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Solutions of the Vlasov equation in a slab with source terms on the boundaries (**)

1 - Introduction

In this paper we study the evolution of a system of particles with mass m and charge q, not subject to scattering and moving in a one dimensional bounded region under the influence of a constant electric field E. We also assume that there are sources of particles at the boundaries of the region. This problem is modeled by the Vlasov equation. Thus, this study may be applied in semiconductors physics when considering a time scale much shorter than the mean time between two consecutive scattering events (see [**7**]).

Our study is based on techniques of analysis for elliptic operators and on the theory of semigroups of linear and affine operators. By means of this kind of approach we are able not only to prove existence and uniqueness of the solution of the problem, but also to write *its explicit form* and consider *its approximations*.

It is worth observing that Bardos ([**1**]) and Beals and Protopopescu ([**4**]) have studied this kind of problem too. The former assumes bounded coefficients for general linear differential equations and solves them by trajectories methods. He also establishes some results on numerical approximations. Bardos studies the problem only for a bounded range of velocity, whereas in our paper the velocity v varies over the whole space $\mathbb{R}$. The latter study the Boltzmann equation for semi-

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conductors by means of trajectories methods, too. Even if their statements are given under general assumptions, they prove the generation of a semigroup only in the case of *dissipative* conditions on the boundary and *without* sources terms. Moreover, the solution we find is different and in some sense more general than the one found by Beals and Protopopescu by means of the trajectories methods because we use abstract techniques.

In section **2** we describe the model and we define the functional space and the operator A_t that we need to write the abstract form of our problem. We also recall the definition of affine operator and we give the general abstract form of the solutions both in the time dependent and time independent boundary source cases ([**2**] and [**3**]). Successively, we introduce a sequence of operators which approximate the operator A_t and we prove convergence theorems for the sequence of operators that we have defined.

In section **3** we prove the fundamental theorem of this paper the generation of a strongly continuous contraction semigroup by using the sequence of operators defined in section **2**. The proof of the theorem is based on a paper by Lunardi and Vespri ([**6**]) about the generation of strongly continuous contraction semigroups by elliptic operators with unbounded coefficients. Following [**6**] and by means of duality arguments we prove also that this theorem holds in L^p spaces with $1 \leq p < \infty$.

In section **4** we show that the assumptions of the approximation theorem due to Trotter ([**9**]) are fulfilled by the sequence of operators defined in section **2**. Hence, we are able to state the convergence of the semigroups generated by the sequence of operators to the semigroup which gives the solution of an abstract form of our problem.

Finally, in section **5** we give the explicit form of the solution of the problem and we show that it is possible to consider also its approximations, as done by Bardos in [**1**].

2 - The problem

We study the evolution of a system of charged particles, with mass m and charge q, moving in a slab under the influence of a constant electric field E. Moreover, we assume that there is some kind of particle injection at the boundaries of the slab. As we disregard scattering events, the problem is modeled by the following Vlasov equation:

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} + a \frac{\partial u}{\partial v} = 0 \,, \tag{1}$$

where $a = qE/m$ is constant and $u = u(x, v, t)$ represents the density of the particles which at time t are in a position $x \in [-b, +b]$ and have velocity $v \in \mathbb{R}$.

The initial and boundary conditions needed to solve this problem read as follows:

$$u(x, v, 0) = u_0(x, v), \qquad x \in [-b, +b], \qquad v \in \mathbb{R}, \tag{2}$$

and

$$\begin{cases} u(-b, v, t) = q_1(v, t), & v > 0, \quad t \geq 0, \\ u(+b, v, t) = q_2(v, t), & v < 0, \quad t \geq 0, \end{cases} \tag{3}$$

where the non negative functions $q_1(v, t)$ and $q_2(v, t)$ represent sources of particles at the boundaries $x = -b$ and $x = +b$, respectively.

In order to write the problem in an abstract form, we introduce the set $\Omega = [-b, +b] \times \mathbb{R}$ and we consider the Banach space $X = L^2(\Omega)$, endowed with the usual L^2 norm, $\|f\|_2 = \|f\|$ for $f \in X$.

