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JAMES ADEDAYO OGUNTUASE (*)

Gronwall-Bellman type integral inequalities for multi-distributions (**)

1 - Introduction

The origin of the results obtained in this paper is the Gronwall-Bellman inequality which plays an important role in the study of the properties of solutions of differential and integral equations (see for example [**1**] and the references cited therein). Due to various motivations, many linear, nonlinear and discrete generalizations of Gronwall-Bellman type inequalities have been obtained and applied extensively (see for example [**1**]).

The purpose of this paper is to further investigate the Gronwall type inequalities for multi-distributions and to extend some of the results obtained in [**6**] and where necessary to obtain improved apriori bounds than those given in [**6**].

In addition, the results obtained in this paper would enable us to study equations of the form

$$Dx = f(t, x)\, Du \tag{1}$$

where Dx and Du denote the derivatives of the functions x and u respectively in the sense of the distribution.

The results obtained in this paper are in the sense of Lebesgue-Stieltjes integral for functions of bounded variation. Throughtout this paper, we shall assume that the functions $u_j(t)$ is right continuous at $t = 0$, $j = 1, \ldots, m$ and that $BV(I)$

(*) Department of Mathematical Sciences, University of Agriculture, Abeokuta, Nigeria.

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will denote the set of all functions of bounded variation defined on $I \subset \Re$ and taking values in $\Re$.

2 - Main results

The following results will be needed in the proof of our main results.

Lemma 2.1 [5]. *Let f and g be two real-valued functions on the real line $\Re$ such that both are of bounded variation on every compact subinterval of $\Re$. Then fg defines a distribution, and the derivative of fg in the sense of the distribution is equal to the locally summable function $(fg)'$ given by*

$$f'(x)\,g(x)+f(x)g'(x)$$

for almost all x. That is

$$D(fg)=(Df)g+f(Dg),$$

where Df and Dg denote the derivatives of the functions f and g respectively in the sense of the distributions.

Proof. See [5], pp. 546-547.

Theorem 2.1. *Suppose that for $j=1,\dots,m.$ and $t,s\in[0,T]$,*

1. *$Q_j(t,s)\geqslant 0$, $y(t)\geqslant 0$ and $Q_j(t,s)$, $y(t)$, $f(t)\in BV[0,T]$.*
2. *$u_j(t)$ are nondecreasing in t.*
3. *$Q_j(t,s)$ and its partial derivatives $\dfrac{\partial}{\partial t}Q_j(t,s)$ are continuous and nondecreasing in its first variable and that $Q_j(t,s)$ and $\dfrac{\partial}{\partial t}Q_j(t,s)$ are nonnegative and integrable with respect to $u_j(t)$ and if the following inequality holds*

$$y(t)\leqslant f(t)+\sum_{j=1}^{m}\int_0^t Q_j(t,s)\,y(s)\,du_j(s). \tag{2}$$

Then

$$
y(t) \leqslant A_m(f) + A_m(1) \int_0^t \left(Q_m(s,\, s)\, A_m(f) + \int_0^s \frac{\partial}{\partial s} Q_m(s,\, \tau)\, A_m(f) \right) \times \exp\left(\int_s^t Q_m(s,\, \tau)\, A_m(1)\, \mathrm{d}u_m(\tau) \right) \mathrm{d}u_m(s)\,, \tag{3}
$$

for all $t,\, s,\, \tau \in [0,\, T]$, *and where* $A_k(v)$ *is defined inductively as follows*

$$
\begin{aligned}
&A_1(v) = v \\
&A_{k+1}(v) = A_k(v) + \int_0^t \left(A_k(Q_k(s,\, s))\, A_k(v) + \int_0^s \frac{\partial}{\partial s} A_k(Q_k(s,\, \tau))\, A_k(v)\, \mathrm{d}u_k(\tau) \right) \\
&\qquad \times \exp\left(\int_s^t A_k(Q_k(s,\, \tau))\, \mathrm{d}u_k(\tau) \right) \mathrm{d}u_k(s)\,.
\end{aligned} \tag{4}
$$

