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Existence of weak solutions for the equations of the electrorheological fluids (**)

1 - Introduction

An electrorheological fluid consists of a suspension of fine dielectric particles in a liquid of low dielectric constant [**6**], [**2**]. Its peculiar property is a dramatic increasing of the viscosity in presence of an electric field. Moreover, when the intensity of the applied field exceeds a critical value, the fluid behaves as a rigid body. The phenomenon is reversible. The transition's time-scale and the intensity of the involved electric currents are respectively of a few milliseconds and microampères. These properties make electrorheological fluids of potential use in industry, especially in the automotive and aerospace sectors.

In this paper we model electrorheological fluids as electrically controlled Bingham fluids. The Einstein's convention on repeated indices is adopted, while an index preceded by a comma denotes the derivative with respect to the corresponding variable. We recall that a Bingham fluid is a visco-plastic material governed by the constitutive equations (see [4], Chapter VI)

$$\mathbf{D} = \begin{cases} 0 & \text{if } \sigma_{\mathrm{II}}^{1/2} < g \\ \dfrac{1}{2\mu}\left(1 - \dfrac{g}{\sigma_{\mathrm{II}}^{1/2}}\right)\boldsymbol{\sigma}^D & \text{if } \sigma_{\mathrm{II}}^{1/2} \geq g\,; \end{cases}$$

here $\mathbf{D}$ is the tensor of the strain velocity, $\boldsymbol{\sigma}^D$ is the deviation of the stress tensor,

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$\sigma_{\rm II} := \dfrac{1}{2}\sigma^D_{ij}\sigma^D_{ij}$ is the second invariant of $\boldsymbol{\sigma}^D$, and the positive constants μ and g are respectively the viscosity of the Bingham fluid and its threshold of plasticity. Thus, according to the magnitude of the function $\sigma^{1/2}_{\rm II}$, we may observe either a classical viscous fluid, or a rigid medium. To take into account of the influence of the electric field E we define a yield limit g as non-negative, continuous increasing function of $|\boldsymbol{E}|$, vanishing when $|\boldsymbol{E}|$ does not exceed a critical value $|\boldsymbol{E}|_c$, and assuming a constant value for $|\boldsymbol{E}|$ large enough (see Figure 1.1 below).

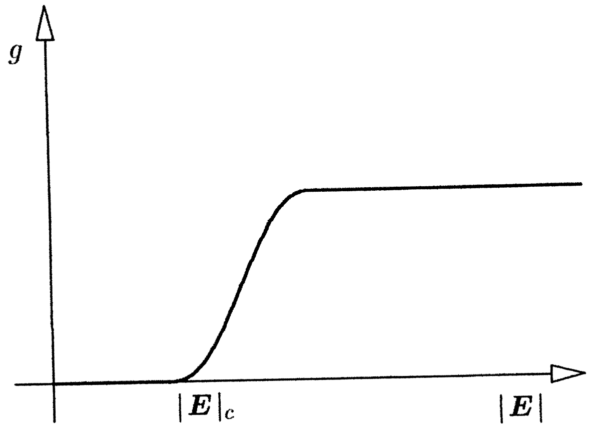

Figure 1.1. The function $g = g(|\boldsymbol{E}|)$.

If $|\boldsymbol{E}| < |\boldsymbol{E}|_c$, then $g(|\boldsymbol{E}|) = 0$ and we have a Newtonian fluid (see [3]). If $|\boldsymbol{E}| \geqslant |\boldsymbol{E}|_c$, the behaviour of the fluid depends on the stress tensor: if the second invariant $\sigma_{\rm II}$ satisfies $\sigma^{1/2}_{\rm II} < g(|\boldsymbol{E}|)$, the system behaves as a rigid medium; if $\sigma^{1/2}_{\rm II} \geqslant g(|\boldsymbol{E}|)$, we have a fluid whose viscosity is an increasing function of $|\boldsymbol{E}|$.

Thus we have

$$\mathbf{D} = \begin{cases} 0 & \text{if } \sigma^{1/2}_{\rm II} < g(|\boldsymbol{E}|) \\ \dfrac{1}{2\mu(|\boldsymbol{E}|)}\left(1 - \dfrac{g(|\boldsymbol{E}|)}{\sigma^{1/2}_{\rm II}}\right)\boldsymbol{\sigma}^D & \text{if } \sigma^{1/2}_{\rm II} \geqslant g(|\boldsymbol{E}|), \end{cases} \tag{1.1}$$

where the dynamic viscosity μ is also an increasing and continuous function of $|\boldsymbol{E}|$, which satisfies

$$0 < \mu_0 \leqslant \mu(\xi) \leqslant \mu_1 < +\infty\,, \qquad \forall \xi \in \mathbb{R}^+ .$$

We recall that the symmetric tensor $\mathbf{D}$ represents the rate of deformation: if $\boldsymbol{u}$ is the velocity field we have

$$D_{ij} := \frac{1}{2}(\boldsymbol{u}_{i,j} + \boldsymbol{u}j,\, i).$$

Let D_{II} denote the second invariant of $\mathbf{D}$:

$$D_{\mathrm{II}} := \frac{1}{2} D_{ij} D_{ij}.$$

It follows from (1.1) that, when $\sigma_{\mathrm{II}}^{1/2} \geq g(|\boldsymbol{E}|)$, the second invariants of $\boldsymbol{\sigma}^D$ and $\mathbf{D}$ are related by the equation

$$\sigma_{\mathrm{II}}^{1/2} = 2\mu(|\boldsymbol{E}|)\, D_{\mathrm{II}}^{1/2} + g(|\boldsymbol{E}|).$$

If we decompose the stress tensor into a spherical and a deviatoric part, we have

$$\sigma_{ij} = -p\delta_{ij} + \sigma_{ij}^D, \tag{1.2}$$

where p is the pressure of the fluid, and by (1.1)

$$\boldsymbol{\sigma}^D = \begin{cases} \text{indeterminate} & \text{if } \sigma_{\mathrm{II}}^{1/2} \leq g(|\boldsymbol{E}|) \\ \left(2\mu(|\boldsymbol{E}|) + \dfrac{g(|\boldsymbol{E}|)}{D_{\mathrm{II}}^{1/2}}\right)\mathbf{D} & \text{if } \sigma_{\mathrm{II}}^{1/2} > g(|\boldsymbol{E}|). \end{cases} \tag{1.3}$$

Since electric currents are very small, all magnetic effects are neglected and the electric field derives from a potential Φ, hence $\boldsymbol{E} = -\nabla\Phi$. The unknown functions describing the behaviour of the system are then the electric potential Φ and the velocity field $\boldsymbol{u}$.

We recall that the Poisson equation

$$-\varepsilon\Delta\Phi = q \tag{1.4}$$

relates the potential Φ to the charge density q, being ε the dielectric constant. Additionally, we assume that the current density is given by

$$\boldsymbol{J} = -k\nabla q + q\boldsymbol{u} + \sigma\boldsymbol{E}, \tag{1.5}$$

where the positive constants k and σ are respectively the diffusion coefficient and the electric conductivity. Thus, taking into account (1.4) and (1.5), the *conservation of charge*

$$q_t + \nabla \cdot \boldsymbol{J} = 0 ,$$

yields

$$\Delta \Phi_t - k\Delta^2 \Phi + \nabla \cdot (\boldsymbol{u} \Delta \Phi) + \varepsilon^{-1\nabla} \cdot (\sigma \nabla \Phi) = 0 . \tag{1.6}$$

The fluid is incompressible, therefore

$$\nabla \cdot \boldsymbol{u} = 0 . \tag{1.7}$$

Finally, we assume that only the electric body force $\boldsymbol{f} := q\boldsymbol{E}$ is acting on the fluid. Therefore, by the conservation of momentum, the *law of motion* is

$$\sigma_{ij,j} + qE_i = \varrho \frac{Du_i}{Dt} , \tag{1.8}$$

where $\dfrac{Du_i}{Dt} := (u_i)_t + \nabla u_i \cdot \boldsymbol{u}$.

