \tilde{g}(\tilde{F}\tilde{X}, \tilde{Y})\,\tilde{P},$$

for arbitrary vector fields $\tilde{X}, \tilde{Y}$ in $M^{n+1}$ where $\tilde{\dot{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric $\tilde{g}$, $\tilde{\pi}$ is a 1-form, $\tilde{F}$ is a tensor of type (1,1) such that $\tilde{g}(\tilde{F}\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{F}\tilde{Y})$, and $\tilde{P}$ the vector field defined by $\tilde{g}(\tilde{P}, \tilde{X}) = \tilde{\pi}(\tilde{X})$, for arbitrary vector field $\tilde{X}$ of $M^{n+1}$.

Let us now put

$$\tilde{P} = BP + \lambda \underset{1}{N} + \mu \underset{2}{N}, \tag{4.1}$$

$P$ being a vector field in the tangent space of $M^{n-1}$ and $\lambda, \mu$ functions of $M^{n-1}$.

We have the following theorem:

Theorem 4.1. *The connection induced on the submanifold $M^{n-1}$ of codimension 2 of the Riemannian manifold $M^{n+1}$ with quarter symmetric metric connection is also quarter symmetric.*

Proof. Let $\dot{\nabla}$ be the connection induced on the submanifold $M^{n-1}$ from the connection $\tilde{\dot{\nabla}}$ on the enveloping manifold $M^{n+1}$, with respect to unit normals $\underset{1}{N}$ and $\underset{2}{N}$. Then we have [4]

$$\tilde{\dot{\nabla}}_{BX}BY = B(\dot{\nabla}_X Y) + h(X, Y)\,\underset{1}{N} + k(X, Y)\,\underset{2}{N} \tag{4.2}$$

for arbitrary vector fields $X, Y$ of $M^{n-1}$, where $h$ and $k$ are second fundamental tensors of $M^{n-1}$. Similarly, if $\nabla$ be connection induced on $M^{n-1}$ from the quarter symmetric metric connection $\tilde{\nabla}$ on $M^{n+1}$, we have

$$\tilde{\nabla}_{BX}BY = B(\nabla_X Y) + m(X, Y)\,\underset{1}{N} + n(X, Y)\,\underset{2}{N}, \tag{4.3}$$

$m$ and $n$ being tensor fields of type $(0, 2)$ of the submanifold $M^{n-1}$. We also have, in view of (1.1)

$$\tilde{\nabla}_{BX} BY = \tilde{\dot{\nabla}}_{BX} BY + \tilde{\pi}(BY)\, \tilde{F}(BX) - \tilde{g}(\tilde{F}(BX), BY)\, \tilde{P}.$$

In view of (4.1), (4.2) and (4.3), we have

$$\begin{aligned} & B(\nabla_X Y) + m(X, Y) \underset{1}{N} + n(X, Y) \underset{2}{N} \\ (4.4) \qquad & = B(\dot{\nabla}_X Y) + h(X, Y) \underset{1}{N} + k(X, Y) \underset{2}{N} \\ & + \pi(Y)\, BFX - g(FX, Y)(BP + \lambda \underset{1}{N} + \lambda \underset{2}{N}) \end{aligned}$$

where $\tilde{\pi}(BX) = \pi(X)$ and $\tilde{F}(BX) = BFX$.

Comparing tangential and normal vector fields to $M^{n-1}$, we get

$$(4.5) \qquad \nabla_X Y = \dot{\nabla}_X Y + \pi(Y)\, FX - g(FX, Y)\, P,$$

where $\lambda$ and $\mu$ are chosen such that

$$(4.6) \qquad \begin{aligned} &(a) \quad m(X, Y) = h(X, Y) - \lambda g(FX, Y) \quad \text{and} \\ &(b) \quad n(X, Y) = k(X, Y) - \mu g(FX, Y). \end{aligned}$$

Thus,

$$(4.7) \qquad \nabla_X Y - \nabla_Y X - [X, Y] = \pi(Y)\, FX - \pi(X)\, FY.$$

Hence the connection $\nabla$ induced on $M^{n-1}$ is quarter symmetric [**3**].