Proof. Let

$$
x_i(t) = \int_0^t Q_i(t,\, s)\, y(s)\, \mathrm{d}u_i(s)\,, \qquad t,\, s \in [0,\, T], \quad i = 1,\, \ldots,\, m\,. \tag{5}
$$

Clearly $x_i(t)$ are functions of bounded variation. We also observed that $x_i(0) = 0$. Hence, in view of (5), inequality (2) becomes

$$
y(t) \leqslant f(t) + \sum_{j=0}^{m} x_j(t)\,. \tag{6}
$$

Thus

$$
Dx_i(t) = Q_i(t,\, t)\, y(t)\, Du_i(t) + \int_0^t \frac{\partial}{\partial t} Q_i(t,\, s)\, y(s)\, Du_i(s)\,. \tag{7}
$$

If we put $i=1$ in (6) and (7), we obtain

$$Dx_1(t) = Q_1(t,t)\,y(t)\,Du_1(t) + \int_0^t \frac{\partial}{\partial t} Q_1(t,s)\,y(s)\,Du_1(s)$$
$$\leqslant \left(Q_1(t,t)\left[f(t) + \sum_{j=1}^{m} x_j(t)\right] + \int_0^t \frac{\partial}{\partial t} Q_1(t,s)\left[f(s) + \sum_{j=1}^{m} x_j(s)\right]\right) Du_1(t).$$

That is

$$Dx_1(t) - \left(Q_1(t,t)x_1(t) + \int_0^t \frac{\partial}{\partial t} Q_1(t,s)\,x_1(s)\right) Du_1(t) \leqslant \left(Q_1(t,t)\left[f(t) + \sum_{j=2}^{m} x_j(t)\right]\right)$$
(8)
$$+ \int_0^t \frac{\partial}{\partial t}\left(Q_1(t,s)\left[f(s) + \sum_{j=2}^{m} x_j(s)\right]\right) Du_1(t).$$

Multiply both sides of (8) by $\exp\left(-\int_0^t Q_1(t,s)\,\mathrm{d}u_1(s)\right)$ we have

$$\left[Dx_1(t) - \left(Q_1(t,t)x_1(t) + \int_0^t \frac{\partial}{\partial t} Q_1(t,s)\,x_1(s)\right) Du_1(t)\right] \exp\left(-\int_0^t Q_1(t,s)\,\mathrm{d}u_1(s)\right)$$

$$\leqslant \left(Q_1(t,t)\left[f(t) + \sum_{j=2}^{m} x_j(t)\right] \int_0^t \frac{\partial}{\partial t} Q_1(t,s)\left[f(s) + \sum_{j=2}^{m} x_j(s)\right]\right)$$

$$\times \exp\left(-\int_0^t Q_1(t,s)\,\mathrm{d}u_1(s)\right) Du_1(t).$$

By Lemma 2.1, we have

$$D\left(x_1(t)\exp\left(-\int_0^t Q_1(t,s)\,\mathrm{d}u_1(s)\right)\right) \leqslant \left(Q_1(t,t)\left[f(t) + \sum_{j=2}^{m} x_j(t)\right]\right.$$

$$\left. + \int_0^t \frac{\partial}{\partial t} Q_1(t,s)\left[f(s) \sum_{j=2}^{m} x_j(s)\right]\right) \exp\left(-\int_0^t Q_1(t,s)\,\mathrm{d}u_1(s)\right) Du_1(t).$$

Integrate with respect to t from 0 to t, we have

$$(x_1(t)-x_1(0))\exp\left(-\int_0^t Q_1(t,s)\,\mathrm{d}u_1(s)\right)\leqslant\int_0^t\left(Q_1(s,s)\left[f(s)+\sum_{j=2}^m x_j(s)\right]\right.$$
$$\left.+\int_0^s\frac{\partial}{\partial s}Q_1(s,\tau)\left[f(\tau)+\sum_{j=2}^m x_j(\tau)\right]\right)\exp\left(-\int_0^s Q_1(s,\tau)\,\mathrm{d}u_1(\tau)\right)\mathrm{d}u_1(s).$$