Goal of the paper is to give a weak formulation and an existence result for the following initial-boundary value problem.

Problem $(\mathcal{P})$. Let Ω be a bounded open subset of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Find Φ and $\boldsymbol{u}$ satisfying equations (1.3), (1.6), (1.7), (1.8) on $\Omega_T := \Omega \times (0, T)$, and the initial-boundary data

$$\Delta \Phi(x, 0) = 0 , \qquad \boldsymbol{u}(x, 0) = \boldsymbol{u}^0(x) \quad \text{on } \Omega ;$$

$$\Phi(x, t) = \Phi_b(x) , \qquad \Delta \Phi(x, t) = 0 , \qquad \boldsymbol{u}(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T) .$$

The remaining of the paper is organized as follows. In Section **2** we give a weak formulation of the initial boundary problem $(\mathcal{P})$ by using Sobolev spaces. Section **3** contains our main existence result. The proof in divided into four steps: in subsection **3.1** we define two approximating sequences of problems $(\mathcal{P}_\varepsilon)$ and $(\mathcal{P}_{\varepsilon m})$; in subsections **3.2** and **3.3** we show respectively the existence of a weak solution to problems $(\mathcal{P}_{\varepsilon m})$ and $(\mathcal{P}_\varepsilon)$; finally, the weak existence theorem for $(\mathcal{P})$ is proved in subsection **3.4**.

2 - Weak formulation of the problem

In this section we present the weak formulation of problem $(\mathcal{P})$ which we adopt. We shall use as basic tool the Sobolev spaces [**1**]. We denote by $(\cdot, \cdot)$, $((\cdot, \cdot))$, and $|\cdot|$, $\|\cdot\|$ respectively the scalar product and the norm in $L^2(\Omega)$ and in

$H_0^1(\Omega)$, as well as in $(L^2(\Omega))^2 := L^2(\Omega;\mathbb{R}^2)$ and in $(H_0^1(\Omega))^2 := H_0^1(\Omega;\mathbb{R}^2)$.

We start with the weak formulation of (1.6).To work with homogeneous boundary conditions, we introduce the solution $\overline{\Phi}$ to the problem

$$\begin{cases} \Delta\overline{\Phi} = 0 & \text{in } \Omega \\ \overline{\Phi} = \Phi_b & \text{on } \partial\Omega\,, \end{cases}$$

where we assume that Φ_b is the trace on $\partial\Omega$ of a function in the Sobolev space $H^{2,p}(\Omega)$, $p>2$. We set $\varphi := \Phi - \overline{\Phi}$. Looking for φ with

$$\varphi \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))\cap L^2(0,T;H^3(\Omega)), \quad \Delta\varphi' \in L^2(0,T;H^{-1}(\Omega)), \tag{2.1}$$

$$\Delta\varphi = 0 \;\text{ on }\; \partial\Omega\times(0,T), \qquad \Delta\varphi(x,0) = 0 \;\text{ on }\; \Omega\,, \tag{2.2}$$

the weak formulation of equation (1.6) reads

$$\begin{aligned}(\Delta\varphi',\xi) - k(\Delta\varphi,\Delta\xi) - (\boldsymbol{u}\Delta\varphi,\nabla\xi) - \varepsilon^{-1}\sigma(\nabla\varphi,\nabla\xi)\\ -\varepsilon^{-1}\sigma(\nabla\overline{\Phi},\nabla\xi) = 0 \qquad \forall\xi\in H_0^1(\Omega)\cap H^2(\Omega).\end{aligned} \tag{2.3}$$

We turn now to the weak formulation of (1.7) and (1.8).

Let $\mathcal{V} := \left\{\boldsymbol{\psi}\in(C_0^\infty(\Omega))^2\colon \nabla\cdot\boldsymbol{\psi} = 0\right\}$, and let H, V be the closures of $\mathcal{V}$ in $(L^2(\Omega))^2$ and in $(H_0^1(\Omega))^2$. For functions $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ defined on Ω such that the following integrals exist, we set

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) := \int_\Omega u_i v_{j,i} w_j\,\mathrm{d}x\,,$$

$$a(\boldsymbol{u},\boldsymbol{v},\boldsymbol{E}) := \int_\Omega 2\mu(|\boldsymbol{E}|)\,D_{ij}(\boldsymbol{u})\,D_{ij}(\boldsymbol{v})\,\mathrm{d}x\,,$$

$$j(\boldsymbol{u},\boldsymbol{E}) := \int_\Omega 2g(|\boldsymbol{E}|)\,(D_{\mathrm{II}}(\boldsymbol{u}))^{1/2}\,\mathrm{d}x\,. \tag{2.4}$$

It follows from the above definitions that, for any $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V$, there holds

$$\begin{aligned}&b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -\,b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}), && b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}) = 0\,,\\ &a(\boldsymbol{u},\boldsymbol{u},\boldsymbol{E}) \geqslant \mu_0\|\boldsymbol{u}\|^2, && a(\boldsymbol{u},\boldsymbol{v},\boldsymbol{E}) \leqslant 2\mu_1\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\,.\end{aligned} \tag{2.5}$$

Furthermore, we recall the inequality (see [7])

$$|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leqslant \sqrt{2}\,|\boldsymbol{u}|^{1/2}\|\boldsymbol{u}\|^{1/2}\|\boldsymbol{v}\|\,|\boldsymbol{w}|^{1/2}\|\boldsymbol{w}\|^{1/2}. \tag{2.6}$$

Take now a test vector field $\boldsymbol{v}$ in $\mathcal{V}$, multiply equation (1.8) by $u_i - v_i$ and integrate on Ω. Recalling that $\sigma_{ij} = \sigma_{ji}$, one gets

$$\int_\Omega \sigma_{ij}(\boldsymbol{u})\, D_{ij}(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}x = \int_\Omega qE_i(v_i - u_i)\,\mathrm{d}x - \int_\Omega \varrho(u_i' + u_{i,j}u_j)(v_i - u_i)\,\mathrm{d}x\,. \tag{2.7}$$

We can also multiply by $u_{i,j} - v_{i,j}$ the equation satisfied by $\boldsymbol{\sigma}$ for $\sigma_{\mathrm{II}}^{1/2} > g(|\boldsymbol{E}|)$ (see (1.2) and (1.3)). It results, summing over i and j, and using $D_{ii} = 0$,

$$\begin{aligned}\int_\Omega \sigma_{ij}(\boldsymbol{u})\, D_{ij}(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}x &= \int_\Omega g(|\boldsymbol{E}|)\,(D_{\mathrm{II}}(\boldsymbol{u}))^{-1/2} D_{ij}(\boldsymbol{u})\, D_{ij}(\boldsymbol{v})\,\mathrm{d}x \\ &- \int_\Omega g(|\boldsymbol{E}|)\,(D_{\mathrm{II}}(\boldsymbol{u}))^{-1/2} D_{ij}(\boldsymbol{u})\, D_{ij}(\boldsymbol{u})\,\mathrm{d}x + a(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}, \boldsymbol{E})\,.\end{aligned} \tag{2.8}$$

Recalling (1.7), assume now that $\boldsymbol{u}$ satisfies

$$\boldsymbol{u} \in L^\infty(0, T; H) \cap L^2(0, T; V), \qquad \boldsymbol{u}' \in L^2(0, T; V')\,, \tag{2.9}$$

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}^0(x) \;\text{ on }\; \Omega\,; \tag{2.10}$$

here and in the following, we let $\boldsymbol{u}^0 \in H$.

