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Orderings and preorderings in rings with involution (**)

1 - Introduction, Definitions and Basic facts

The notion of an ordering of a field was studied by Artin and Schreier. This notion was extended to division rings with involution in [**1**], [**2**] and [**3**]. One can ask now if this can be generalized to noncommutative rings with involution. In this paper, the notions of a preordering and an ordering of a ring R with involution is investigated. An algebraic condition for the existence of an ordering of R is given. Also, a condition for enlarging an ordering of R to an overring is given. As for the case of a field, any preordering of R can be extended to some ordering. Finally, we establish a classification theorem for archimedean ordered rings with involution. We should remark that the orderings as defined in this work can only exist for rings without zero-divisors.

Now, we state some definitions and basic facts that will be needed in this work. Hereafter R will be a not necessarily commutative ring with unity with involution * (an anti- automorphism of period 2). By a norm in R we mean an element of the form xx^* for some $x \in R$. Let $S = \{s \in R: s = s^*\}$ be the set of all symmetric elements of R. Let X be the set of all finite products of elements of the set $\{x_i, x_i^* / 0 \neq x_i \in R\}$ in some arbitrary but fixed order, and we write P for the subset of R consisting of sums of elements of X. P is called the *-core of R. This generalizes the notion of a *-core given in [**1**] for the case of a ring with involution.

Clearly X contains the set of all products of norms of R and P contains the set of all sums of products of norms, in particular $X \subset P$. Also, it is clear that X is *-

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closed, multiplicatively closed and contains 1; and P is *-closed and closed under sums and products. When $* =$ identity, then R is commutative and P will be the set of all sums of products of squares of R. Our goal is to show that R has an ordering if and only if $0 \notin P$. First, we give the definition of an ordering.

Definition. A *-closed subset $M \subset R$ is called a preordering of R if:

(1) $M + M \subseteq M$;

(2) $M \cdot M \subseteq M$;

(3) $0 \notin M$, $1 \in M$; and

(4) $a_1, a_2, \ldots, a_t \in M$; $x_1 x_2, \ldots, x_r \in R$ implies that any product of the $2r+t$ elements a_j, x_i, x_i^* in some arbitrary but fixed order belongs to M (where $x_i \neq 0$).

A preordering M is called an ordering of R if:

(5) For $0 \neq s = s^* \in R$, $s \in M \cup -M$, i.e. S is a totally ordered (additive) group.

If R is commutative, then condition (4) above is equivalent to the condition:

$$a \in M, \qquad x \in R \Rightarrow axx^* \in M .$$

The above definition of an ordering of R generalizes the notion of a strong ordering of a division ring with involution given in [**2**]. Also, $M \cap S$ will be a Jordan ordering in the sense given in [**3**] in the case of a division ring with involution. When $* =$ identity, then R is commutative, and the definition of an ordering reduces to that of a classical Artin-Schreier ordering.

Proposition 1. *Let M be an ordering on R. Then*

$$M \cap -M = \phi ,$$

and R is a domain with characteristic zero.

Proof. If $a \in M \cap -M$, then $0 = a + (-a) \in M + M \subseteq M$, contradicting Property (3) above. Since $1 \in M$, it follows that, for any natural number n,

$$n \cdot 1 = 1 + \ldots + 1 \in M .$$

Therefore, char $R = 0$. Finally, if $x, y \in R \backslash \{0\}$ and $xy = 0$, then $0 = x^* xyy^* \in M$, a contradiction. This shows that R is a domain.

Proposition 2. *Let M be a preordering, then*

(1) $s = s^* \in M$, *s invertible* $\Rightarrow s^{-1} \in M$.
(2) $s \in R$, *s invertible* $\Rightarrow sMs^{-1} \subset M$.

Proof.

(1) We note that $s^{-1} = s(s^{-1}s^{-1^*}) \in M$
(2) $sMs^{-1} = sMs^{-1}(s^{-1^*}s^*) \subset M$ (by Property (4)).

If we are given an ordering M of R, then M defines an order relation on R by:

$$b \geqslant a \Leftrightarrow b - a \in M \cup \{0\}.$$

The ring of integers Z, the field of rational numbers Q and the field of real numbers R, with their usual orderings and the identity as involution are examples of ordered commutative rings. The field of complex numbers C with conjugation as involution, is ordered by the set $M = \mathbf{R}^+$ (the positive real numbers).

An example of a non commutative ordered ring is the Weyl algebra generated over R by x and y with relation $xy - yx = 1$, i.e., $R = \mathbf{R}[x, y]/(xy - yx - 1)$, relative to the involution making x symmetric and y skew. Elements of R have the canonical form

$$r = r_0(x) + r_1(x)\,y + \ldots + r_n(x)\,y^n,$$

where each $r_i(x) \in \mathbf{R}[x]$, $r_n(x) \neq 0$. Let $M \subset R$ be the set of all non zero elements $r \in R$ as above for which $r_n(x)$ has a positive leading coefficient. One can show that M is an ordering of R.

