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Asymptotic formulae concerning arithmetical functions defined by cross-convolutions, VIII. On the product and the quotient of $\sigma_{A,s}$ and $\phi_{A,s}$ (**)

1 - Introduction

Let $\mathbb{N}$ denote the set of positive integers and let $A(n)$ be a subset of the positive divisors of n for each $n \in \mathbb{N}$. The A-convolution of the arithmetical functions f and g is given by

$$(1) \qquad (f *_A g)(n) = \sum_{d \in A(n)} f(d)\, g(n/d).$$

W. NARKIEWICZ [**5**] defined the A-convolution (1) to be regular if

(a) the set of arithmetical functions is a commutative ring with unity with respect to ordinary addition and the A-convolution,

(b) the A-convolution of multiplicative functions is multiplicative,

(c) the function I, defined by $I(n) = 1$ for all $n \in \mathbb{N}$, has an inverse μ_A with respect to the A-convolution and $\mu_A(p^a) \in \{-1, 0\}$ for every prime power p^a.

For example, the Dirichlet convolution D, where $D(n) = \{d \in \mathbb{N}: d|n\}$, and the unitary convolution U, where $U(n) = \{d \in \mathbb{N}: d|n, (d, n/d) = 1\}$, are regular.

It can be proved, see [**5**], that an A-convolution is regular if and only if

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(i) $A(mn) = \{de\colon d \in A(m),\ e \in A(n)\}$ for every $m, n \in \mathbb{N}, (m, n) = 1$,

(ii) for every prime power p^a there exists a divisor $t = t_A(p^a)$ of a, called the type of p^a with respect to A, such that $A(p^{it}) = \{1, p^t, p^{2t}, \ldots, p^{it}\}$ for every $i \in \{0, 1, \ldots, a/t\}$.

The elements of the set $A(n)$ are called the A-divisors of n.

Let $\sigma_{A,s}(n) = \sum_{d \in A(n)} d^s$ denote the sum of s-th powers of the A-divisors of n and let $\phi_{A,s}(n) = \sum_{d \in A(n)} d^s \mu_A(n/d)$ be the generalized Euler function. Note that for $s = k \in \mathbb{N}$, $\phi_{A,k}(n)$ represents the number of integers $x \pmod{n^k}$ such that $(x, n^k)_{A,k} = 1$, where $(a, b)_{A,k}$ stands for the greatest k-th power divisor of a which belongs to $A(b)$.

For $A = D$, $\sigma_{D,s}(n) \equiv \sigma_s(n)$ and $\phi_{D,s}(n) \equiv \phi_s(n)$ are the usual divisor-sum and Euler-type functions. For $A = U$, $\sigma_{U,s}(n) \equiv \sigma_s^*(n)$ and $\phi_{U,s}(n) \equiv \phi_s^*(n)$ are the unitary analogues of these functions, investigated by E. COHEN [1], K. NAGESWARA RAO [4] and others.

For other properties of regular convolutions see also P. J. MCCARTHY [3] and V. SITA RAMAIAH [6].

In [8] we introduced the notion of *cross-convolution* of arithmetical functions as a special case of Narkiewicz's regular convolution as follows. We say that the regular convolution A is a cross-convolution if for every prime p we have either $t_A(p^a) = 1$, i.e. $A(p^a) = \{1, p, p^2, \ldots, p^a\} \equiv D(p^a)$ for every $a \in \mathbb{N}$ or $t_A(p^a) = a$, i.e. $A(p^a) = \{1, p^a\} \equiv U(p^a)$ for every $a \in \mathbb{N}$. Let $P_A = P$ and $Q_A = Q$ be the sets of the primes of the first and second kind of above, respectively, where $P \cup Q = \mathbb{P}$ is the set of all primes. For $P = \mathbb{P}$ and $Q = \emptyset$ we have the Dirichlet convolution D and for $P = \emptyset$ and $Q = \mathbb{P}$ we obtain the unitary convolution U.

Furthermore, let (P) and (Q) denote the multiplicative semigroups generated by $\{1\} \cup P$ and $\{1\} \cup Q$, respectively. Every $n \in \mathbb{N}$ can be written uniquely in the form $n = n_P n_Q$, where $n_P \in (P)$, $n_Q \in (Q)$.

