Riv. Mat. Univ. Parma (6) 3 (2000), 1-10

M. PASTEKA and R. F. TICHY (*)

Uniformly distributed sequences and rings of complex numbers (**)

1 - Introduction

In this paper we want to study the distribution properties of special sequences related to the factorization of elements in rings of complex numbers. Our work is inspired by E. Hlawka's article [H1990], where the author investigated sequences in connection with Gaussian integers. We will extend these results to rings with divisibility theory.

As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the positive integers, the ring of integers, the field of rational, of real and of complex numbers, respectively. Let $\{x_n\}_{n=1}^N = \{x_1, \ldots, x_N\}$ be a finite sequence of points x_n in the s-dimensional unit cube $[0, 1)^s$. Then its discrepancy is defined by

$$D_N(\{x_n\}_{n=1}^N) = \sup_I \left| \frac{1}{N} \{n \leq N : x_n \in I\} - \lambda(I) \right|,$$

where the supremum is extended over all subintervals $I = [a_1, b_1) \times \ldots \times [a_s, b_s) \subseteq [0, 1)^s$ of s-dimensional Lebesgue measure $\lambda(I)$. An infinite sequence $\{x_n\}_{n=1}^\infty$ in $[0, 1)^s$ is called uniformly distributed if $\lim\limits_{N \to \infty} D_N(\omega_N) = 0$, where ω_N denotes the initial string $\{x_1, \ldots, x_N\}$ of the sequence $\{x_n\}$. The most classical example of a uniformly distributed sequence is given by the multiples $x_n = n\alpha (\mathrm{mod}\ 1)$ of a point $\alpha = (\alpha_1, \ldots, \alpha_s) \in [0, 1)^s$, where $1, \alpha_1, \ldots, \alpha_s$ are linearly independent

(*) M. PASTEKA: Mathematical Institute of Slovak Academy of Science, Štefànikova 49, SK - 814 73 Bratislava, Slovakia; R. F. TICHY: Technical University of Graz, Dept. of Mathematics, Steyrergasse 30, 8010 Graz, Austria.
(**) Received May 24, 1999. AMS classification 11 K 38, 11 K 31.

over $\mathbb{Q}$ and where (mod 1) means the componentwise taken fractional part. For an introduction to the theory of uniformly distributed sequences we refer to the monographs of L. Kuipers and H. Niederreiter [KN1974] and E. Hlawka [H1979]; for more recent developments in this theory and various applications see [DT1997].

In section 2 of the present paper we establish quantitative refinements and extensions to rings with divisibility theory (in the sense of Borevic-Shafarevic [BS1966]). In the final section 3 applications to diophantine approximation on the unit circle and to the construction of uniformly distributed points on the sphere are presented.

2 - Sequences in $[0, 1)^s$ and rings with divisibility theory

In the following we consider a ring $R \subset \mathbb{C}$ with divisibility theory and canonical homomorphismus $\psi : R \to G$, where G is the semigroup with unique factorization. For more details see the book of Borevic-Shafarevic [BS1966]. By (x, y) we will denote as usual the greatest common divisor in G and by $\overline{\alpha}$ the complex conjugate of α. Our first simple result is the following:

Theorem 1. *Let* $\alpha_1, \dots, \alpha_s \in R$ *be such that* $\overline{\alpha}_j \in R$, $j = 1, \dots, s$, *and let* $(\psi(\alpha_j), \psi(\alpha_k)) = 1$ *for* $j \neq k$ *and* $(\psi(\alpha_j), \psi(\overline{\alpha}_k)) = 1$ *for arbitrary* j, k.

Put for $j = 1, \dots, s$

$$\frac{\alpha_j}{|\alpha_j|} = e^{2\pi i \varphi_j}, \qquad \varphi_j \in [0, 1). \tag{1}$$

Then the numbers $1, \varphi_1, \dots, \varphi_s$ *are linearly independent over* $\mathbb{Z}$.