Since $x_1(0)=0$, we obtain

$$\begin{aligned}x_1(t)\leqslant&\int_0^t\left(Q_1(s,s)\left[f(s)+\sum_{j=2}^m x_j(s)\right]\right.\\ &\left.+\int_0^s\frac{\partial}{\partial s}Q_1(s,\tau)\left[f(\tau)+\sum_{j=2}^m x_j(\tau)\right]\right)\exp\left(-\int_s^t Q_1(s,\tau)\,\mathrm{d}u_1(\tau)\right)\mathrm{d}u_1(s).\end{aligned}\tag{9}$$

If we put (9) into (6) and using the fact that $X_j(t)$ are nondecreasing, we obtain

$$\begin{aligned}y(t)\leqslant& f(t)+\int_0^t\left(Q_1(s,s)\left[f(s)+\sum_{j=2}^m x_j(s)\right]\right.\\ &\left.+\int_0^s\frac{\partial}{\partial s}Q_1(s,\tau)\left[f(\tau)+\sum_{j=2}^m x_j(\tau)\right]\right)\exp\left(-\int_s^t Q_1(s,\tau)\,\mathrm{d}u_1(\tau)\right)\mathrm{d}u_1(s)+\sum_{j=2}^m x_j(t)\\ &\leqslant f(t)+\int_0^t\left(Q_1(s,s)\,f(s)+\int_0^s\frac{\partial}{\partial s}Q_1(s,\tau)\,f(\tau)\right)\exp\left(-\int_0^s Q_1(s,\tau)\,\mathrm{d}u_1(\tau)\right)\mathrm{d}u_1(s)\\ &+\sum_{j=2}^m\left[1+\int_0^t\left(Q_1(s,s)+\int_0^s\frac{\partial}{\partial s}Q_1(s,\tau)\right)\exp\left(-\int_s^t Q_1(s,\tau)\,\mathrm{d}u_1(\tau)\right)\mathrm{d}u_1(s)\right]x_j(t)\\ &=A_2(f)+\sum_{j=2}^m A_2(1)\,x_j(t)\end{aligned}\tag{10}$$

where $A_2(f)$ and $A_2(1)$ are as defined in (3).

When $i=2$, inequalities (7) and (10) gives

$$Dx_2(t)=Q_2(t,t)\,y(t)\,Du_2(t)+\int_0^t\frac{\partial}{\partial t}Q_2(t,s)\,y(s)\,Du_2(s)$$
$$\leqslant\left(Q_2(t,t)\left[A_2(f)+\sum_{j=2}^m A_2(1)\,x_j(t)\right]+\int_0^t\frac{\partial}{\partial t}Q_2(t,s)\left[A_2(f)+\sum_{j=2}^m A_2(f)\,x_j(s)\right]\right)Du_2(t).$$

That is

$$
\tag{11}
\begin{aligned}
&Dx_2(t)-\left(Q_2(t,t)\,A_2(1)\,x_2(t)+\int_0^t \frac{\partial}{\partial t}\,Q_2(t,s)\,A_2(1)\,x_2(s)\right)Du_2(t)\\
&\leqslant\left(Q_2(t,t)\left[A_2(f)+\sum_{j=3}^{m}A_2(1)\,x_j(t)\right]+\int_0^t \frac{\partial}{\partial t}\,Q_2(t,s)\left[A_2(f)+\sum_{j=3}^{m}A_2(1)\,x_j(s)\right]\right)Du_2(t)\,.
\end{aligned}
$$