Moreover, let us define the operator:

$$A_t f = -v \frac{\partial f}{\partial x} - a \frac{\partial f}{\partial v}, \tag{4}$$

with domain and range:

$$\begin{gathered} D(A_t) = \left\{ f \in X, \ v \frac{\partial f}{\partial x} \in X, \ \frac{\partial f}{\partial v} \in X, \ f \text{ satisfies (3)} \right\}, \\ R(A_t) \subset X. \end{gathered} \tag{5}$$

We remark that A_t is a nonlinear time dependent operator because of the nonhomogeneous boundary conditions which appear in the definition of its domain. Furthermore, the source $q_1(\cdot, t) \in L^2(0, +\infty)$ for every $t \geq 0$ and analogously $q_2(\cdot, t) \in L^2(-\infty, 0)$ for every $t \geq 0$.

The abstract form of problem (1), (2) and (3) reads as follows:

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = A_t u(t), & t > 0 \\ u(0) = u_0 \in D(A_0), \end{cases} \tag{6}$$

where $u(\cdot, \cdot, t)$ is now a function defined on $[0, +\infty)$ with values in X.

In order to prove that (6) has a unique strongly continuous solution, we shall consider the following auxiliary linear operator L:

$$Lf = -v \frac{\partial f}{\partial x} - a \frac{\partial f}{\partial v}, \tag{7}$$

with domain and range:

$$D(L) = \left\{ f \in X,\ v \frac{\partial f}{\partial x} \in X,\ \frac{\partial f}{\partial v} \in X,\ f(x, v)\,|_{\partial \Omega} = 0 \right\}, \qquad R(L) \subset X. \tag{8}$$

It is easy to prove that A_t is an affine operator associated to L. In fact,

$$f_1 - f_2 \in D(L), \quad \forall f_1,\ f_2 \in D(A_t), \qquad A_t(f+g) = A_t\, f + Lg, \quad \forall f \in D(A_t),\ g \in D(L), \tag{9}$$

(see [**2**], [**3**] for details and for (10)-(13)).

We remark that the physical meaning of the operator L has no relevance. As a matter of fact, we need it in order to apply the theory of affine operators, to prove that the problem (6) has a unique solution and to derive its explicit form.

In fact, if we prove that the operator L is the generator of a semigroup $T(t)$, then the solution of the Cauchy problem (6), in the case of time independent sources q_1 and q_2, can be written as follows:

$$u(t) = u_0 + \int_0^t T(s)\, A_t u_0 \, \mathrm{d}s, \tag{10}$$

where we recall that $u_0 \in D(A_t)$.

Moreover, if there exists a function $p = p(x, v) \in D(A_t)$ such that $A_t p = 0$, then (10) simplifies to:

$$u(t) = p + T(t)(u_0 - p). \tag{11}$$

On the other hand, if the source terms $q_1(t)$ and $q_2(t)$ are time dependent, then the solution of problem (6) can be written as follows:

$$u(t) = p(t) + T(t)[u_0 - p(0)] + \int_0^t T(t-s)[A_t p(s) - p'(s)]\, \mathrm{d}s, \tag{12}$$

where $p(t) = p(\cdot, \cdot, t)$ is a function from $\Omega \times [0, t_0)$ in X, $(t_0 \leq +\infty)$ such that $p(t) \in D(A_t)$. Furthermore, if $p(t)$ is such that $A_t p(s) - p'(s) = 0$, where p' is a strong derivative, then (12) becomes:

$$u(t) = p(t) + T(t)[u_0 - p(0)]. \tag{13}$$

We shall give an example of the function p in the case of time independent sources q_1 and q_2 in section **5**.

By using (9) and the choice axiom, it is possible to prove that every affine operator A_t associated to a linear operator L has the representation

$$D(A_t) = p(t) + D(L), \tag{14}$$

where $p(t)$ is a suitable function belonging to $D(A_t)$.

The representation (14) is not unique because we might have $D(A_t) = p(t) + D(L)$ as well as $D(A_t) = p_1(t) + D(L)$ for some other function $p_1(t)$. However, it can be proved that this fact does not affect all the results we have quoted.

Moreover, if the operator L, which A_t is affine to, is such that its closure $\overline{L}$ generates a strongly continuous semigroup, we can define $\widetilde{A}_t$, an extension of A_t, in the following way

$$\begin{aligned} &D(\widetilde{A}_t) = p(t) + D(\overline{L}) = \{f \in X,\ f = p(t) + f_0,\ f_0 \in D(\overline{L})\} \\ &\widetilde{A}_t f = A_t p(t) + \overline{L} f_0 \end{aligned} \tag{15}$$

where $p(t)$ is the function used to represent A_t by means of (14).