Proof. Suppose

$$h_1 \varphi_1 + \dots + h_s \varphi_s + h_{s+1} = 0, \qquad h_j \in \mathbb{Z}.$$

Then

$$2\pi i h_1 \varphi_1 + \dots + 2\pi i h_s \varphi_s + 2\pi i h_{s+1} = 0,$$

thus by (1) we obtain

$$\left(\frac{\alpha_1}{|\alpha_1|} \right)^{h_1} \dots \left(\frac{\alpha_s}{|\alpha_s|} \right)^{h_s} = 1.$$

Therefore

$$\alpha_1^{h_1} \dots \alpha_s^{h_s} = \overline{\alpha}_1^{h_1} \dots \overline{\alpha}_s^{h_s},$$

and applying the homomorphism ψ to the last equality we obtain $h_1 = \dots = h_s = 0$. Hence also $h_{s+1} = 0$, which proves linear independence. ■

From now on let us assume that R is a discrete ring, i.e. $|\alpha| \geqslant \sigma > 0$, for arbitrary $\alpha \neq 0$. For more examples we refer to [BS1966] or [N1990]. We will use the following notation: $a_j = |\alpha_j|$, $j = 1, \dots, s$, $A_s = a_1 \dots a_s$, where α_j denote numbers in R as defined in (1). Furthermore we suppose that $|a_j| > 1$, $j = 1, \dots, s$.

Theorem 2. *Let $\varphi = (\varphi_1, \dots, \varphi_s)$ be a point in $[0, 1)^s$, where φ_j is given as in* (1) *and consider the finite sequence $\omega_N = \{2k\varphi \,(\mathrm{mod}\ 1), k = 1 \dots, N\}$ of points in $[0, 1)^s$.*

Then the discrepancy $D_N = D_N(\omega_N)$ of the sequence ω_N satisfies

$$D_N = O\left(\frac{1}{\log N}\right).$$

Proof. We start from the Erdös-Turán-Koksma inequality (see [DT1997])

$$D_N \leqslant C_s \left(\frac{1}{M} + \sum_{0 < \|h\| \leqslant M} R(h)^{-1} |W_N(h)| \right), \tag{2}$$

where $\|h\| = \max\{|h_j|; j = 1, \dots, s\}$ and $R(h) = \prod_{j=1}^{s} \max(1, |h_j|)$, $M \geqslant 1$ and

$$W_N(h) = \frac{1}{N} \sum_{k=1}^{N} e^{4\pi i k \langle h, \varphi \rangle},$$

$\langle h, \varphi \rangle$ denoting the standard scalar product. Using Theorem 1 we obtain

$$|W_N(h)| \leqslant \frac{2}{N|e^{4\pi i \langle h, \varphi \rangle} - 1|},$$

and

$$e^{4\pi i \langle h, \varphi \rangle} = \left(\frac{\alpha_1}{\overline{\alpha}_1}\right)^{h_1} \dots \left(\frac{\alpha_s}{\overline{\alpha}_s}\right)^{h_s} = \frac{\alpha}{\beta}; \qquad \alpha, \beta \in R,$$

where $|\beta| = a_1^{|h_1|} \dots a_s^{|h_s|}$. Thus

$$|e^{4\pi i \langle h,\, \varphi \rangle} - 1| = \left| \frac{\alpha}{\beta} - 1 \right| = \left| \frac{\alpha - \beta}{\beta} \right| \geqslant \frac{\sigma}{|\beta|}\,,$$

which yields

$$|W_n(h)| \leqslant \frac{2|\beta|}{N\sigma} = \frac{2a_1^{|h_1|} \dots a_s^{|h_s|}}{N\sigma}\,.$$

From $R(h) \geqslant 1$ and (2) we obtain

$$D_N \leqslant C_s \left(\frac{1}{M} + \frac{2}{N\sigma} \sum_{\|h\| \leqslant M} a_1^{|h_1|} \dots a_s^{|h_s|} \right).$$

Now it holds

$$\sum_{\|h\| \leqslant M} a_1^{|h_1|} \dots a_s^{|h_s|} \leqslant 2^s \prod_{j=1}^{s} \sum_{h_j=0}^{M} a_j^{h_j} = 2^s \prod_{j=1}^{s} \frac{a_j^{M+1} - 1}{a_j - 1}\,. \tag{3}$$