2 - Existence of Orderings

For a preordering M and $0 \neq s = s^* \in R$, we define $M(s)$ to be the set of all sums of products of elements of M; elements of $\{x_i, x_i^* / 0 \neq x_i \in R\}$, and s in some arbitrary but fixed order. If R is commutative, then clearly $M(s) = Ms$. For $R = D$ a division ring, also $M(s) = Ms$.

Lemma 3. *$M \cup M(s) \cup M + M(s)$ is a preordering iff $0 \notin M + M(s)$.*

Proof. Let $M' = M \cup M(s) \cup M + M(s)$ then clearly $M' + M' \subset M'$. By the definition of $M(s)$ and Property (4) of a preordering, we have

$$\begin{aligned} M' \cdot M' &= M \cdot M(s) + M(s) \cdot M + M \cdot M + M(s) \cdot M(s) \\ &\subset M(s) + M(s) + M + M \\ &\subset M + M(s) \subset M'. \end{aligned}$$

Also M' satisfies Property (4) and $1 \in M'$. Since $0 \notin M \cup M(s)$, then M' is a preordering iff $0 \notin M + M(s)$.

Lemma 4. *If M is a preordering and $0 \neq s = s^* \in R$, then*

$$M_1 = M \cup M(s) \cup M + M(s) \quad \textit{or} \quad M_2 = M \cup M(-s) \cup M + M(-s),$$

is a preordering containing M.

Proof. We first note that any element of $M(-s)$ is of the form $-x$ where $x \in M(s)$ and hence every element of $M + M(-s)$ is of the form $t - x$; where $t \in M$, $x \in M(s)$. Assume now that the lemma is false, then by Lemma 3, $0 \in M + M(s)$ and $0 \in M + M(-s)$. Hence $t_1 + x_1 = 0 = t_2 - x_2$ where $t_1, t_2 \in M$; x_1, $x_2 \in M(s)$, and $x_1 = -t_1$, $x_2 = t_2$. Since $x_1 x_2 \in M(s) \cdot M(s) \subset M$ and $t_1 t_2 \in M$; and $t_1 t_2 = -x_1 x_2$ then $0 = x_1 x_2 + t_1 t_2 \in M$ which is a contradiction. Thus M_1 or M_2 is a preordering.

Proposition 5. *If M is a maximal preordering with respect to inclusion, then M is an ordering.*

Proof. We need to show that $S \subset M \cup -M$. For $0 \neq s = s^* \in S$,

$$M_1 = M \cup M(s) \cup M + M(s) \quad \text{or} \quad M_2 = M \cup M(-s) \cup M + M(-s),$$

is a preordering containing M. But M is maximal, then $M = M_1$ or $M = M_2$ and hence M contains s or $-s$ as desired.

Theorem 6. *Let R be a ring with involution, then R has an ordering if and only if $0 \notin P$.*

Proof. If R has an ordering M, then $P \subset M$ and $0 \notin P$. Conversely, if $0 \notin P$, then P is a preordering. By Zorn's Lemma, we have a maximal preordering M. By Proposition 5, M is an ordering of R.

Theorem 7. *Any preordering M_0 of R can be extended to some ordering M.*

Proof. By Zorn's Lemma, the set of all preorderings extending M_0 contains some maximal preordering M. By Proposition 5, M is an ordering contains M_0.

We note that, any intersection of orderings of R is a preordering of R. If R is orderable, i.e., $0 \notin P$, then the *-core P is a preordering with the following features $P \subset M$ and $M \cdot P = P \cdot M = M$ for each preordering M. Throughout the rest of this section, we will assume that $0 \notin P$. By Sym (A) we mean the subset of symmetric elements of A.

Corollary 8. $\mathrm{Sym}(P) = \mathrm{Sym}(\bigcap_i M_i)$, *where the intersection runs over all orderings* M_i *of* R.

Proof. Clearly $\mathrm{Sym}(P) \subseteq \mathrm{Sym}(\bigcap_i M_i)$. Conversely, we show that $s = s^* \notin P$ implies $s \notin M$ for some ordering M. Since P is a preordering, then by Lemma 4, $M_1 = P \cup P(-s) \cup P + P(-s)$ is a preordering containing P and $-s$. By Theorem 7, M_1 can be extended to some ordering M. Since $-s \in M_1 \subset M$ and M is an ordering, it follows that $s \notin M$.

Corollary 9. *Let* M_0 *be any preordering. Then* $\mathrm{Sym}(M_0) = \mathrm{Sym}(\bigcap_i M_i)$, *where the intersection runs over all orderings* M_i *containing* M_0.

Lemma 10. *Let* M_1 *and* M_2 *be two orderings of* R. *If* $M_1 \subset M_2$, *then*

$$\mathrm{Sym}(M_1) = \mathrm{Sym}(M_2).$$

Proof. If there is $s = s^* \in M_2 - M_1$, then from $s \notin M_1$ follows $-s \in M_1 \subset M_2$, so both s and $-s$ are in M_2 which is nonsense.