If A is a cross-convolution, then

$$A(n) = \{d \in \mathbb{N}\colon d \mid n, (d, n/d) \in (P)\} \tag{2}$$

and (1) can be written in the form

$$(f *_A g)(n) = \sum_{\substack{d \mid n \\ (d, n/d) \in (P)}} f(d)\, g(n/d). \tag{3}$$

In [8] we gave asymptotic formulae for $\sum_{n \leq x} \sigma_{A,s}(n)$ and $\sum_{n \leq x} \phi_{A,k}(n)$, where $s > 0$ and $k \in \mathbb{N}$, assuming that A is a cross-convolution.

In [**9**] we established asymptotics for $\sum_{n \leq x} \sigma_s(n)\phi_s(n)$, $\sum_{n \leq x} \sigma_s(n)/\phi_s(n)$ in case $s > 0$ and for $\sum_{n \leq x} \phi_s(n)/\sigma_s(n)$ in case $s \geq 1$.

The aim of this paper is to extend the results of [**9**] for $\sigma_{A,s}$ and $\phi_{A,s}$ instead of σ_s and ϕ_s, in case of cross-convolutions. We obtain, as a particular case, the corresponding results for σ_s^* and ϕ_s^* which are also believed to be new.

2 - On the product $\sigma_{A,s}(n)\,\phi_{A,s}(n)$

If A is a regular convolution and $s > 1/2$, then inequalities

$$\frac{n^{2s}}{\zeta(2s)} < \sigma_{A,s}(n)\,\phi_{A,s}(n) \leq n^{2s}$$

hold for every $n \in \mathbb{N}$, where ζ is the Riemann zeta function, cf. [**2**]. Hence we expect that the average order of $\sigma_{A,s}(n)\,\phi_{A,s}(n)$ is $B_{A,s}\,n^{2s}$, where $1/\zeta(2s) \leq B_{A,s} \leq 1$. Let

$$\zeta_P(z) = \prod_{p \in P}\left(1 - \frac{1}{p^z}\right)^{-1}, \qquad \zeta_Q(z) = \prod_{p \in Q}\left(1 - \frac{1}{p^z}\right)^{-1}.$$

Theorem 1. *If A is a cross-convolution and $s > 0$, then*

$$\sum_{n \leq x} \sigma_{A,s}(n)\,\phi_{A,s}(n) = \frac{B_{A,s}}{2s+1}\,x^{2s+1} + O(R_s(x, Q)), \tag{4}$$

where

$$B_{A,s} = \zeta_P(s+1)\,\zeta_Q(2s+1)\prod_{p \in P}\left(1 - \frac{1}{p^{s+1}} - \frac{1}{p^{2s+1}} + \frac{1}{p^{2s+2}}\right)\prod_{p \in Q}\left(1 - \frac{2}{p^{2s+1}} + \frac{1}{p^{2s+2}}\right)$$

and $R_s(x, Q) = x^{2s}(s > 1)$, $x^2\log^2 x(s = 1$ *and* Q *infinite set*$)$, $x^2 \log x(s = 1$ *and* Q *finite set*$)$, $x^{s+1}(s < 1)$.

Proof. Let $f_{A,s}$ be the arithmetical function defined by

$$\sigma_{A,s}(n)\,\phi_{A,s}(n) = \sum_{d \in A(n)} d^{2s} f_{A,s}(n/d),$$

for every $n \in \mathbb{N}$.

It is easy to check that $f_{A,s}$ is multiplicative and

$$f_{A,s}(p^a) = \begin{cases} -1, & \text{if } p \in P,\ a = 1 \text{ or } p \in Q,\ a \in \mathbb{N}, \\ \phi_s(p^a) = p^{as}\left(1 - \dfrac{1}{p^s}\right), & \text{if } p \in P,\ a \geqslant 2, \end{cases}$$

for every prime power p^a, where s is a fixed real number, cf. [**9**], Lemma 1. Therefore

$$S_1(x) \equiv \sum_{n \leqslant x} \sigma_{A,s}(n)\, \phi_{A,s}(n) = \sum_{d \leqslant x} f_{A,s}(d) \sum_{\substack{e \leqslant x/d \\ (e,d) \in (P)}} e^{2s}.$$