Multiply both sides of (11) by $\exp\left(-\int_0^t Q_1(t,\,s)\,\mathrm{d}u_1(s)\right)$ we have

$$
\left[Dx_2(t)-\left(Q_2(t,\,t)\,A_2(1)\,x_2(t)+\int_0^t \frac{\partial}{\partial t}\,Q_2(t,\,s)\,A_2(1)\,x_2(t)\right)Du_2(t)\right]
$$

$$
\times\exp\left(-\int_0^t Q_2(t,\,s)\,A_2(1)\,\mathrm{d}u_2(s)\right)
$$

$$
\leqslant\left(Q_2(t,\,t)\left[A_2(f)+\sum_{j=3}^{m}A_2(1)\,x_j(t)\right]+\int_0^t \frac{\partial}{\partial t}\,Q_2(t,\,s)\left[A_2(1)+\sum_{j=3}^{t}A_2(1)\,x_j(s)\right]\right)
$$

$$
\times\exp\left(-\int_0^t Q_2(t,\,s)\,A_2(1)\,\mathrm{d}u_2(s)\right)Du_2(t)\,.
$$

By Lemma 2.1, we have

$$
D\left(x_2(t)\exp\left(-\int_0^t Q_2(t,\,s)\,A_2(1)\,\mathrm{d}u_2(s)\right)\right)\leqslant\left(Q_2(t,\,t)\left[A_2(f)+\sum_{j=3}^{m}A_2(1)\,x_j(t)\right]\right.
$$

$$
\left.+\int_0^t \frac{\partial}{\partial t}\,Q_2(t,\,s)\left[A_2(f)+\sum_{j=3}^{m}A_2(1)\,x_j(s)\right]\right)\exp\left(-\int_0^t Q_2(t,\,s)\,A_2(1)\,\mathrm{d}u_2(s)\right)Du_2(t)\,.
$$

Integrate with respect to t from 0 to t and noting that $x_2(0)=0$ we have

$$
\tag{12}
\begin{aligned}
x_2(t) \leqslant \int_0^t \Bigg(& Q_2(s,\, s)\left[A_2(f) + \sum_{j=3}^{m} A_2(1)\, x_j(s)\right] + \int_0^s \frac{\partial}{\partial s}\, Q_2(s,\, \tau) \\
& \times \left[A_2(f) + \sum_{j=3}^{m} A_2(1)\, x_j(\tau)\right]\Bigg) \exp\left(-\int_s^t Q_2(s,\, \tau)\, A_2(1)\, \mathrm{d}u_2(\tau)\right) \mathrm{d}u_2(s).
\end{aligned}
$$

On putting (12) into (10) and using the fact that $x_j(t)$ are nondecreasing, we obtain

$$
\tag{13}
\begin{aligned}
y(t) \leqslant{}& A_2(f) + A_2(1) \int_0^t \Bigg(Q_2(s,\, s)\left[A_2(f) + \sum_{j=3}^{m} A_2(1)\, x_j(s)\right] \\
& + \int_0^s \frac{\partial}{\partial s}\, Q_2(s,\, \tau)\left[A_2(f) + \sum_{j=3}^{m} A_2(1)\, x_j(\tau)\right]\Bigg) \\
& \times \exp\left(-\int_s^t Q_2(s,\, \tau)\, A_2(1)\, \mathrm{d}u_2(\tau)\right) \mathrm{d}u_2(s) + \sum_{j=3}^{m} A_2(1)\, x_j(t) \\
\leqslant{}& A_2(f) + A_2(1) \int_0^t \left(Q_2(s,\, s)\, A_2(f) + \int_0^s \frac{\partial}{\partial s}\, Q_2(s,\, \tau)\, A_2(f) \right) \\
& \times \exp\left(-\int_0^s Q_2(s,\, \tau)\, A_2(1)\, \mathrm{d}u_2(\tau)\right) \mathrm{d}u_2(s) \\
& + \sum_{j=1}^{m} \Bigg[A_2(1) + A_2(1) \int_0^t \left(Q_2(s,\, s)\, A_2(1) + \int_0^s \frac{\partial}{\partial s}\, Q_2(s,\, \tau)\, A_2(1) \right) \\
& \times \exp\left(-\int_0^s Q_2(s,\, \tau)\, A_2(1)\, \mathrm{d}u_2(\tau)\right) \mathrm{d}u_2(s) \Bigg] x_j(t) \\
={}& A_3(f) + \sum_{j=3}^{m} A_3(1)\, x_j(t)
\end{aligned}
$$

where $A_3(f)$ and $A_3(1)$ are as defined in (4).