If we couple equations (2.7) and (2.8), using the definition of the functional $j(\boldsymbol{u}, \boldsymbol{E})$, the Schwarz inequality, and the identity $b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) = 0$, we get the following weak formulation for (1.8)

$$\begin{aligned}&(\boldsymbol{u}', \boldsymbol{v}-\boldsymbol{u}) - b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}) + a(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}, \boldsymbol{E}) + j(\boldsymbol{v}, \boldsymbol{E}) - j(\boldsymbol{u}, \boldsymbol{E}) \\ &\qquad \geqslant \varepsilon(\Delta\varphi\nabla\varphi, \boldsymbol{v}-\boldsymbol{u}) + \varepsilon(\Delta\varphi\nabla\overline{\Phi}, \boldsymbol{v}-\boldsymbol{u}) \qquad \forall \boldsymbol{v} \in V\,,\end{aligned} \tag{2.11}$$

where we set $\varrho = 1$ for the constant density of the fluid.

Summing up, we can reformulate problem $(\mathcal{P})$, having introduced suitable non-dimensional constants a_j, $j = 1, 2, 3$.

Problem $(\mathcal{P})$, weak formulation. Find φ and $\boldsymbol{u}$, with

$$\varphi \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^2(0, T; H^3(\Omega)), \qquad \Delta\varphi' \in L^2(0, T; H^{-1}(\Omega))\,, \tag{2.12}$$

$$\Delta\varphi = 0 \;\text{ on }\; \partial\Omega \times (0, T), \qquad \Delta\varphi(x, 0) = 0 \;\text{ on }\; \Omega \tag{2.13}$$

$$\boldsymbol{u} \in L^\infty(0, T; H) \cap L^2(0, T; V), \qquad \boldsymbol{u}' \in L^2(0, T; V')\,, \tag{2.14}$$

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}^0(x) \;\text{ on }\; \Omega\,, \tag{2.15}$$

satisfying on Ω_T the equation

$$\begin{aligned}(2.16)\qquad &(\Delta\varphi', \xi) - a_2(\Delta\varphi, \Delta\xi) - (\boldsymbol{u}\Delta\varphi, \nabla\xi) - a_3(\nabla\varphi, \nabla\xi)\\ &\qquad - a_3(\nabla\overline{\Phi}, \nabla\xi) = 0 \qquad \forall\xi \in H_0^1(\Omega) \cap H^2(\Omega),\end{aligned}$$

and the variational inequality

$$\begin{aligned}(2.17)\qquad &(\boldsymbol{u}', \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}) + a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}, \boldsymbol{E}) + j(\boldsymbol{v}, \boldsymbol{E}) - j(\boldsymbol{u}, \boldsymbol{E})\\ &\qquad \geqslant (\Delta\varphi\nabla\varphi, \boldsymbol{v} - \boldsymbol{u}) + (\Delta\varphi\nabla\overline{\Phi}, \boldsymbol{v} - \boldsymbol{u}) \qquad \forall \boldsymbol{v} \in V.\end{aligned}$$

3 - Existence of weak solutions

In this section we show that there exists a solution to problem $(\mathcal{P})$. The proof consists of a double approximation. We construct a sequence of problems $(\mathcal{P}_\varepsilon)_{\varepsilon > 0}$ in which the inequality (2.17) is replaced by a sequence of equations. Then, we approximate each problem $(\mathcal{P}_\varepsilon)$ by a sequence of finite-dimensional problems $(\mathcal{P}_{\varepsilon m})_{m\in\mathbb{N}}$. Standard results in the linear theory will give the existence of a solution to $(\mathcal{P}_{\varepsilon m})$ for any $\varepsilon > 0$ and $m \in \mathbb{N}$. Then, passing to the limit as $m \to +\infty$ and $\varepsilon \to 0^+$, we shall find a solution to the limit problem $(\mathcal{P})$.

3.1 - *Definition of the approximating problems*

We replace the non-differentiable functional j defined in (2.4) and appearing in (2.17), by a sequence of differentiable functionals j_ε. For every $\varepsilon > 0$, and for any function $\boldsymbol{u}$ on Ω such that the integral makes sense, we set

$$j_\varepsilon(\boldsymbol{u}, \boldsymbol{E}) = \frac{2}{1+\varepsilon}\int_\Omega g(|\boldsymbol{E}|)\, D_{\mathrm{II}}(\boldsymbol{u})^{1+\varepsilon/2}\,\mathrm{d}x.$$

The Fréchet differential of j_ε on V exists and is given by

$$(3.1)\qquad (j_\varepsilon'(\boldsymbol{u}, \boldsymbol{E}), \boldsymbol{v}) = \int_\Omega g(|\boldsymbol{E}|)\, D_{\mathrm{II}}(\boldsymbol{u})^{\varepsilon - 1/2} D_{ij}(\boldsymbol{u})\, D_{ij}(\boldsymbol{v})\,\mathrm{d}x.$$

We then approximate inequality (2.17) with a sequence of equations, considering the following problem, where we set $\boldsymbol{E}_\varepsilon = -\nabla(\varphi_\varepsilon + \overline{\Phi})$.

Problem $(\mathcal{P}_\varepsilon)$. Find $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)$ solutions to (2.12)-(2.15), and to the following system of PDE's

$$(3.2)\qquad \begin{aligned}&(\Delta\varphi'_\varepsilon, \xi) - a_2(\Delta\varphi_\varepsilon, \Delta\xi) - (\boldsymbol{u}_\varepsilon\Delta\varphi_\varepsilon, \nabla\xi) - a_3(\nabla\varphi_\varepsilon, \nabla\xi)\\ &\qquad - a_3(\nabla\overline{\Phi}, \nabla\xi) = 0 \qquad \forall\xi \in H_0^1(\Omega)\cap H^2(\Omega),\end{aligned}$$

$$(3.3)\qquad \begin{aligned}&(\boldsymbol{u}'_\varepsilon, \boldsymbol{v}) - b(\boldsymbol{u}_\varepsilon, \boldsymbol{v}, \boldsymbol{u}_\varepsilon) + a(\boldsymbol{u}_\varepsilon, \boldsymbol{v}, \boldsymbol{E}_\varepsilon) + (j'_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon), \boldsymbol{v})\\ &\qquad = (\Delta\varphi_\varepsilon\nabla\varphi_\varepsilon, \boldsymbol{v}) + (\Delta\varphi_\varepsilon\nabla\overline{\Phi}, \boldsymbol{v}) \qquad \forall\boldsymbol{v}\in V.\end{aligned}$$

Each problem $(\mathcal{P}_\varepsilon)$ is approximated by a sequence of finite-dimensional problems, using the Faedo-Galerkin method. Let $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}}$ be the base of V given by the normalized eigenfunctions of the canonical isomorphism $\Lambda: V\to V'$, i.e.

$$(\nabla\boldsymbol{w}_i, \nabla\boldsymbol{v}) = \lambda_i(\boldsymbol{w}_i, \boldsymbol{v}) \qquad \forall\boldsymbol{v}\in V, \qquad |\boldsymbol{w}_i| = 1.$$

We set $V_m := \mathrm{span}\{\boldsymbol{w}\, i,\ i = 1, \ldots, m\}$, $m\in\mathbb{N}$.