## 5 - Totally geodesic and totally umbilical submanifolds

Let $X_1, X_2, \dots, X_{n-1}$ be $(n-1)$ orthonormal vector fields on the submanifold $M^{n-1}$. Then the function

$$\frac{1}{2(n-1)} \sum_{i=1}^{n-1} \{h(X_i, X_i) + k(X_i, X_i)\}$$

is the mean curvature of $M^{n-1}$ with respect to the Riemannian connection $\dot{\nabla}$ and

$$\frac{1}{2(n-1)} \sum_{i=1}^{n-1} \{m(X_i, X_i) + n(X_i, X_i)\}$$

is the mean curvature of $M^{n-1}$ with respect to $\nabla$ [**5**].

Now we have the following definitions:

Definition 5.1. If $h$ and $k$ vanish separately the submanifold $M^{n-1}$ is called *totally geodesic with respect to the Riemannian connection* $\dot{\nabla}$.

Definition 5.2. The submanifold $M^{n-1}$ is called *totally umbilical with respect to the connection* $\dot{\nabla}$ if $h$ and $k$ are proportional to the metric tensor $g$.

We call $M^{n-1}$ is totally geodesic and totally umbilical with respect to the quarter symmetric connection $\nabla$ according as the functions $m$ and $n$ vanish separately and are proportional as metric tensor $g$ respectively.

We now prove the following theorem:

Theorem 5.1. *In order that the mean curvature of $M^{n-1}$ with respect to connection $\dot{\nabla}$ may coincide with that of $M^{n-1}$ with respect to the connection $\nabla$, it is necessary and sufficient that $\tilde{P}$ is in the tangent space of $M^{n+1}$.*

Proof. In view of (4.6), we have

$$m(X_i, X_i) + n(X_i, X_i) = h(X_i, X_i) + k(X_i, X_i) - (\lambda + \mu)\, g(FX_i, X_i).$$

Summing up for $i = 1, 2, \ldots, (n-1)$ and dividing by $2(n-1)$, we get

$$\frac{1}{2(n-1)} \sum_{i=1}^{n-1} \{m(X_i, X_i) + n(X_i, X_i\} = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} \{h(X_i, X_i) + k(X_i, X_i\}$$

if and only if,

$$\lambda = \mu = 0\,.$$

Hence from (4.1), it follows that $\tilde{P} = BP$. Thus the vector field $\tilde{P}$ is in the tangent space of $M^{n+1}$.

Theorem 5.2. *The submanifold $M^{n-1}$ is totally umbilical with respect to the Riemannian connection $\dot{\nabla}$ if and only if it is totally umbilical with respect to the quarter symmetric connection $\nabla$.*

Proof. The proof follows easily from equations (4.6(a) and (b)).

## 6 - Curvature tensor and Weingarten equations

For the Riemannian connection $\dot{\nabla}$, the Weingarten equations are given by [4]

$$(6.1)\qquad \begin{aligned} &(a)\quad \tilde{\dot{\nabla}}_{BX}\underset{1}{N} = -BHX + 1(X)\,\underset{2}{N} \quad \text{and}\\ &(b)\quad \tilde{\dot{\nabla}}_{BX}\underset{2}{N} = -BKX + 1(X)\,\underset{1}{N} \end{aligned}$$

where $H$ and $K$ are tensor fields of type (1,1) such that

$$(6.2)\qquad \begin{aligned} &(a)\quad g(HX,\, Y) = h(X,\, Y) \quad \text{and}\\ &(b)\quad g(KX,\, Y) = k(X,\, Y)\,. \end{aligned}$$

Also, making use of (1.1) and (6.2) (a), we get

$$\tilde{\nabla}_{BX}\underset{1}{N} = -BHX + 1(X)\,\underset{2}{N} + \tilde{\pi}(\underset{1}{N})\,BFX - \tilde{g}(BFX,\,\underset{1}{N})\,\tilde{P}\,.$$

Since

$$\tilde{\pi}(\underset{1}{N}) = \tilde{g}(\tilde{P},\,\underset{1}{N}) = \lambda$$

and

$$\tilde{g}(BFX,\,\underset{1}{N}) = 0\,.$$

We have

$$(6.3)\qquad \tilde{\nabla}_{BX}\underset{1}{N} = -B(H-\lambda F)\,X + 1(X)\,\underset{2}{N}\,.$$