Let us take a constant $K = K(a_1, \dots, a_s)$ such that

$$\frac{a_j - a_j^{-M}}{a_j - 1} \leqslant K\,.$$

Thus we obtain from (3)

$$\sum_{\|h\| \leqslant M} a_1^{|h_1|} \dots a_s^{|h_s|} \leqslant 2^s K^s A_s^M,$$

and so we have

$$D_N \leqslant C_s \left(\frac{1}{M} + \frac{2}{N\sigma} 2^s K^s A_s^M \right). \tag{4}$$

Put now $M = \dfrac{\log N}{2 \log A_s}$. Inequality (4) yields

$$D_N \leqslant C_s \left(\frac{2 \log A_s}{\log N} + \frac{2}{\sigma N} 2^s K^s N^{1/2} \right) = O\left(\frac{1}{\log N} \right),$$

which completes the proof. ■

In the following we will sketch a continuous analagon. Let $\omega(t)$ denote a continuous curve on the s-dimensional torus $\mathbb{R}^s/\mathbb{Z}^s \cong [0, 1)^s$. The corresponding discrepancy

$$D_T(\omega(t)) = \sup_I \left| \frac{1}{T} \int_0^T 1_I(\omega(t))\, dt - \lambda(I) \right|$$

is defined as the maximal deviation of the mean values with respect to time and with respect to space; 1_I denotes the characteristic function of the s-dimensional interval I. For more details on continuous discrepancy we refer to [DT1997].

Theorem 3. *Let us consider the function*

$$\omega(t) = 2t\varphi \pmod 1 \qquad t \in [0, T],$$

where φ is chosen as in Theorem 2. Then we have

$$D_T(\omega(t)) = O\left(\frac{1}{\log T}\right).$$

Proof. In this case we use the continuous version of Erdös-Turán-Koksma's inequality, see [DT1997], page 279. Put

$$W_T(h) = \frac{1}{T} \int_0^T e^{4\pi i t\langle h, \varphi\rangle}\, dt\,,$$

thus

$$|W_T(h)| \leqslant \frac{2}{T|\langle h, \varphi\rangle|}\,.$$

For each t we have $|t| \geqslant |e^{2\pi i t} - 1|$, and so

$$|W_T(h)| \leqslant \frac{2}{T|e^{2\pi i\langle h, \varphi\rangle} - 1|} = \frac{2}{T} a_1^{|h_1|} \ldots a_s^{|h_s|}\,.$$

Following the proof of Theorem 2 we obtain the result. ∎

3 - Applications

In the following let $A(N, J)$ denote the number of positive integers $k \leqslant N$ such that $2k\varphi$ (mod 1) belongs to the subinterval J of $[0, 1)^s$, where φ is a point in $[0, 1)^s$ chosen as in Theorem 2. Obviously, $A(N, J) \geqslant N(V(J) - D_N)$. Therefore the inequality

$$V(J) > D_N$$

yields that $2k\varphi$ (mod 1) $\in J$ for some k.

Put $\alpha_j = u_j + iv_j$, $j = 1, \dots, s$. Then

$$\frac{\alpha_j}{\overline{\alpha}_j} = \frac{u_j + iv_j}{u_j - iv_j} = e^{4\pi i \varphi_j}, \qquad j = 1, \dots, s\,,$$

and so

$$e^{4\pi i k \varphi_j} = \left(\frac{\alpha_j}{\overline{\alpha}_j} \right)^{h_j} = \frac{u_{j,k} + iv_{j,k}}{u_{j,k} - iv_{j,k}} = p_{j,k} + ir_{j,k}, \qquad j = 1, \dots, s\,,$$

where

$$p_{j,k} = \frac{u_{j,k}^2 - v_{j,k}^2}{u_{j,k}^2 + v_{j,k}^2}\,,\ r_{j,k} = \frac{2u_{j,k} v_{j,k}}{u_{j,k}^2 + v_{j,k}^2}\,.$$

Using this notation we will establish a result for simultaneous diophantine approximation on the unit circle.

Theorem 4. *Let $(A_1, B_1), \dots, (A_s, B_s)$ be points on the unit circle. Then there exist positive integers $k \leqslant N$ such that*

$$|A_j - p_{j,k}| \leqslant \left(\frac{c}{\log N} \right)^{\frac{1}{s}}, \qquad |B_j - r_{j,k}| \leqslant \left(\frac{c}{\log N} \right)^{\frac{1}{s}},$$

for some constant $c > 0$.