Thus, we can apply formulas (10), (11), (12), (13), where now $T(t)$ is the semigroup generated by $\overline{L}$ and the abstract evolution problem is (6) with A_t replaced by $\widetilde{A}_t$.

We now consider the following sequence of linear operators L_n which approximate L. For every $n \in \mathbb{N}$:

$$L_n f = \frac{k}{n} \Delta f + Lf = \frac{1}{n} \Delta f + Lf, \tag{16}$$

with domain and range:

$$\begin{aligned} D(L_n) = D(L) \cap \{f \in X,\ \Delta f \in X,\ f(x, v)\,|_{\partial\Omega} = 0\}, \\ R(L_n) \subset X, \end{aligned} \tag{17}$$

where Δ is to be considered with respect to x and v, and where k is a dimensional constant which for simplicity we consider equal to 1.

Let us now define also the operators $A_{t,n}$ which are the affine operators associated to L_n and which approximate the operator A_t. For every $n \in \mathbb{N}$:

$$A_{t,n} f = \frac{1}{n} \Delta f + A_t f, \tag{18}$$

with domain and range:

$$\begin{gathered} D(A_{t,n}) = D(A_t) \cap \{ f \in X,\ \Delta f \in X,\ f \text{ satisfies (3)} \}, \\ R(A_{t,n}) \subset X. \end{gathered} \tag{19}$$

It is worth to remark that $D = D(L_n)$ and $\mathcal{D}_t = D(A_{t,n})$ are *independent* of n. The following two lemmas hold:

Lemma 2.1. *Let L and L_n be defined by* (7) *and* (16), *then we have*:

$$\lim_{n \to \infty} \| L_n f - Lf \| = 0 \qquad \forall f \in D = D(L_n) \subset D(L).$$

Proof. For every $f \in D$, we have:

$$\| L_n f - Lf \| = \left\| \frac{1}{n} \Delta f + Lf - Lf \right\| = \frac{1}{n} \| \Delta f \|,$$

which tends to 0 as n goes to ∞. ■

Lemma 2.2. *Let A_t and $A_{t,n}$ be defined by* (4) *and* (18), *then for every* $t \geq 0$:

$$\lim_{n \to \infty} \| A_{t,n} f - A_t f \| = 0 \qquad \forall f \in D(A_{t,n}) \subset D(A_t).$$

Proof. As in the previous lemma, the proof follows from the definition of $D(A_{t,n})$. ■

3 - Generation of the semigroup

In this section we prove the generation of a strongly continuous semigroup of contractions by the operator L_n by using the techniques of [**6**]. By using this result, it will follow that the closure of the linear operator L, $\overline{L}$ is the generator of a strongly continuous semigroup of contractions (i.e. $\overline{L} \in \mathcal{G}(1, 0; X)$), [**5**] and [**8**]).

Theorem 3.1. *The operator* L_n *defined by* (16), (17) *is the generator of a strongly continuous semigroup of contractions, that is* $L_n \in \mathcal{G}(1, 0; X)$.

Proof. The proof is a rather simplified version of the proofs of [**6**] with $\Omega = \mathbb{R}^n$. We give it here for sake of completeness. Let us fix $n \in \mathbb{N}$ and consider the following bilinear form associated to the operator L_n:

$$\widehat{a}(f, \varphi) = \langle L_n f, \varphi \rangle = -\frac{1}{n} \int_{-b}^{+b} \int_{\mathbb{R}} \left(\frac{\partial f}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial \varphi}{\partial v} \right) \mathrm{d}x \, \mathrm{d}v + \int_{-b}^{+b} \int_{\mathbb{R}} \left(-v \frac{\partial f}{\partial x} - a \frac{\partial f}{\partial v} \right) \varphi \, \mathrm{d}x \, \mathrm{d}v \tag{20}$$

for every $f = f(x, v) \in D(L_n)$ and $\varphi = \varphi(x, v) \in W_0^{1,2}(\Omega) \subset D(L_n)$.