Similary, from (1.1) and (6.2) (b), we get

$$(6.4)\qquad \tilde{\nabla}_{BX}\underset{2}{N} = -B(K-\mu F)\,X - 1(X)\,\underset{1}{N}\,.$$

Putting

$$H - \lambda F = M_1$$

and

$$K - \mu F = M_2\,.$$

We get

$$(6.5)\qquad \begin{aligned} &\text{(a)}\quad \tilde{\nabla}_{BX}\underset{1}{N} = -BM_1X + 1(X)\underset{2}{N} \quad \text{and} \\ &\text{(b)}\quad \tilde{\nabla}_{BX}\underset{2}{N} = -BM_2X - 1(X)\underset{1}{N} \end{aligned}$$

(6.5) (a), (b) are equations of Weingarten with respect to the quarter symmetric metric connection $\tilde{\nabla}$.

The Riemannian curvature tensor for quarter symmetric metric connection can be obtained as follows.

Let $\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}$ be the Riemannian curvature tensor of the enveloping manifold $M^{n+1}$ with respect to quarter symmetric metric connection $\tilde{\nabla}$.

Then

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]}\tilde{Z}.$$

Replacing $\tilde{X}$ by $BX$, $\tilde{Y}$ by $BY$ and $\tilde{Z}$ by $BZ$, wet get

$$\tilde{R}(BX, BY)BZ = \tilde{\nabla}_{BX}\tilde{\nabla}_{BY}BZ - \tilde{\nabla}_{BY}\tilde{\nabla}_{BX}BZ - \tilde{\nabla}_{[BXBY]}BZ\,.$$

Using (4.3), we have

$$\begin{aligned} \tilde{R}(BX, BY)BZ = {} & \tilde{\nabla}_{BX}\{B(\nabla_Y Z) + m(Y, Z)\underset{1}{N} + n(Y, Z)\underset{2}{N}\} \\ & - \tilde{\nabla}_{BY}\{B(\nabla_X Z) + m(X, Z)\underset{1}{N} + n(X, Z)\underset{2}{N}\} \\ & - \{B(\nabla_{[X, Y]}Z) + m([X, Y], Z)\underset{1}{N} + n([X, Y], Z)\underset{2}{N}\}\,. \end{aligned}$$

Again by virtue of (4.3), (6.5) (a) and (b) and the condition (4.7), we get

$$\begin{aligned} \tilde{R}(BX, BY)BZ = {} & BR(X, Y)Z + m(\pi(Y)F(X) - \pi(X)F(Y), Z)\underset{1}{N} \\ & + n(\pi(Y)F(X) - \pi(X)F(Y), Z)\underset{2}{N} \\ & + \{(\nabla_X m)(Y, Z) - (\nabla_Y m)(X, Z)\}\underset{1}{N} \\ & + \{(\nabla_X n)(Y, Z) - (\nabla_Y n)(X, Z)\}\underset{2}{N} \\ & + B\{m(X, Z)M_1Y - m(Y, Z)M_1X + n(X, Z)M_2Y \end{aligned}$$

$$-n(Y, Z)\, M_2 X\} + 1(X)\{m(Y, Z)\, \underset{2}{N} - n(Y, Z)\, \underset{1}{N}\}$$

$$-1(Y)\{m(X, Z)\, \underset{2}{N} - n(X, Z)\, \underset{1}{N}\},$$

where $R(X, Y)\, Z$ being the Riemannian curvature tensor of the submanifold with respect to the quarter symmetric connection $\nabla$.

## References


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## Summary


*Semi symmetric metric connections have been studied by Imai* [2]. *Submanifolds of a Riemaniann manifold with semi-symmetric metric connection have been studied by Ram Nivas* [5] *and others. Professors Mishra and Pandey* [3] *defined the notion of quarter symmetric connection in a differentiable manifold. The aim of the present paper is to study hypersurfaces and submanifolds of a manifold admitting quarter symmetric connections. Some interesting results have been established on such manifolds.*


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