Now using the estimate

$$\sum_{\substack{n \leqslant x \\ (n,a) \in (P)}} n^r = \frac{\phi(a_Q)\, x^{r+1}}{a_Q(r+1)} + O(x^{r+\varepsilon} \eta_Q(a)) \tag{5}$$

valid for every $r \geqslant 0$, $a \in \mathbb{N}$ and for every ε with $0 \leqslant \varepsilon < 1$, where $\phi \equiv \phi_1$ is the Euler function, $\eta_Q(a) = 1$ (Q finite set), $\sigma_{-\varepsilon}(a)$ (Q infinite set), cf. [8], Lemma 7 and [**7**], Lemma 2.1, we get

$$S_1(x) = \sum_{d \leqslant x} f_{A,s}(d) \left(\frac{\phi(d_Q)}{d_Q(2s+1)} (x/d)^{2s+1} + O(\eta_Q(d)(x/d)^{2s+\varepsilon}) \right)$$

$$= \frac{x^{2s+1}}{2s+1} \sum_{d \leqslant x} \frac{f_{A,s}(d)\, \phi(d_Q)}{d^{2s+1} d_Q} + O\left(x^{2s+\varepsilon} \sum_{d \leqslant x} \frac{|f_{A,s}(d)|\, \eta_Q(d)}{d^{2s+\varepsilon}} \right)$$

$$= \frac{x^{2s+1}}{2s+1} \sum_{d=1}^{\infty} \frac{f_{A,s}(d)\, \phi(d_Q)}{d^{2s+1} d_Q} + O\left(x^{2s+1} \sum_{d > x} \frac{1}{d^{s+1}} \right) + O\left(x^{2s+\varepsilon} \sum_{d \leqslant x} \frac{\eta_Q(d)}{d^{s+\varepsilon}} \right),$$

taking into account that $|f_{A,s}(n)| \leqslant n^s$ for every $n \in \mathbb{N}$. It also yields that the series appearing here is absolutely convergent and its sum is $B_{A,s}$, by Euler's product formula. The first O-term is $O(x^{2s+1}/x^s) = O(x^{s+1})$ and the second O-term is $O(x^{2s})$ if $s > 1$, choosing $\varepsilon = 0$; it is $O(x^2 \log x)$ for $s = 1$ and Q finite; $O(x^2 \log^2 x)$ for $s = 1$ and Q infinite, in both cases with $\varepsilon = 0$; finally it is $O(x^{2s+\varepsilon} x^{1-s-\varepsilon}) = O(x^{s+1})$ if $0 < s < 1$ and $\varepsilon < 1 - s$, see [**7**], Lemma 2.2.

Corollary 1. *For $s > 0$ the average order of the product $\sigma_{A,s}(n)\, \phi_{A,s}(n)$ is $B_{A,s} n^{2s}$.*

Corollary 2 ($A = U$). *If* $s > 0$, *then*

$$\sum_{n \leqslant x} \sigma_s^*(n)\, \phi_s^*(n) = \frac{B_s^* x^{2s+1}}{2s+1} + O(R_s^*(x)),$$

where

$$B_s^* = \zeta(2s+1) \prod_p \left(1 - \frac{2}{p^{2s+1}} + \frac{1}{p^{2s+2}}\right)$$

and $R_s^*(x) = x^{2s} (s > 1)$, $x^2 \log^2 x (s = 1)$, $x^{s+1} (s < 1)$.

3 - On the quotient $\sigma_{A,s}(n)/\phi_{A,s}(n)$.

Theorem 2. *If* A *is a cross-convolution and* $s > 0$, *then*

(6) $$\sum_{n \leqslant x} \frac{\sigma_{A,s}(n)}{\phi_{A,s}(n)} = C_{A,s}\, x + O(S_s(x)),$$

where

$$C_{A,s} = \zeta_P(s+1) \prod_{p \in P} \left(1 + \frac{p^{s+1} + p - 1}{p^{s+2}(p^s - 1)}\right) \prod_{p \in Q} \left(1 + 2\left(1 - \frac{1}{p}\right) \sum_{j=1}^{\infty} \frac{1}{p^j (p^{js} - 1)}\right)$$

and $S_s(x) = 1 (s > 1)$, $x^{1-s+\varepsilon} (0 < s \leqslant 1)$ *for every* $\varepsilon > 0$.