As $W_0^{1,2}(\Omega)$ is dense in X, for every $f \in D(L_n)$ the map $\varphi \mapsto \widehat{a}(f, \varphi)$ can be extended with continuity to X in such a way that there exists one and only one $h \in X$ such that $\widehat{a}(f, \varphi) = \langle h, \varphi \rangle$; thus $\widehat{a}(f, \varphi) = 0 \ \forall \varphi$ if and only if $f = 0$. It follows that, for every $\lambda > 0$ and for every $g \in X$, $f \in D(L_n)$ is a solution of the resolvent equation:

$$\lambda f - L_n f = g \, , \tag{21}$$

if and only if for every $\varphi \in W_0^{1,2}(\Omega)$ we have:

$$\frac{1}{n} \int_{-b}^{+b} \int_{\mathbb{R}} \left(\frac{\partial f}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial \varphi}{\partial v} \right) \mathrm{d}x \, \mathrm{d}v + \int_{-b}^{+b} \int_{\mathbb{R}} \left(v \frac{\partial f}{\partial x} + a \frac{\partial f}{\partial v} + \lambda f \right) \varphi \, \mathrm{d}x \, \mathrm{d}v = \int_{-b}^{+b} \int_{\mathbb{R}} g \varphi \, \mathrm{d}x \, \mathrm{d}v \, . \tag{22}$$

Following [**6**], we now approximate $v \in \mathbb{R}$ by means of some bounded v_m as follows, given $m \in \mathbb{N}$:

$$v_m = \begin{cases} v & \text{if } |v| \leq m \\ \dfrac{mv}{|v|} & \text{otherwise} \, . \end{cases} \tag{23}$$

If we define the bilinear form $\widehat{a}_m(f_m, \varphi)$ as the bilinear form $\widehat{a}(f, \varphi)$ (20) in which v is replaced by v_m, then $\widehat{a}_m$ is continue and coercive on $H^1(\Omega) = W^{1,2}(\Omega)$.

Moreover, replacing in equation (21) v by v_m and L_n by $L_{n,m}$, where $L_{n,m}$ is defined as L_n with v replaced by v_m and $D(L_{n,m}) = D(L_n)$, we have that the resolvent equation:

$$\lambda f_m - L_{n,\,m} f_m = g \ ,$$

thanks to the Lax-Millgram theorem, has a unique solution $f_m \in D(L_{n,\,m})$ for every $g \in X$, where f_m are the solutions corresponding to the problem with the truncated velocities v_m. Therefore, replacing φ with the solution f_m in (22), we obtain:

$$\begin{aligned}(24)\qquad & \frac{1}{n} \int_{-b}^{+b}\int_{\mathbb{R}} \left(\frac{\partial f_m}{\partial x} \frac{\partial f_m}{\partial x} + \frac{\partial f_m}{\partial v} \frac{\partial f_m}{\partial v} \right) \mathrm{d}x\, \mathrm{d}v \\ & + \int_{-b}^{+b}\int_{\mathbb{R}} \left(v_m \frac{\partial f_m}{\partial x} + a \frac{\partial f_m}{\partial v} + \lambda f_m \right) f_m \,\mathrm{d}x\,\mathrm{d}v = \int_{-b}^{+b}\int_{\mathbb{R}} g f_m \,\mathrm{d}x\,\mathrm{d}v \ .\end{aligned}$$

Since f_m must belong to $D(L_{n,\,m})$, we have from (24):

$$(25)\qquad \frac{1}{n} \|\nabla f_m\|^2 + \lambda \|f_m\|^2 \leq \|g\| \, \|f_m\| \ .$$

Following the same arguments of [**6**] it is possible to prove that, being f_m equi-bounded functions in $H^1(\Omega)$, they converge weakly to f, which is the solution of (21). Therefore we also have:

$$(26)\qquad \frac{1}{n} \|\nabla f\|^2 + \lambda \|f\|^2 \leq \|g\| \, \|f\| \ ,$$

from which we get:

$$(27)\qquad \|f\| \leq \frac{1}{\lambda} \|g\| \ ,$$

which is the well known Hille-Yosida estimate. Thus, $L_n \in \mathcal{G}(1,\,0;\,X)$. ■

We remark that by the same techniques of [**6**] it is possible to prove the generation of a strongly continuous contraction semigroup also when the coefficients of the first order derivatives appearing in the definition of L_n are unbounded functions of x and v with not more than linear growth at ∞.

It is also possible to prove that L_n generates a strongly continuous contraction semigroup in $L^p(\Omega)$ with $p > 2$. On the other hand, in $L^\infty(\Omega)$ there is not generation of semigroup because the domain is not dense in X, even if estimate (27) still holds (see [**6**]).