Theorem 11. *Let* $R \subseteq R'$ *be rings with involution. Let* M *be an ordering of* R. *Let* M' *be the set of all sums of products of* $2r + t$ *elements* a_j, x_i, x_i^* *in some arbitrary but fixed order, where* $a_1, a_2, \ldots, a_t \in M$ *and* $x_1, x_2, \ldots, x_r \in R' - \{0\}$. *If* $0 \notin M'$, *then* M *can be enlarged to some ordering of* R'.

Proof. Since $0 \notin M'$, it follows that $0 \notin P'$ (the *-core of R') and R' is ordered. It is easy to show that M' is a preordering of R'. By Theorem 7, M' can be enlarged to some ordering $M_1 \supset M' \supset M$.

It is known that any archimedean ordered ring is commutative. In the rest of this work, we shall give a classification theorem for archimedean ordered rings with involution. Let $s = s^*$ be a positive element in an ordered ring R with involution. We say that s is infinitely large if $s > n$ for any integer $n \geqslant 1$, and that s is infinitely small if $n \cdot s < 1$ for any integer $n \geqslant 1$.

Lemma 12. *For any ordered ring* R, *the following two properties are equivalent*:

(1) *For any positive elements* $s = s^*$, $d = d^*$ *in* R, *there exists an integer* $n \geqslant 1$ *such that* $n \cdot s > d$.

(2) R *has neither infinitely large nor infinitely small elements.*

Proof. Assume (2) holds and consider $s, d > 0$. By (2), there exist integers $m, n \geqslant 1$ such that $d < n$ and $m \cdot s > 1$. Then $m \cdot n \cdot s > n > d$ as desired. Now, assume (1) holds, and $s = s^* > 0$. Since $1, s > 0$, then by (1) there exist integers $m, n \geqslant 1$ such that $m = m \cdot 1 > s$ and $n \cdot s > 1$, so that s is neither infinitely large nor infinitely small.

An ordered ring with involution is called archimedean if it satisfies any of the two conditions of Lemma 12. We note that, if $R = D$ is an ordered division ring, then for $s = s^* > 0$, s is infinitely large if and only if s^{-1} is infinitely small. Thus D is archimedean if and only if D has no infinitely large elements, if and only if D has no infinitely small elements.

Theorem 13. *Let* R *be an archimedean ordered ring with involution. Then all symmetric elements in* R *mutually commute.*

Proof. Let b, d and s be three symmetric elements of R. Let k be the skew symmetric element $[b, d] = bd - db$ and form the symmetric element $[k, s] = [[b, d], s]$. From $(s-k)^*(s-k) \geqslant 0$ and $(s-k)(s-k)^* \geqslant 0$ one can get the inequality $0 \leqslant |[k, s]| \leqslant s^2 - k^2$ where $|[k, s]|$ means the absolute value symbol in its usual sense. We assume that $s > 0$ (if $s < 0$ we replace s by $-s$). Since R is archimedean, then for each $n \geqslant 1$ there exists an integer m such that $1 > ns - m \geqslant 0$ so that $(ns - m)^2 < 1$. Now, replace s by $ns - m$ in the above inequality we get $0 \leqslant n|[k, s]| \leqslant 1 - k^2$, $n = 1, 2, \ldots$; which implies $[k, s] = 0$ (since both $|[k, s]|$ and $1 - k^2$ are positive symmetric elements), i.e. $k = [b, d]$ commutes with s for all symmetrics b, d, and s. This says that all commutators $[b, d]$; $b, d \in S$; commutes with all symmetric elements. From the identity

$$2b[b, d] = [b^2, d] + [b, [b, d]] = [b^2, d],$$

$2b[b, d]$ also commutes with all symmetric elements, for $b, d \in S$. Thus both $[b, d]$ and $2b[b, d]$ commute with all symmetric elements. As R is a domain, b must commute with all symmetric elements. Hence all symmetric elements mutually commute.

Corollary 14. *Let* R *be an archimedean ordered ring with involution where the set of symmetric elements* S *generates* R. *Then* R *is a commutative domain.*

In the case of a division ring R with involution, it is known that S generates R, unless R is of dimension 4 over its centre. Hence

Corollary 15. *If R is an archimedean ordered division ring with involution, then R is commutative or of dimension* 4 *over its centre.*

References

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Abstract

The notion of an ordering of a field was studied by Artin and Schreler. One can ask now if this can be generalized to noncommutative rings with involution. In this paper, the notions of a preordering and an ordering of a ring R with involution is investigated. An algebraic condition for the existence of an ordering of R is given. Also, a condition for enlarging an ordering of R to an overring is given. As for the case of a field, any preordering of R can be extended to some ordering. Finally, we establish a classification theorem for archimedean ordered rings with involution. We should remark that the orderings as defined in this work can only exist for rings without zero-divisors.

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