Problem $(\mathcal{P}_{\varepsilon m})$. Find $(\varphi_{\varepsilon m}, \boldsymbol{u}_{\varepsilon m})$ solutions to (2.12)-(2.15) and to the system

$$(3.4)\qquad \begin{aligned}&(\Delta\varphi'_{\varepsilon m}, \xi) - a_2(\Delta\varphi_{\varepsilon m}, \Delta\xi) - (\boldsymbol{u}_{\varepsilon m}\Delta\varphi_{\varepsilon m}, \nabla\xi) - a_3(\nabla\varphi_{\varepsilon m}, \nabla\xi)\\ &\qquad - a_3(\nabla\overline{\Phi}, \nabla\xi) = 0 \qquad \forall\xi\in H_0^1(\Omega)\cap H^2(\Omega).\end{aligned}$$

$$(3.5)\qquad \begin{aligned}&(\boldsymbol{u}'_{\varepsilon m}, \boldsymbol{w}_i) - b(\boldsymbol{u}_{\varepsilon m}, \boldsymbol{w}_i, \boldsymbol{u}_{\varepsilon m}) + a(\boldsymbol{u}_{\varepsilon m}, \boldsymbol{w}_i, \boldsymbol{E}_{\varepsilon m}) + (j'_\varepsilon(\boldsymbol{u}_{\varepsilon m}, \boldsymbol{E}_{\varepsilon m}), \boldsymbol{w}_i)\\ &\qquad = (\Delta\varphi_{\varepsilon m}\nabla\varphi_{\varepsilon m}, \boldsymbol{w}_i) + (\Delta\varphi_{\varepsilon m}\nabla\overline{\Phi}, \boldsymbol{w}_i) \qquad \forall i = 1, \ldots, m.\end{aligned}$$

3.2 - *Existence of a weak solution to* $(\mathcal{P}_{\varepsilon m})$

For any given function $\boldsymbol{v}_m\in V_m$, consider the initial-boundary value problem in the unknown φ given by (2.12), (2.13) and by the equation

$$(3.6)\qquad \begin{aligned}&(\Delta\varphi', \xi) - a_2(\Delta\varphi, \Delta\xi) - (\boldsymbol{v}_m\Delta\varphi, \nabla\xi) - a_3(\nabla\varphi, \nabla\xi)\\ &\qquad - a_3(\nabla\overline{\Phi}, \nabla\xi) = 0 \qquad \forall\xi\in H_0^1(\Omega)\cap H^2(\Omega).\end{aligned}$$

Such problem admits a unique solution, which will be denoted by $\tilde{\varphi}(\boldsymbol{v}_m)$. Indeed, notice that the parabolic problem

$$\begin{cases} q' - a_2\Delta q + \boldsymbol{v}_m\cdot\nabla q - a_3 q = 0 \qquad \text{on } \Omega_T\\ q = 0 \ \text{ on } \partial\Omega\times(0, T), \qquad q(x, 0) = 0 \ \text{ on } \Omega\end{cases}$$

admits an unique solution $q := \tilde{q}(\boldsymbol{v}_m)\in L^\infty(0, T; L^2(\Omega))\cap L^2(0, T; H^1(\Omega))$.

So $\tilde{\varphi}(\boldsymbol{v}_m)$ will be the unique function φ satisfying (2.12), (2.13), and solving $-\Delta\varphi = \tilde{q}(\boldsymbol{v}_m)$ on Ω_T.

Let us now look for a solution $(\varphi_{\varepsilon m}, \boldsymbol{u}_{\varepsilon m})$ to $(\mathcal{P}_{\varepsilon m})$ with $\boldsymbol{u}_{\varepsilon m} \in C^1(0, T; V_m)$, that is $\boldsymbol{u}_{\varepsilon m}(x, t) = \sum_{i=1}^{m} g_{\varepsilon m}^i(t)\, \boldsymbol{w}_i(x)$, with $g_{\varepsilon m}^i \in C^1(0, T; \mathbb{R})$ for $i = 1, \ldots, m$. This is equivalent to find $\boldsymbol{u}_{\varepsilon m} \in C^1(0, T; V_m)$ satisfying (2.14), (2.15), and

$$
\begin{aligned}
(3.7) \qquad & (\boldsymbol{u}'_{\varepsilon m}, \boldsymbol{w}_i) - b(\boldsymbol{u}_{\varepsilon m}, \boldsymbol{w}_i, \boldsymbol{u}_{\varepsilon m}) + a(\boldsymbol{u}_{\varepsilon m}, \boldsymbol{w}_i, \boldsymbol{E}_{\varepsilon m}) + (j'_\varepsilon(\boldsymbol{u}_{\varepsilon m}, \boldsymbol{E}_{\varepsilon m}), \boldsymbol{w}_i) \\
& = (\Delta\varphi_{\varepsilon m} \nabla\varphi_{\varepsilon m}, \boldsymbol{w}_i) + (\Delta\varphi_{\varepsilon m} \nabla\overline{\Phi}, \boldsymbol{w}_i) \qquad \forall i = 1, \ldots, m\,,
\end{aligned}
$$

where $\varphi_{\varepsilon m}$ and $\boldsymbol{E}_{\varepsilon m}$ equal respectively $\tilde{\varphi}(\boldsymbol{u}_{\varepsilon m})$ and $-\nabla[\tilde{\varphi}(\boldsymbol{u}_{\varepsilon m}) + \overline{\Phi}]$.

Then, the existence of a solution $\boldsymbol{u}_{\varepsilon m}$ to (3.7) follows from the observation that, writing $\boldsymbol{u}_{\varepsilon m}$ as $\boldsymbol{u}_{\varepsilon m}(x, t) = \sum_{i=1}^{m} g_{\varepsilon m}^i(t)\, \boldsymbol{w}_i(x)$, (3.7) becomes a first order ODE's system in the unknown $g_{\varepsilon m}^i(t)$, $i = 1, \ldots, n$, while the initial condition (2.15) can be reformulated as

$$
(3.8) \qquad g_{\varepsilon m}^i(0) = g_0^i, \qquad i = 1, \ldots, m\,,
$$

being $\{g_0^i\}_{i\in\mathbb{N}}$ a sequence such that $\lim_{m\to+\infty} \left\| \sum_{i=1}^{m} g_0^i \boldsymbol{w}_i - \boldsymbol{u}^0 \right\|_V = 0$.

3.3 - *Existence of a weak solution to* $(\mathcal{P}_\varepsilon)$

In the following we shall denote for simplicity by $(\varphi_m, \boldsymbol{u}_m)$ a solution to problem $(\mathcal{P}_{\varepsilon m})$. In the next lemma, we collect some a priori-estimates on $(\varphi_m, \boldsymbol{u}_m)$.