Nevertheless, by using estimate (27) in $L^\infty(\Omega)$ we are able to state that L_n

generate a strongly continuous semigroup also in $L^1(\Omega)$. This fact has a precise physical meaning because in $L^1(\Omega)$ the norm of the density function $u(x, v, t)$ gives the total number of particles present at time t in the region Ω. Hence, from the generation of a strongly continuous contraction semigroup, follows that it is possible to have bounds of the total number of particles which are in Ω at time t by means of the total number of particles which are in Ω at time 0, $\|u_0\|$.

Define the linear operator $L_{n;1}$:

$$L_{n;1} f = \frac{1}{n} \Delta f + Lf \qquad \forall f \in L^1(\Omega)$$

$$D(L_{n;1}) = D(L) \cap \{ f \in L^1(\Omega),\ \Delta f \in L^1(\Omega), \quad f(x, v)\,|_{\partial\Omega} = 0 \}, \tag{28}$$

$$R(L_{n;1}) \subset L^1(\Omega).$$

we remark that $L_{n;1} f = L_n f$, where now in the definition of $D(L_n)$ the space X is $L^1(\Omega)$.

Theorem 3.2. *The operator $L_{n;1}$ defined as in* (28) *is the generator of a strongly continuous semigroup of contractions, i.e.* $L_{n;1} \in \mathcal{G}(1, 0; L^1(\Omega))$.

Proof. Define the dual operator of $L_{n;1}$, L_n^*, as follows:

$$L_n^* f = \frac{1}{n} \Delta f - Lf, \tag{29}$$

with domain and range:

$$\begin{aligned} D(L_n^*) &= D(L) \cap \{ f \in L^\infty(\Omega),\ \Delta f \in L^\infty(\Omega) \}, \\ R(L_n^*) &\subset L^\infty(\Omega). \end{aligned} \tag{30}$$

We remark that in definition (30) the space X in $D(L)$ is $L^\infty(\Omega)$.

By the same procedure used to show that equation (21) has a solution in $L^p(\Omega)$ we can prove that the equation $\lambda f - L_n^* f = g$ has a solution f in $L^\infty(\Omega)$ for every $g \in L^\infty(\Omega)$ and that estimate (27) holds (see [**6**]).

Let $h \in L^1(\Omega)$ and, for every $\lambda > 0$, consider the following resolvent equation for the operator $L_{n;1}$:

$$\lambda w - L_{n;1} w = h\,. \tag{31}$$

If $h \in C_0^\infty(\Omega)$, by using again the results of [**6**], there exists a unique solution

$w \in C_0^\infty(\Omega)$ of the resolvent equation (31). Thus, we have the existence of the solution of (31) in $L^1(\Omega)$, because $C_0^\infty(\Omega)$ is dense in $L^1(\Omega)$. If f is the solution of $\lambda f - L_n^* f = g$, by means of duality arguments we have:

$$\|w\|_1 = \sup_{g \in L^\infty(\Omega),\, \|g\|_\infty \leq 1} \int_{-b}^{+b} \int_{\mathbb{R}} wg \,\mathrm{d}x\,\mathrm{d}v$$

$$= \int_{-b}^{+b} \int_{\mathbb{R}} w(\lambda - L_n^*)\, f \,\mathrm{d}x\,\mathrm{d}v$$

$$= \int_{-b}^{+b} \int_{\mathbb{R}} (\lambda - L_{n,1})\, wf \,\mathrm{d}x\,\mathrm{d}v$$

$$= \int_{-b}^{+b} \int_{\mathbb{R}} hf \,\mathrm{d}x\,\mathrm{d}v$$

$$\leq \|h\|_1 \|f\|_\infty$$

$$\leq \|h\|_1 \frac{1}{\lambda} \|g\|_\infty \leq \frac{1}{\lambda} \|h\|_1,$$

which is the Hille-Yosida estimate in $L^1(\Omega)$. ∎

We remark that every statement that we shall prove holds in $L^p(\Omega)$ spaces for every p such that $1 \leq p < \infty$.

4 - Convergence of the semigroup generated by L_n

In order to prove the convergence of the semigroup generated by L_n to the semigroup generated by the closure $\overline{L}$ of L, we first prove the lemmas below.