If we set $i = m-1$, then we easily obtain

$$
\tag{14}
y(t) \leqslant A_m(f) + A_m(1)\, x_m(t).
$$

Next, suppose $i = m$, then (7) and (12) implies

$$Dx_m(t) = Q_m(t,\, t)\, y(t)\, Du_m(t) + \int_0^t \frac{\partial}{\partial t} Q_m(t,\, s)\, y(s)\, Du_m(s)$$

$$\leqslant \left(Q_m(t,\, t)\,[A_m(f) + A_m(1)\, x_m(t)] + \int_0^t \frac{\partial}{\partial t} Q_m(t,\, s)[A_m(f) + A_m(1)\, x_m(s)] \right) Du_m(t).$$

Thus

$$\tag{15} \begin{aligned} Dx_m(t) - &\left(Q_m(t,\, t)\, A_m(1)\, x_m(t) + \int_0^t \frac{\partial}{\partial t} Q_m(t,\, s)\, A_m(1)\, x_m(s) \right) Du_m(t) \\ &\leqslant \left(Q_m(t,\, t) A_m(f) + \int_0^t \frac{\partial}{\partial t} Q_m(t,\, s)\, A_m(f) \right) Du_m(t). \end{aligned}$$

Multiply both sides of (11) by $\exp\left(-\int_0^t Q_m(t,\, s)\, A_m(1)\, \mathrm{d}u_m(s) \right)$ and integrate with respect to t from 0 to t and noting that $x_m(0) = 0$, we have

$$\tag{16} \begin{aligned} x_m(t) \leqslant \int_0^t &\left(Q_m(s,\, s)\, A_m(f) + \int_0^s \frac{\partial}{\partial s} Q_m(s,\, \tau)\, A_m(f) \right) \\ &\times \exp\left(-\int_s^t Q_m(t,\, \tau)\, A_m(1)\, \mathrm{d}u_m(\tau) \right) \mathrm{d}u_m(s). \end{aligned}$$

Substituting (15) into (14) and noting that $x_m(t)$ is nondecreasing, we obtain

$$\begin{aligned} y(t) \leqslant A_m(f) + A_m(1) \int_0^t &\left(Q_m(s,\, s)\, A_m(f) + \int_0^s \frac{\partial}{\partial s} Q_m(s,\, \tau)\, A_m(f) \right) \\ &\times \exp\left(-\int_s^t Q_m(t,\, \tau)\, A_m(1)\, \mathrm{d}u_m(\tau) \right) \mathrm{d}u_m(s). \end{aligned}$$

This completes the proof of the theorem.

As an immediate consequence of Theorem 2.1, we have the following result if we set $Q_j(t,\, s) = g_j(t) h_j(s)$, $j = 1, 2, \ldots, m$.

Theorem 2.2. *Suppose that for* $j = 1, \dots, m$ *and* $t, s \in [0, T]$,

1. $g_j(t) \geq 0$, $y(t) \geq 0$ *and* $g_j(t)$, $y(t)$, $f(t) \in BV[0, T]$.
2. $u_j(t)$ *are nondecreasing in* t.
3. $h_j(t)$ *are nonnegative and integrable with respect to* $u_j(t)$.

If the following inequality

$$y(t) \leq f(t) + \sum_{j=1}^{m} g_j(t) \int_0^t h_j(s)\, y(s)\, \mathrm{d}u_j(s) \tag{17}$$

holds, then

$$y(t) \leq A_m(f) + A_m(1) \int_0^t g_m(s)\, h_m(s)\, A_m(f) \exp\left(\int_s^t g_m(\tau)\, h_m(\tau)\, \mathrm{d}u_m(\tau) \right) \mathrm{d}u_m(s). \tag{18}$$

Proof. This follows directly from the proof of Theorem 2.1 and the details are omitted.