Proof. Let $Z_j = A_j + iB_j = e^{2\pi i k \gamma_j}$ and consider the interval $J = \times_{j=1}^{s} [\gamma_j - D_N^{\frac{1}{s}}, \gamma_j + D_N^{\frac{1}{s}}]$. Then $V(J) = 2^s D_N > D_N$, hence $2k\varphi$ (mod 1) $\in J$ for some k. This yields $|\gamma_j - 2k\varphi_j| \leqslant 2\pi D_N^{\frac{1}{s}}$, and so $|Z_j - e^{4\pi i k \varphi_j}| \leqslant 2\pi D_N^{\frac{1}{s}}$. Thus the assertion follows. ■

In the following we present an application to the construction of uniformly distributed points on the sphere. Consider the unit sphere $K : X_1^2 + X_2^2 + X_3^2 = 1$ in $\mathbb{R}^3$ with surface area 4π and equipped with a probability measure

$$P(S) = \frac{1}{4\pi} \iint_S dK$$

(for each measurable set $S \subset K$). Let F be a real valued continuous function defined on this sphere, then

$$\int_K F dP = \frac{1}{4\pi} \iint_K F dK .$$

Consider now the parametrization of K given by

$$X_1 = \sqrt{1 - (1 - 2t_1)^2} \cos 2\pi t_2 =: g_1(t_1, t_2)$$

$$X_2 = \sqrt{1 - (1 - 2t_1)^2} \sin 2\pi t_2 =: g_2(t_1, t_2)$$

$$X_3 = 1 - 2t_1 =: g_3(t_1, t_2),$$

for $t_1, t_2 \in [0, 1]$.

Setting

$$f(t_1, t_2) = F(g_1, g_2, g_3) \tag{5}$$

we obtain after simple calculations

$$\iint_K F dK = 4\pi \int_0^1 \int_0^1 f(t_1, t_2)\, dt_1\, dt_2 .$$

Thus we derive

$$\int_K F dP = \int_0^1 \int_0^1 f(t_1, t_2)\, dt_1\, dt_2 . \tag{6}$$

Let $I^2 = [0, 1] \times [0, 1]$ and $(u_1, v_1), \ldots, (u_N, v_N) \in I^2$ be a sequence in $[0, 1)^2$ with discrepancy D_N. Suppose that f given by (5) is of bounded variation $V(f)$ in the sense of Hardy and Krause. Then by Koksma-Hlawka's inequality (see [DT1997]) we have

$$\left| \frac{1}{N} \sum_{k=1}^{N} f(u_k, v_k) - \int_0^1 \int_0^1 f(t_1, t_2)\, dt_1\, dt_2 \right| \leq V(f) \cdot D_N. \tag{7}$$

Put now $X(k) = (g_1(u_k, v_k), g_2(u_k, v_k), g_3(u_k, v_k))$ for $k = 1, \ldots, N$. Then $X(k)$ are points on the sphere K, and $F(X(k)) = f(u_k, v_k)$. Thus by (6) and (7) we obtain

$$\left| \frac{1}{N} \sum_{k=1}^{N} F(X(k)) - \int_K F dP \right| \leq V(f) \cdot D_N. \tag{8}$$

Let $\varphi = (\varphi_1, \ldots, \varphi_s) \in [0, 1)^s$ such as in Theorem 2. Consider the vector $\overline{\varphi} = \left(\varphi_1, \ldots, \varphi_s, \frac{1}{2N}\right) = \left(\varphi, \frac{1}{2N}\right)$, and the sequence $\overline{\omega}_N = \{2k\overline{\varphi} (\mathrm{mod}\, 1),\ k = 1 \ldots N\}$. For abitrary sequence $\{x_n\}_{n=1}^N$ it holds $ND_N\left(\left\{\left(x_n, \frac{n}{N}\right)\right\}_{n=1}^N\right) \ll \max_{m \leq N} mD_m(\{x_n\}_{n=1}^m)$ and so by Theorem 2 we have

$$D_N(\overline{\omega}_N) = O\left(\frac{1}{\log N}\right). \tag{9}$$

Let α be a Gaussian integer relatively prime with its complex conjugate $\overline{\alpha}$. Consider $\varphi \in [0, 1)$ such that $\frac{\alpha}{|\alpha|} = e^{2\pi i \varphi}$. Then

$$\frac{\alpha}{\overline{\alpha}} = e^{4\pi i \varphi}, \left(\frac{\alpha}{\overline{\alpha}}\right)^k = e^{4k\pi i \varphi},$$

and moreover

$$\left(\frac{\alpha}{\overline{\alpha}}\right)^k = \frac{a_k^2 - b_k^2}{a_k^2 + b_k^2} + i\,\frac{2a_k b_k}{a_k^2 + b_k^2}.$$