Lemma 4.1. *The resolvent sequence $\{R(\lambda, L_n)\}$ strongly converges as n goes to ∞ for any given $\lambda > 0$.*

Proof. Let $\varepsilon > 0$ be fixed, $f \in D$ and $n, m \in \mathbb{N}$ such that $n > m$. Given any

$\lambda > 0$, since $D(L_n) = D$ does not depend on n, we have:

$$\|(\lambda I - L_n)^{-1} f - (\lambda I - L_m)^{-1} f\|$$

$$= \|(\lambda I - L_n)^{-1}[f - (\lambda I - L_n)(\lambda I - L_m)^{-1} f]\|$$

$$\leqslant \frac{1}{\lambda} \|[(\lambda I - L_m) - (\lambda I - L_n)](\lambda I - L_m)^{-1} f\|$$

$$= \frac{1}{\lambda} \|(\lambda I - L_m - \lambda I + L_n)(\lambda I - L_m)^{-1} f\|$$

$$= \frac{1}{\lambda} \|(\lambda I - L_m)^{-1}(L_n - L_m) f\|$$

$$\leqslant \frac{1}{\lambda} \frac{1}{\lambda} \|L_n f - L_m f\| < \frac{\varepsilon}{\lambda^2},$$

where the last inequality holds because $\{L_n f\}$ is convergent in a Banach space (as proved in Lemma 2.1), and so it is a Cauchy sequence. Hence, it follows that $\{R(\lambda, L_n) f\}$ is a Cauchy sequence, and therefore it converges if $f \in D$. Since D is dense in X and the resolvent operator $R(\lambda, L_n)$ are uniformily bounded, this result holds for any $f \in X$. ■

Lemma 4.2. *The operator $(I - \alpha L_n)^{-1}$, where $\alpha > 0$, strongly converges to the identity operator I as $\alpha \to 0^+$, uniformly with respect to n.*

Proof. By taking into account estimate (27), we have for every $f \in D = D(L_n)$:

$$\|(I - \alpha L_n)^{-1} f - f\|$$

$$= \|(I - \alpha L_n)^{-1}[f - (I - \alpha L_n) f]\|$$

$$\leqslant \|(I - \alpha L_n)^{-1}\| \|f - f + \alpha L_n f\|$$

$$= \left\| \left[\alpha \left(\frac{I}{\alpha} - L_n \right)\right]^{-1} \right\| \|\alpha L_n f\|$$

$$= \left\| \frac{1}{\alpha} \left(\frac{1}{\alpha} I - L_n \right)^{-1} \right\| \|\alpha L_n f\|$$

$$\leqslant \frac{1}{\alpha}\, \alpha \|\alpha L_n f\| = \alpha \left(\left\| \frac{1}{n} \Delta f \right\| + \|Lf\| \right) \leqslant \alpha (\|\Delta f\| + \|Lf\|)\,,$$

which approaches 0 as $\alpha \to 0^+$, uniformly in $n \in \mathbb{N}$. Since D is dense in X, the above result holds for every $f \in X$. ■

Theorem 4.1. *The closure $\overline{L}$ of the operator L generates a strongly continuous semigroup of contractions in X. Moreover, for every $t \geqslant 0$, we have:*

$$\lim_{n \to \infty} \|\exp(tL_n)\, f - \exp(t\overline{L})\, f\| = 0 \qquad \forall f \in X\,,$$

and the above limit is uniform in t for t in bounded intervals.

Proof. From Lemma 4.1 and 4.2, it follows that there exists an operator $\widehat{L}$ such that $\widehat{L} \in \mathcal{G}(1,\, 0;\, X)$ and the semigroup generated by L_n, $\exp(tL_n)$, strongly converges to the semigroup $\exp(t\widehat{L})$ generated by $\widehat{L}$, (see [**5**] and [**9**]).

If we prove that $(\lambda I - L)\, D$ is dense in X for a fixed $\lambda > 0$, we can apply a theorem of [8], (Theorem 4.5, page 88) and have that $\widehat{L} = \overline{L}$.