Lemma 3.1. *The following sequences are bounded in the corresponding functional spaces*:

$$
(3.9) \qquad \boldsymbol{u}_m \ \textit{in}\ L^\infty(0, T; H) \cap L^2(0, T; V),
$$

$$
(3.10) \qquad \boldsymbol{u}'_m \ \textit{in}\ L^2(0, T; V'),
$$

$$
(3.11) \qquad \Delta\varphi_m \ \textit{in}\ L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)),
$$

$$
(3.12) \qquad \Delta\varphi'_m \ \textit{in}\ L^2(0, T; H^{-1}(\Omega)),
$$

$$
(3.13) \qquad \nabla\varphi_m \ \textit{in}\ L^\infty(0, T; H^1(\Omega)).
$$

Proof. Taking $\boldsymbol{u}_m$ and φ_m as test functions in (3.5) and (3.4) respectively, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}_m(t)|^2+a(\boldsymbol{u}_m,\boldsymbol{u}_m,\boldsymbol{E}_m)+(j_\varepsilon'(\boldsymbol{u}_m,\boldsymbol{E}_m),\boldsymbol{u}_m) = (\Delta\varphi_m\nabla\varphi_m,\boldsymbol{u}_m)+(\Delta\varphi_m\nabla\overline{\Phi},\boldsymbol{u}_m) \tag{3.14}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\varphi_m(t)|^2+a_2|\Delta\varphi_m(t)|^2+(\boldsymbol{u}_m\Delta\varphi_m,\nabla\varphi_m) +a_3(\nabla\varphi_m,\nabla\varphi_m)+a_3(\nabla\overline{\Phi},\nabla\varphi_m)=0\,. \tag{3.15}$$

We add now (3.14) and (3.15), taking into account that $a(\boldsymbol{u}_m,\boldsymbol{u}_m,\boldsymbol{E}_m) \geqslant \mu_0\|\boldsymbol{u}_m(t)\|^2$ by $(2.5)_1$, and $(j_\varepsilon'(\boldsymbol{u}_m,\boldsymbol{E}_m),\boldsymbol{u}_m)\geqslant 0$ by (3.1). Using the inequality

$$ab\leqslant\frac{a^2}{2\delta}+\frac{\delta}{2}b^2, \tag{3.16}$$

to estimate the terms $(\Delta\varphi_m\nabla\overline{\Phi},\boldsymbol{u}_m)$ and $a_3(\nabla\overline{\Phi},\nabla\varphi_m)$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[|\boldsymbol{u}_m(t)|^2+|\nabla\varphi_m(t)|^2]+\mu_0\|\boldsymbol{u}_m(t)\|^2+a_2|\Delta\varphi_m(t)|^2+a_3|\nabla\varphi_m(t)|^2\leqslant$$

$$\Big(\sup_{\overline{\Omega_T}}|\nabla\overline{\Phi}|\Big)\Big[\frac{\delta}{2}|\Delta\varphi_m(t)|^2+\frac{1}{2\delta}|\boldsymbol{u}_m(t)|^2\Big]+a_3\Big[\frac{\delta'}{2}|\nabla\varphi_m(t)|^2+\frac{1}{2\delta'}|\nabla\overline{\Phi}|^2\Big].$$

If we choose δ such that $a_2-\dfrac{\delta}{2}\sup_{\overline{\Omega_T}}|\nabla\overline{\Phi}|\geqslant 1$, and $\delta'=3$, we infer

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[|\boldsymbol{u}_m(t)|^2+|\nabla\varphi_m(t)|^2]+\mu_0\|\boldsymbol{u}_m(t)\|^2+|\Delta\varphi_m(t)|^2$$

$$\leqslant C_1[|\boldsymbol{u}_m(t)|^2+|\nabla\varphi_m(t)|^2+|\nabla\overline{\Phi}|^2],$$

with $C_1:=\max\Big\{\dfrac{1}{2\delta}\sup_{\overline{\Omega_T}}|\nabla\overline{\Phi}|,\ \dfrac{a_3}{2}\Big\}$. By the Gronwall inequality, it follows that $\boldsymbol{u}_m$ remains bounded in $L^\infty(0,T;H)\cap L^2(0,T;V)$, and we have proved (3.9). Additionally, we have proved that

$$\nabla\varphi_m \text{ remains in a bounded subset of } L^\infty(0,T;L^2(\Omega)), \tag{3.17}$$

Let us take $\xi = \Delta\varphi_m$ in (3.4). Noticing that $(\boldsymbol{u}_m \Delta\varphi_{\varepsilon m}, \nabla\Delta\varphi_{\varepsilon m}) = 0$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\varphi_{\varepsilon m}(t)|^2 + a_2|\nabla\Delta\varphi_{\varepsilon m}(t)|^2 - a_3(\nabla(\varphi_{\varepsilon m} + \overline{\Phi}), \nabla\Delta\varphi_{\varepsilon m}) = 0 .$$

Using the elementary inequality (3.16), we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\varphi_{\varepsilon m}(t)|^2 + |\nabla\Delta\varphi_{\varepsilon m}(t)|^2 \leqslant \frac{a_3}{\alpha}\left[\frac{\delta}{2}|\nabla\Delta\varphi_{\varepsilon m}(t)|^2 + \frac{1}{\delta}(|\nabla\varphi_{\varepsilon m}(t)|^2 + |\nabla\overline{\Phi}|^2)\right],$$

with $\alpha := \min\{1, a_2\}$. If we choose $\delta > 0$ such that $1 - \frac{a_3\delta}{2\alpha} \geqslant \frac{1}{2}$ and we set $C_2 := \frac{2a_3}{\alpha\delta}$, it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\varphi_{\varepsilon m}(t)|^2 + |\nabla\Delta\varphi_{\varepsilon m}(t)|^2 \leqslant C_2(|\nabla\varphi_{\varepsilon m}(t)|^2 + |\nabla\overline{\Phi}|^2). \tag{3.18}$$

By (3.17) and (2.12), this implies (3.11) and (3.13).

Let us write (3.4) as $\Delta\varphi'_m = \nabla\cdot F_m$, where

$$F_m = a_2\nabla(\Delta\varphi_m) - \boldsymbol{u}_m\Delta\varphi_m - a_3\nabla(\varphi_m + \overline{\Phi}).$$

Then (3.9), (3.11) and (3.13) give (3.12).

It only remains to prove (3.10). To this aim we write (3.5) as

$$(\boldsymbol{u}'_m, \boldsymbol{v}) + \langle B_m + A_m + K_m - H_m, \boldsymbol{v}\rangle_{(V', V)} = 0, \qquad \forall \boldsymbol{v} \in V_m, \tag{3.19}$$

where the functionals B_m, A_m, K_m and H_m are defined on V as follows.

Let $B_m = B_m(\boldsymbol{u}_m(t))$ be the linear functional defined on V by

$$\langle B_m, \boldsymbol{v}\rangle_{(V', V)} = -\,b(\boldsymbol{u}_m(t), \boldsymbol{v}, \boldsymbol{u}_m(t)).$$

From (2.6), we have

$$|b(\boldsymbol{u}_m(t), \boldsymbol{v}, \boldsymbol{u}_m(t))| \leqslant \sqrt{2}\|\boldsymbol{u}_m(t)\|\,|\boldsymbol{u}_m(t)|\,\|\boldsymbol{v}\|;$$

since $\boldsymbol{u}_m$ is bounded in $L^\infty(0, T; H) \cap L^2(0, T; V)$, B_m is bounded in $L^2(0, T; V')$.

Let $A_m = A_m(\boldsymbol{u}_m(t))$ be the linear functional defined on V by

$$\langle A_m(t), \boldsymbol{v}\rangle_{(V', V)} = a(\boldsymbol{u}_m(t), \boldsymbol{v}, \boldsymbol{E}_m).$$

From $(2.5)_2$, we have

$$|a(\boldsymbol{u}_m(t), \boldsymbol{v}, \boldsymbol{E}_m)| \leqslant 2\mu_1\|\boldsymbol{u}_m(t)\|\,\|v\|;$$

then, since $\boldsymbol{u}_m$ is bounded in $L^2(0, T; V)$, it turns out that the sequence A_m belongs to a bounded subset of $L^2(0, T; V')$.