Thus

$$\cos 4\pi k\varphi = \frac{a_k^2 - b_k^2}{a_k^2 + b_k^2}, \qquad \sin 4\pi k\varphi = \frac{2a_k b_k}{a_k^2 + b_k^2},$$

where a_k, b_k are given by linear reccurence relations $a_1 = a$, $b_1 = b$, $a_{k+1} = aa_k - bb_k$, $b_{k+1} = ba_k + ab_k$. Hence, by (9) the two-dimensional sequence of points $\left\{\left(2k\varphi\, (\mathrm{mod}\, 1), \frac{k}{N}\right)\right\}$, $k = 1, \ldots, N$ has a discrepancy $D_N = O\left(\frac{1}{\log N}\right)$. Therefore, for the points

$$X(k,N)=\left(\sqrt{\frac{2k}{N}-\frac{k^2}{N^2}}\cdot\frac{a_k^2-b_k^2}{a_k^2+b_k^2},\sqrt{\frac{2k}{N}-\frac{k^2}{N^2}}\cdot\frac{2a_kb_k}{a_k^2+b_k^2},1-\frac{2k}{N}\right),\quad k=1,\dots,N$$

we obtain

$$\left|\frac{4\pi}{N}\sum_{k=1}^{N}F(X(k,N))-\iint_K F dK\right|=O\left(\frac{1}{\log N}\right). \tag{10}$$

Now, let us consider the pointset $w(N)=\{X(k,N),\ k=1,\dots,N\}$ and a Jordan measurable set $S\subset K$. Then (10) yields

$$\lim_{N\to\infty}\frac{4\pi}{N}\,card\,(w(N)\cap S)=area\,S\,. \tag{11}$$

We conclude with the following result.

Theorem 5. *Let $\{X(k)\}$ be the sequence $\{w(1),w(2),w(3)\dots\}$ consisting of the blocks $w(N)$, $N=1,2,\dots$. Then $\{X(k)\}_{k=1}^{\infty}$ is uniformly distributed on K.*

Proof. For $S\subset K$ put $A(S,N)=|\{k\leqslant N;\ X(k)\in S\}|$. Let $k(N)$ be defined by a number that $|w(1)|+\dots+|w(k(N))|\leqslant N<|w(1)|+\dots+|w(k(N)+1)|$. Then $A(S,N)=|w(1)\cap S|+\dots+|w(k(N))\cap S|+O(k(N))$. Thus we obtain from (11) that for any Jordan measurable set $S\subset K$

$$\lim_{N\to\infty}\frac{A(S,N)}{N}=\frac{area\,(S)}{4\pi},$$

which proves uniform distribution of the sequence $\{X(k)\}_{k=1}^{\infty}$. ■

References

[BS1966] S. I. Borevic and I. R. Shafarevic, *Zahlentheorie*, Birkhäuser-Verlag, Basel, 1966.

[DT1997] M. Drmota and R. F. Tichy, *Sequences, discrepancies and applications*, Lecture Notes in Math. **1651**, Springer-Verlag, 1997.

[H1990] E. Hlawka, *Über eine Klasse von gleichverteilten Folgen*, Acta Arith. **53** (1990), 389-402.

[KN1974] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley-Interscience Publ., New York, London, Sidney 1974.

[N1990] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*, Springer Verlag, Berlin, Heidelberg, New York, London, Paris, Tokio, Hong Kong, 1990.

Abstract

The discrepancy of special sequences related to the factorization of elements in rings with divisibility theory is estimated. Applications to diophantine approximation on the unit circle and to the construction of uniformly distributed sequences on the sphere are established.

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