Proof. Let $g_{A,s}$ be the arithmetical function given by

$$\frac{\sigma_{A,s}(n)}{\phi_{A,s}(n)} = \sum_{d \in A(n)} g_{A,s}(d)$$

for every $n \in \mathbb{N}$. We obtain that $g_{A,s}$ is multiplicative and

$$g_{A,s}(p^a) = \begin{cases} \dfrac{2}{p^{as} - 1}, & \text{if } p \in P,\ a = 1 \text{ or } p \in Q,\ a \in \mathbb{N}, \\ 1/\phi_s(p^a) = \dfrac{1}{p^{as}} \left(1 - \dfrac{1}{p^s}\right)^{-1}, & \text{if } p \in P,\ a \geqslant 2, \end{cases}$$

for every prime power p^a, where s is real and $s \neq 0$. Consequently, applying (5) for $r = 0$ and $\varepsilon = 0$ we have

$$\sum_{n \leqslant x} \frac{\sigma_{A,s}(n)}{\phi_{A,s}(n)} = \sum_{d \leqslant x} g_{A,s}(d) \sum_{\substack{e \leqslant x/d \\ (e,d) \in (P)}} 1 = \sum_{d \leqslant x} g_{A,s}(d) \left(\frac{\phi(d_Q)\, x}{d_Q d} + O(\eta_Q(d)) \right)$$

$$= x \sum_{d \leqslant x} \frac{g_{A,s}(d)\, \phi(d_Q)}{d d_Q} + O\Big(\sum_{d \leqslant x} g_{A,s}(d)\, \eta_Q(d) \Big)$$

$$= x \sum_{d=1}^{\infty} \frac{g_{A,s}(d)\, \phi(d_Q)}{d d_Q} + O\left(x \sum_{d > x} \frac{g_{A,s}(d)}{d} \right) + O\Big(\sum_{d \leqslant x} g_{A,s}(d)\, \eta_Q(d) \Big).$$

Here for $s > 0$, $g_{A,s}(n) \leqslant 2^{\omega(n)}/\phi_s(n) = O(n^{\varepsilon - s})$ for every $\varepsilon > 0$, cf. [**9**], Lemma 6, where $\omega(n)$ stands for the number of distinct prime factors of n. Therefore the above series is absolutely convergent and its sum is $C_{A,s}$, using the Euler product formula. The first O-term is $O(x \sum_{d > x} 1/d^{1+s-\varepsilon}) = O(x^{1-s+\varepsilon})$ for every $s > 0$ and the second O-term is, by $\tau(n) = O(n^{\varepsilon/2})$, $O(\sum_{d \leqslant x} \tau(d)/d^{s-\varepsilon/2}) = O(\sum_{d \leqslant x} 1/d^{s-\varepsilon}) = O(1)$ for $s > 1$ (choosing $\varepsilon < s - 1$) and it is $O(x^{1-s+\varepsilon})$, with $\varepsilon < s$, for $s \leqslant 1$ and the proof of (6) is complete.

Corollary 3. *For $s > 0$ the mean value of $\sigma_{A,s}(n)/\phi_{A,s}(n)$ is $C_{A,s}$.*

Corollary 4 ($A = U$). *If $s > 0$, then*

$$\sum_{n \leqslant x} \frac{\sigma_s^*(n)}{\phi_s^*(n)} = C_s^* x + O(S_s(x)),$$

where

$$C_s^* = \prod_p \left(1 + 2\left(1 - \frac{1}{p}\right) \sum_{j=1}^{\infty} \frac{1}{p^j (p^{js} - 1)} \right)$$

and $S_s(x)$ is defined in Theorem 2.

4 - On the quotient $\phi_{A,s}(n)/\sigma_{A,s}(n)$.