Thus, let $\lambda > 0$ be fixed, let $\overline{g} = \overline{g}(x,\, v) \in C = C_0^\infty(\Omega)$ and define the following function

$$\text{(32)} \quad \begin{aligned} &\overline{f}(x,v) \\ &= + \frac{1}{a} \exp\left(\frac{-\lambda v}{a} \right) \int_{-\gamma}^{v} \overline{g}\left(x - \frac{v^2}{2a} + \frac{v'^2}{2a}, v' \right) \exp\left(\frac{\lambda v'}{a} \right) \mathrm{d}v', \; v<0 \\ &\overline{f}(x,v) \\ &= - \frac{1}{a} \exp\left(\frac{-\lambda v}{a} \right) \int_{v}^{+\gamma} \overline{g}\left(x - \frac{v^2}{2a} + \frac{v'^2}{2a}, v' \right) \exp\left(\frac{\lambda v'}{a} \right) \mathrm{d}v', \; v>0, \end{aligned}$$

where $\gamma = \sqrt{2a \left| x + b - \dfrac{v^2}{2a} \right|}$. It is easy to see that $\overline{f} \in D$ and

$$\text{(33)} \qquad (\lambda I - L)\, \overline{f} = \overline{g}\,.$$

Thus, $(\lambda I - L)\, D \supset C$ and $(\lambda I - L)\, D$ is dense in X because C has this property. ■

5 - Approximation and solution

In section 4 we have proved that the linear operator $\overline{L}$ generates a strongly continuous semigroup of contractions.

On the other hand, the operator A_t is an affine operator associated to the operator L, so it is possible to define an extension $\widetilde{A}_t$ of A_t, such that $\widetilde{A}_t$ is affine to $\overline{L}$, see (15). Then, relations (9) holds for the operator $\overline{L}$ and $\widetilde{A}_t$ and we can use (10) and (13).

For example, in the time independent source case, the solution of problem (6) with $\widetilde{A}_t$ in place of A_t is given by

$$u(t) = p + \exp(t\overline{L})(u_0 - p). \tag{34}$$

It follows, from Theorem 4.1, that $\exp(tL_n)$ approximates $\exp(t\overline{L})$. Hence, given p such that $A_t p = 0$, if we find a sequence $\{p_n\} \subset D(A_{t,\,n})$ converging to p, we have that the following sequence

$$u_n(t) = p_n + \exp(tL_n)(u_0 - p_n) \tag{35}$$

converges to the solution (34). A similar result can be proved in the time dependent source case.

As regards the explicit form of the function p appearing in (11) we have the following proposition:

Proposition 5.1. *In the time independent sources case, if* $q_1 = q_1(\cdot) \in C^1(0, \infty) \cap L^2(0, \infty)$ *and if* $q_2 = q_2(\cdot) \in C^1(-\infty, 0) \cap L^2(-\infty, 0)$, *then the solution of* (6) *is written explicitly as in* (34), *where* p *is given by*

$$p(x, v) = \begin{cases} q_1(\sqrt{2a|-x-b+v^2/2a|}) & \text{if } v > 0 \\ q_2(-\sqrt{2a(-x+b+v^2/2a)}) & \text{if } v < 0\,. \end{cases} \tag{36}$$

Proof. It is easy to check that the function $p(x, v)$ given by (36) satisfies the equation $A_t p = 0$. This can be done by using the following transformations:

$$\begin{cases} \hat{x} = |x + b - v^2/2a| \\ \hat{v} = v \end{cases} \tag{37}$$

for $v > 0$, and

$$\begin{cases} \hat{x} = x - b - v^2/2a \\ \hat{v} = v \end{cases} \tag{38}$$

for $v < 0$.

We consider only the case $v > 0$ with $x + b - v^2/2a \geq 0$, the cases $v > 0$ with $x + b - v^2/2a \leq 0$ and $v < 0$ are analogous. With the above change of variables, defining $\Phi(\hat{x}, \hat{v}) = p(\hat{x} - b + \hat{v}^2/2a, \hat{v})$, equation $A_t p = 0$ reads as follows:

$$a \frac{\partial \Phi}{\partial \hat{v}} = 0 , \tag{39}$$

with boundary condition:

$$\Phi(-v^2/2a, v) = q_1(v) , \qquad v > 0 .$$

By integrating (39) with respect to $\hat{v}$ and considering the boundary condition and the change of variable (37) we have that the function $p(x, v)$ solution of $A_t p = 0$ is given by (36). We remark that $p(x, v) \in D(A_t)$ owing to the regularity assumptions made on q_1 and q_2. ■

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Abstract

We consider the one-dimensional Vlasov equation in a slab with source terms on the boundaries. The study is based on techniques of analysis for elliptic operators and on the theory of semigroups of linear and affine operators. Existence and uniqueness of the solution of the problem are proved and an approximation to this solution is also given.

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