Remark 2.1. *We observed that Theorem* 2.2 *is not essentially the same as Theorem* 2.1 *in* [**6**] *in the sense that Theorem* 2.2 *contains Theorem* 2.1 *in* [**6**] *as a special case. Indeed Theorem* 2.2 *is more general than Theorem* 2.1 *in* [**6**].

If we set $m = 1$ in Theorem 2.2 and noting that $A_m(v) = v$, then we have the following result

Theorem 2.3. *Let* $Q(t, s) = g(t)\, h(s)$ *for all* $t, s \in [0, T]$ *and assume that*

1. $g(t) \geq 0$, $y(t) \geq 0$ *and* $g(t)$, $y(t)$, $f(t) \in BV[0, T]$.
2. $u(t)$ *is nondecreasing in* t.
3. $h(t)$ *is nonnegative and integrable with respect to* $u(t)$.

If the following inequality

$$y(t) \leq f(t) + g(t) \int_0^t h(s)\, y(s)\, \mathrm{d}u(s) \tag{19}$$

holds, then

$$y(t) \leqslant f(t) + \int_0^t g(s)\, h(s)\, f(s) \exp\left(\int_s^t g(\tau)\, h(\tau)\, \mathrm{d}u(\tau) \right) \mathrm{d}u(s)\,. \tag{20}$$

Proof. Follow directly from the proof of Theorem 2.1 and the details are omitted.

Remark 2.2. *In Theorem* 2.3, *setting* $g(t) = 1$, *then Theorem* 2.3 *reduces to Theorem* 3.2 *of* [**5**] *in the sense that Theorem* 2.2 *contains Theorem* 2.1 *in* [**6**] *as a special case. Indeed Theorem* 2.2 *is more general than Theorem* 2.1 *in* [**6**].

3 - Application

In this section, we shall give one application of our results which is sufficient to convey the usefulness of our results in the study of differential equations.

Let us consider the differential equation (1) where $f: [0, \infty) \times \Re^n \to \Re^n$. Suppose $S \subset \Re^n$ is an open set and $I \subset \Re$ is an interval with the left endpoint $t \geqslant 0$.

We shall say that the function $x(.) = x(. , t_0 . x_0)$ is a solution of (1) through the point $(t_0, x_0) \in I$ if $x(.)$ is a right continuous function of bounded variation in S with $x(t_0) = x_0$ and the distributional derivative of $x(.)$ on (t_0, α) for any $\alpha \in I$ satisfies (1).

Under the hypothesis that for each $x(.) \in BV(I, S)$, $f(t, x(t))$ is integrable in the sense of the Lebesgue-Stieltjes measure du, then it easily follows that $x(.)$ is a solution of (1) through the point (t_0, x_0) on $J = [t_0, t_0 + a]$ if and only if it satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))\, \mathrm{d}s, \qquad t \in J\,. \tag{21}$$

As an application of our results in the study of the uniqueness property of the solution of differential equations, we shall assume that f is Lipschitz continuous and that $x(t)$ and $y(t)$ are two possible solutions of (1) through the same point (t_0, x_0), i.e. $x(t_0) = x_0 = y(t_0)$.

If we put $z(t) = |x(t) - y(t)|$, then it easily follows that $z(t_0) = 0$ and we obtain from (21) that $z(t) \leq L \int_{t_0}^{t} z(s)\, d\nu_u(s)$, where ν_u is the total variation of u and L is the Lipschitz constant.

The proof is complete if in Theorem 2.3, we set $f(t) = 0$, $g(t) = 1$ and $h(t) = L$.

4 - Acknowledgement

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Abstract

The object of this paper is to establish a new Gronwall-Bellman type integral inequalities for multi-distributions. These inequalities generalize some results of Zhihong and Yongquing obtained in [**6**].

* * *