Let $H_m = H_m(\varphi_m(t))$ be the linear functional defined on V by

$$\langle H_m, \boldsymbol{v}\rangle = (\Delta\varphi_m \nabla(\varphi_m + \overline{\Phi}), \boldsymbol{v}).$$

For a suitable positive constant C_3 it holds

$$|(\Delta\varphi_m \nabla(\varphi_m + \overline{\Phi}), \boldsymbol{v})| \leq |\Delta\varphi_m \nabla(\varphi_m + \overline{\Phi})|\,|\boldsymbol{v}| \leq C_3\,|\Delta\varphi_m \nabla(\varphi_m + \overline{\Phi})|\,\|\boldsymbol{v}\|;$$

then each H_m is continuous on V. Moreover, by (3.11) and (3.13), $\Delta\varphi_m$ and $\nabla\varphi_m$ are bounded respectively in $L^2(0, T; L^p(\Omega))$ and $L^\infty(0, T; L^p(\Omega))$ for any $p < +\infty$, hence

$$\Delta\varphi_m \nabla\varphi_m \text{ is bounded in } L^2(0, T; L^2(\Omega)), \tag{3.20}$$

which proves that H_m is bounded in $L^2(0, T; V')$.

Finally, let K_m be the linear functional defined on V as

$$\langle K_m, \boldsymbol{v}\rangle_{(V', V)} = (j_\varepsilon'(\boldsymbol{u}_m, \boldsymbol{E}_m), \boldsymbol{v}).$$

Using the Cauchy-Schwarz and the Hölder inequalities, we get, for a positive constant C_4,

$$\begin{aligned} |(j_\varepsilon'(\boldsymbol{u}_m, \boldsymbol{E}_m), \boldsymbol{v})| &= \left| \int_\Omega g(|\boldsymbol{E}_m|) D_{\mathrm{II}}(\boldsymbol{u}_m)^{\varepsilon - 1/2} D_{ij}(\boldsymbol{u}_m)\, D_{ij}(\boldsymbol{v})\, \mathrm{d}x \right| \\ &\leq 2 \int_\Omega g(|\boldsymbol{E}_m|)\, D_{\mathrm{II}}(\boldsymbol{u}_m)^{\varepsilon/2} D_{\mathrm{II}}(\boldsymbol{v})^{1/2}\, \mathrm{d}x \qquad (3.21) \\ &\leq C_4 \left[\int_\Omega D_{\mathrm{II}}(\boldsymbol{u}_m)^\varepsilon\, dx \right]^{1/2} \|\boldsymbol{v}\|. \end{aligned}$$

Thus each K_m is a continuous functional, and by (3.9) the sequence K_m is bounded in $L^2(0, T; V')$.

Let π_m be the orthogonal projection of V' onto $V_m' := \operatorname{span}\{\Lambda \boldsymbol{w}_i, i = 1, \ldots, m\}$; we have $\pi_m \boldsymbol{u}_m' = \boldsymbol{u}_m'$, so that (3.19) gives $\boldsymbol{u}_m' = \pi_m(H_m - K_m - A_m - B_m)$. By the choice of the basis $\{\boldsymbol{w}_i\}_{i \in \mathbb{N}}$, it is easy to check that π_m is a contraction on V'. Thus the boundedness of $H_m - K_m - A_m - B_m$ in $L^2(0, T; V')$ yields the boundedness of $\boldsymbol{u}_m'$ in the same space. ■

We are now in a position to prove the existence of a solution to problem $(\mathcal{P}_\varepsilon)$.

Lemma 3.2. *For every $\varepsilon > 0$, there exists a solution to problem $(\mathcal{P}_\varepsilon)$.*

Proof. By Lemma 3.1, we can assume that, when $m \to +\infty$, possibly passing to a subsequence, we have

(3.22) $\boldsymbol{u}_m \to \boldsymbol{u}_\varepsilon$ weakly star in $L^\infty(0, T; H)$, weakly in $L^2(0, T; V)$,

strongly in $L^2(0, T; H)$ and a.e. (by components) on Ω_T,

(3.23) $\boldsymbol{u}'_m \to \boldsymbol{u}'_\varepsilon$ weakly in $L^2(0, T; V')$,

(3.24) $\Delta\varphi_m \to \Delta\varphi_\varepsilon$ weakly star in $L^\infty(0, T; L^2(\Omega))$, weakly in $L^2(0, T; H^1_0(\Omega))$,

strongly in $L^2(0, T; L^2(\Omega))$ and a.e. on Ω_T,

(3.25) $\Delta\varphi'_m \to \Delta\varphi'_\varepsilon$ weakly in $L^2(0, T; H^{-1}(\Omega))$,

(3.26) $\nabla\varphi_m \to \nabla\varphi_\varepsilon$ weakly star in $L^\infty(0, T; H^1(\Omega))$,

strongly in $L^\infty(0, T; L^2(\Omega))$ and a.e. on Ω_T,

(3.27) $\Delta\varphi_m \nabla\varphi_m \to \Delta\varphi_\varepsilon \nabla\varphi_\varepsilon$ weakly in $L^2(0,T;L^2(\Omega))$ and a.e. on Ω_T,

Indeed, the weak and weak star convergence in (3.22)-(3.26) is satisfied (possibly passing to a subsequence) by Lemma 3.1, respectively by the reflexivity of the corresponding functional spaces, and by the Alaoglu-Bourbaki compactness theorem. The compact embedding $H^1_0(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ (see [**1**]) then yields the strong convergence in (3.22), (3.24), and (3.26).

Since the sequence $\Delta\varphi_m \nabla\varphi_m$ is bounded in $L^2(0, T; L^2(\Omega))$ (see (3.20)), possibly passing to a subsequence we have $\Delta\varphi_m \nabla\varphi_m \to g$ weakly in $L^2(0, T; L^2(\Omega))$; by (3.24) and (3.26), it must be $g = \Delta\varphi_\varepsilon \nabla\varphi_\varepsilon$, and (3.27) holds.

Finally, the pointwise convergence almost everywhere on Ω_T in (3.22), (3.24), (3.26), and (3.27), is fulfilled possibly passing to subsequences, because of the strong convergence of the involved sequences.

We can now pass to the limit, as $m \to +\infty$, in (3.5) and (3.4). Using (3.25), (3.24), (3.22), and (3.26), if we pass to the limit in (3.4), we obtain

$$(\Delta\varphi'_{\varepsilon},\, \xi) - a_2(\Delta\varphi_{\varepsilon},\, \Delta\xi) - (\boldsymbol{u}_{\varepsilon}\Delta\varphi_{\varepsilon},\, \nabla\xi) - a_3(\nabla\varphi_{\varepsilon},\, \nabla\xi)$$

$$- a_3(\nabla\overline{\Phi},\, \nabla\xi) = 0 \qquad \forall\xi \in H_0^1(\Omega) \cap H^2(\Omega).$$