Theorem 3. *If A is a cross-convolution and $s \geqslant 1$, then*

$$\sum_{n \leqslant x} \frac{\phi_{A,s}(n)}{\sigma_{A,s}(n)} = D_{A,s}\, x + O(S_s(x)), \tag{7}$$

where

$$D_{A,s} = \prod_{p \in P} \left(1 - \frac{2}{p(p^s+1)} - \frac{(p^s-1)^3}{p^{2s}} \sum_{j=2}^{\infty} \frac{p^{j(s-1)}}{(p^{js}-1)(p^{(j+1)s}-1)}\right)$$

$$\times \prod_{p \in Q} \left(1 - 2\left(1 - \frac{1}{p}\right) \sum_{j=1}^{\infty} \frac{1}{p^j(p^{js}+1)}\right)$$

and $S_s(x)$ is given in Theorem 2.

Proof. Define the function $h_{A,s}$ by

$$\frac{\phi_{A,s}(n)}{\sigma_{A,s}(n)} = \sum_{d \in A(n)} h_{A,s}(d)$$

for every $n \in \mathbb{N}$. We obtain that $h_{A,s}$ is multiplicative,

$$h_{A,s}(p^a) = \begin{cases} -\dfrac{p^{(a-2)s}(p^s-1)^3}{(p^{(a+1)s}-1)(p^{as}-1)}, & \text{if } p \in P,\ a \geqslant 2, \\ -\dfrac{2}{p^{as}+1}, & \text{if } p \in P,\ a = 1 \text{ or } p \in Q,\ a \in \mathbb{N}, \end{cases}$$

for every prime power p^a and the proof is like that of Theorem 2, see also [**9**], Theorem 3.

Corollary 5. *For $s \geqslant 1$ the mean value of $\phi_{A,s}(n)/\sigma_{A,s}(n)$ is $D_{A,s}$.*

Corollary 6 ($A = U$). *If $s \geqslant 1$, then*

$$\sum_{n \leqslant x} \frac{\phi_s^*(n)}{\sigma_s^*(n)} = D_s^*\, x + O(S_s(x)),$$

where

$$D_s^* = \prod_p \left(1 - 2\left(1 - \frac{1}{p}\right) \sum_{j=1}^{\infty} \frac{1}{p^j(p^{js}+1)}\right)$$

and $S_s(x)$ *is given in Theorem* 2.

References

[1] E. Cohen, *Arithmetical functions associated with the unitary divisors of an integer*, Math. Z. **74** (1960), 66-80.

[2] P. Haukkanen, *On an inequality for* $\sigma(n)\,\phi(n)$, Octogon Math. Mag. **4** (1996), 3-5.

[3] P. J. McCarthy, *Introduction to arithmetical functions*, Springer Verlag, New York-Berlin-Heidelberg-Tokyo 1986.

[4] K. Nageswara Rao, *On the unitary analogues of certain totients*, Monatsh. Math. **70** (1966), 149-154.

[5] W. Narkiewicz, *On a class of arithmetical convolutions*, Colloq. Math. **10** (1963), 81-94.

[6] V. Sita Ramaiah, *Arithmetical sums in regular convolutions*, J. Reine Angew. Math. **303/304**(1978), 265-283.

[7] L. Tóth, *The unitary analogue of Pillai's arithmetical function*, Collect. Math. **40** (1989), 19-30.

[8] L. Tóth, *Asymptotic formulae concerning arithmetical functions defined by cross-convolutions, I. Divisor-sum functions and Euler-type functions*, Publ. Math. Debrecen **50** (1997), 159-176.

[9] L. Tóth, *Asymptotic formulae concerning the product and the quotient of the arithmetical functions* σ_s *and* ϕ_s, Tatra Mountains Math. Publ. **11** (1997), 167-175.

Abstract

Let A be a regular convolution of Narkiewicz type, $\sigma_{A,s}(n)$ *denote the sum of s-th powers of A-divisors of n and let* $\phi_{A,s}(n) = \sum_{d \in A(n)} d^s \mu_A(n/d)$ *be the generalized Euler function. In this paper we establish asymptotic formulae for* $\sum_{n \leq x} \sigma_{A,s}(n)\phi_{A,s}(n)$, $\sum_{n \leq x} \sigma_{A,s}(n)/\phi_{A,s}(n)$ *in case* $s > 0$ *and for* $\sum_{n \leq x} \phi_{A,s}(n)/\sigma_{A,s}(n)$ *in case* $s \geq 1$, *assuming that A is a cross-convolution, investigated in our previous papers.*

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