Let us pass to the limit in (3.5). By (3.21) and (3.9), $j'_{\varepsilon}(\boldsymbol{u}_m, \boldsymbol{E}_m)$ is bounded in $L^2(0, T; V')$, hence we can assume that $j'_{\varepsilon}(\boldsymbol{u}_m, \boldsymbol{E}_m) \to \chi$ weakly in $L^2(0, T; V')$. Moreover, by (3.23), (3.22), (3.27), and (3.24), for every $\boldsymbol{v} \in V_m$ we have

$$(\boldsymbol{u}'_m, \boldsymbol{v}) \to (\boldsymbol{u}'_{\varepsilon}, \boldsymbol{v}), \qquad a(\boldsymbol{u}_m, \boldsymbol{v}, \boldsymbol{E}_m) \to a(\boldsymbol{u}_{\varepsilon}, \boldsymbol{v}, \boldsymbol{E}_{\varepsilon}),$$

$$(\Delta\varphi_m \nabla(\varphi_m + \overline{\Phi}), \boldsymbol{v}) \to (\Delta\varphi_{\varepsilon}\nabla(\varphi_{\varepsilon} + \overline{\Phi}), \boldsymbol{v}),$$

and $b(\boldsymbol{u}_m, \boldsymbol{v}, \boldsymbol{u}_m) \to b(\boldsymbol{u}_{\varepsilon}, \boldsymbol{v}, \boldsymbol{u}_{\varepsilon})$. Therefore, by the completeness of the system $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}}$, passing to the limit in (3.5), we obtain

$$\begin{aligned}(\boldsymbol{u}'_{\varepsilon}, \boldsymbol{v}) - b(\boldsymbol{u}_{\varepsilon}, \boldsymbol{v}, \boldsymbol{u}_{\varepsilon}) + a(\boldsymbol{u}_{\varepsilon}, \boldsymbol{v}, \boldsymbol{E}_{\varepsilon}) + (\chi, \boldsymbol{v}) \\ = (\Delta\varphi_{\varepsilon}\nabla\varphi_{\varepsilon}, \boldsymbol{v}) + (\Delta\varphi_{\varepsilon}\nabla\overline{\Phi}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V.\end{aligned} \tag{3.28}$$

It remains to prove that $\chi = j'_{\varepsilon}(\boldsymbol{u}_{\varepsilon}, \boldsymbol{E}_{\varepsilon})$. Let $\boldsymbol{f} \in L^2(0, T; V)$, $\boldsymbol{f}' \in L^2(0, T; V')$, with $\boldsymbol{f}(0) = \boldsymbol{u}^0$, and set

$$\begin{aligned}X_m = \int_0^T [(j'_{\varepsilon}(\boldsymbol{u}_m, \boldsymbol{E}_m) - j'_{\varepsilon}(\boldsymbol{f}, \boldsymbol{E}_m), \boldsymbol{u}_m - \boldsymbol{f}) \\ + a(\boldsymbol{u}_m - \boldsymbol{f}, \boldsymbol{u}_m - \boldsymbol{f}, \boldsymbol{E}_m) + (\boldsymbol{u}'_m - \boldsymbol{f}', \boldsymbol{u}_m - \boldsymbol{f})]\, dt.\end{aligned} \tag{3.29}$$

Notice that $X_m \geq 0$, because j_{ε} is a convex functional, $a(\boldsymbol{u}_m - \boldsymbol{f}, \boldsymbol{u}_m - \boldsymbol{f}, \boldsymbol{E}_m)$ is non-negative by definition, and $\int_0^T (\boldsymbol{u}'_m - \boldsymbol{f}', \boldsymbol{u}_m - \boldsymbol{f})\, dt = \frac{1}{2}|\boldsymbol{u}_m(T) - \boldsymbol{f}(T)|^2$.

Let us insert into (3.29) the equation

$$(\boldsymbol{u}'_m, \boldsymbol{u}_m) + a(\boldsymbol{u}_m, \boldsymbol{u}_m, \boldsymbol{E}_m) + (j'_{\varepsilon}(\boldsymbol{u}_m, \boldsymbol{E}_m), \boldsymbol{u}_m)$$

$$= (\Delta\varphi_m\nabla\varphi_m, \boldsymbol{u}_m) + (\Delta\varphi_m\nabla\overline{\Phi}, \boldsymbol{u}_m)$$

(obtained by taking $\boldsymbol{u}_m$ as a test function in (3.5)), and pass to the limit as

$m \to +\infty$. We get $X_m \to X$, with $X \geq 0$ given by

$$X = \int_0^T [(\Delta\varphi_\varepsilon \nabla\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) + (\Delta\varphi_\varepsilon \nabla\overline{\Phi}, \boldsymbol{u}_\varepsilon) - (\chi, \boldsymbol{f}) - (j_\varepsilon'(\boldsymbol{f}, \boldsymbol{E}_\varepsilon), \boldsymbol{u}_\varepsilon - \boldsymbol{f})$$

$$- a(\boldsymbol{u}_\varepsilon, \boldsymbol{f}, \boldsymbol{E}_\varepsilon) - a(\boldsymbol{f}, \boldsymbol{u}_\varepsilon - \boldsymbol{f}, \boldsymbol{E}_\varepsilon) - (\boldsymbol{u}_\varepsilon', \boldsymbol{f}) - (\boldsymbol{f}', \boldsymbol{u}_\varepsilon - \boldsymbol{f})]\, dt$$

$$= \int_0^T [(\boldsymbol{u}_\varepsilon' - \boldsymbol{f}', \boldsymbol{u}_\varepsilon - \boldsymbol{f}) + a(\boldsymbol{u}_\varepsilon - \boldsymbol{f}, \boldsymbol{u}_\varepsilon - \boldsymbol{f}, \boldsymbol{E}_\varepsilon) + (\chi - j_\varepsilon'(\boldsymbol{f}, \boldsymbol{E}_\varepsilon), \boldsymbol{u}_\varepsilon - \boldsymbol{f})]\, dt\,,$$

where the last equality has been obtained using (3.28), with $\boldsymbol{v} = \boldsymbol{u}_\varepsilon$.

Now, choosing $\boldsymbol{f} = \boldsymbol{u}_\varepsilon - \lambda\boldsymbol{\psi}$, where λ is a positive parameter, and $\boldsymbol{\psi} \in L^2(0, T; V)$, $\boldsymbol{\psi}' \in L^2(0, T; V')$ with $\boldsymbol{\psi}(0) = 0$, we get

$$X = \lambda^2 \int_0^T [(\boldsymbol{\psi}, \boldsymbol{\psi}') + a(\boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{E}_\varepsilon)]\, dt + \lambda \int_0^T (\chi - j_\varepsilon'(\boldsymbol{u}_\varepsilon - \lambda\boldsymbol{\psi}, \boldsymbol{E}_\varepsilon), \boldsymbol{\psi})\, dt\,.$$

Dividing by λ, letting λ tend to zero, and recalling that X must be non-negative, we conclude

$$\int_0^T (\chi - j_\varepsilon'(\boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon), \boldsymbol{\psi})\, dt \geq 0\,.$$

Hence, by the arbitrariness of $\boldsymbol{\psi}$, it follows $\chi = j_\varepsilon'(\boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon)$. ∎

3.4 - *Existence of a weak solution to* ($\mathcal{P}$)

Our main existence result can now be proved using Lemma 3.2, and passing to the limit as $\varepsilon \to 0^+$.

Theorem 3.3. *The initial-boundary value problem* ($\mathcal{P}$) *admits a solution.*

Proof. By Lemma 3.1 and Lemma 3.2, for any $\varepsilon > 0$ there exists a solution $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)$ to problem $(\mathcal{P}\varepsilon)$ such that the sequences φ_ε and $\boldsymbol{u}_\varepsilon$ satisfy the estimates (3.9)-(3.13), with the index m replaced by the index ε. In particular, possibly passing to a subsequence, (3.22)-(3.27) hold (again replacing m by ε). Then, passing to the limit in (3.2) as $\varepsilon \to 0^+$, one gets (2.16). The passage to the limit in (3.3) can be done similarly as in [4]: we set, for a fixed $\boldsymbol{v} \in L^2(0, T; V)$,

$$Z_\varepsilon := \int_0^T [(\boldsymbol{u}_\varepsilon', \boldsymbol{v} - \boldsymbol{u}_\varepsilon) - b(\boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon) + a(\boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon)$$

$$+ j_\varepsilon(\boldsymbol{v}, \boldsymbol{E}_\varepsilon) - j_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon) - (\Delta\varphi_\varepsilon \nabla\varphi_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon) - (\Delta\varphi_\varepsilon \nabla\overline{\Phi}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon)]\, dt\,.$$

By (3.3), we have

$$Z_\varepsilon = \int_0^T [j_\varepsilon(\boldsymbol{v}, \boldsymbol{E}_\varepsilon) - j_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon) - (j_\varepsilon'(\boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon), \boldsymbol{v} - \boldsymbol{u}_\varepsilon)]\, \mathrm{d}t \geq 0\,,$$

hence

$$\int_0^T [(\boldsymbol{u}_\varepsilon', \boldsymbol{v}) - b(\boldsymbol{u}_\varepsilon, \boldsymbol{v}, \boldsymbol{u}_\varepsilon) + a(\boldsymbol{u}_\varepsilon, \boldsymbol{v}, \boldsymbol{E}_\varepsilon) + j_\varepsilon(\boldsymbol{v}, \boldsymbol{E}_\varepsilon)$$
$$-(\Delta\varphi_\varepsilon \nabla\varphi_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon) - (\Delta\varphi_\varepsilon \nabla\overline{\Phi}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon)]\, \mathrm{d}t$$
$$\geq \frac{1}{2}\,|\boldsymbol{u}_\varepsilon(T)|^2 - \frac{1}{2}\,|\boldsymbol{u}^0|^2 + \int_0^T a(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon)\, dt + \int_0^T j_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon)\, \mathrm{d}t\,.$$

Passing to the liminf as $\varepsilon \to 0^+$ in the above inequality, we obtain

$$\int_0^T [(\boldsymbol{u}', \boldsymbol{v}) - b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}) + a(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{E}) + j(\boldsymbol{v}, \boldsymbol{E})$$

$$-(\Delta\varphi\nabla\varphi, \boldsymbol{v} - \boldsymbol{u}) - (\Delta\varphi\nabla\overline{\Phi}, \boldsymbol{v} - \boldsymbol{u})]\, \mathrm{d}t \geq \liminf_{\varepsilon\to 0} \frac{1}{2}\,|\boldsymbol{u}_\varepsilon(T)|^2 - \frac{1}{2}\,|\boldsymbol{u}^0|^2 \tag{3.30}$$

$$+ \liminf_{\varepsilon\to 0} \int_0^T a(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon)\, \mathrm{d}t + \liminf_{\varepsilon\to 0} \int_0^T j_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon)\, \mathrm{d}t\,.$$

We claim that

$$\liminf_{\varepsilon\to 0} \frac{1}{2}\,|\boldsymbol{u}_\varepsilon(T)|^2 \geq |\boldsymbol{u}(T)|^2, \tag{3.31}$$

$$\liminf_{\varepsilon\to 0} \int_0^T a(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon)\, \mathrm{d}t \geq \int_0^T a(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{E})\, \mathrm{d}t\,, \tag{3.32}$$

$$\liminf_{\varepsilon\to 0} \int_0^T j_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon)\, \mathrm{d}t \geq \int_0^T j(\boldsymbol{u}, \boldsymbol{E})\, \mathrm{d}t\,. \tag{3.33}$$

The inequality (3.31) is a straightforward consequence of the weak convergence of $\boldsymbol{u}_\varepsilon(T)$ to $\boldsymbol{u}(T)$ in H. In order to prove (3.32), we use the continuity assumption on the function $\mu = \mu(\xi)$ on $\mathbb{R}^+$ and the lower semicontinuity of the function $\boldsymbol{v} \mapsto \int_0^T a(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{E})\, dt$ on $L^2(0, T; V)$; we get

$$\liminf_{\varepsilon \to 0} \int_0^T a(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon)\, dt \geqslant \liminf_{\varepsilon \to 0} \int_0^T [a(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon) - a(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{E})]\, dt$$

$$+ \liminf_{\varepsilon \to 0} \int_0^T a(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{E})\, dt \geqslant \int_0^T a(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{E})\, dt\,.$$

Similarly, the continuity assumption on the function $g = g(\xi)$ on $\mathbb{R}^+$ gives

$$\liminf_{\varepsilon \to 0} \int_0^T j_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon)\, dt \geqslant \liminf_{\varepsilon \to 0} \int_0^T [j_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{E}_\varepsilon) - j_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{E})]\, dt$$

$$+ \liminf_{\varepsilon \to 0} \int_0^T j_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{E})\, dt \geqslant \liminf_{\varepsilon \to 0} \int_0^T j_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{E})\, dt\,.$$

By the Hölder inequality, we infer

$$\int_0^T j_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{E})\, dt \geqslant \frac{1}{(1+\varepsilon)(2g_1\,|\Omega_T|)^\varepsilon} \left[\int_0^T j(\boldsymbol{u}_\varepsilon, \boldsymbol{E})\, dt\right]^{1+\varepsilon}.$$

Then, using the weak lower semicontinuity of the convex functional $\boldsymbol{v} \mapsto \int_0^T j(\boldsymbol{v}, \boldsymbol{E})\, dt$ on $L^2(0, T; V)$, we get

$$\liminf_{\varepsilon \to 0} \int_0^T j_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{E})\, dt \geqslant \liminf_{\varepsilon \to 0} \int_0^T j(\boldsymbol{u}_\varepsilon, \boldsymbol{E})\, dt \geqslant \int_0^T j(\boldsymbol{u}, \boldsymbol{E})\, dt\,,$$

and (3.33) is proved. From (3.30), (3.31), (3.32) and (3.33), it follows

$$\int_0^T [(\boldsymbol{u}', \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}) + a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}, \boldsymbol{E}) + j(\boldsymbol{v}, \boldsymbol{E}) - j(\boldsymbol{u}, \boldsymbol{E})$$

$$- (\Delta\varphi \nabla\varphi, \boldsymbol{v} - \boldsymbol{u}) - (\Delta\varphi \nabla\overline{\Phi}, \boldsymbol{v} - \boldsymbol{u})]\, dt \geqslant 0 \qquad \forall \boldsymbol{v} \in V\,.$$

Choosing

$$v = v_n := \begin{cases} \bar{v} & \text{if } t \in \left[\bar{t} - \frac{1}{n}, \bar{t} + \frac{1}{n}\right] \\ u(t) & \text{otherwise}, \end{cases}$$

where $\bar{v}$ is a fixed function of V and $\bar{t}$ is arbitrary in $[0, T]$, and passing to the limit as $n \to \infty$ we deduce that the integrand function in the above inequality must be non-negative for a.e. $t \in [0, T]$. Hence (2.17) holds, and this completes the proof.

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Abstract

Electrorheological fluids are characterized by a relevant increasing of the viscosity under the action of an electric field. We state a constitutive law which consists in viewing them as Bingham fluids controlled by a yield function depending on the intensity of the applied electric field. We couple this constitutive equation with the fundamental conservation laws which govern the motion of the fluid, and prove an existence result for a weak formulation of the corresponding initial-boundary value problem.

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