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Stability of the origin of scalar comparison equations, I (**)

Dedicated to the memory of G. L. Braglia

Introduction

In this paper we consider the scalar differential equation

$$\dot{u} = a(t)\, f(u)\,, \tag{1.1}$$

where $a\colon \mathbb{R}^+ \mapsto \mathbb{R}$ and $f\colon [0,\, U) \mapsto \mathbb{R}^+$, $0 < U \leqslant \infty$, are continuous functions, and $f(u) > 0$ on $(0,\, U)$.

Equation (1.1) is the most usual scalar comparison equation that we obtain by applying the comparison method [**2**], [**3**] to the Liapunov's direct method [**7**], [**8**]. This justifies the search for sufficient and necessary conditions for the Liapunov stability of the origin $u = 0$ of equation (1.1), i.e. for the stability of the set $\{0\} \subset \mathbb{R}^+$. This paper is one of a series on the stability of scalar comparison equations.

Throughout the paper we denote by $u(t,\, t_0,\, u_0)$ a solution for the Cauchy problem

$$\dot{u} = a(t)\, f(u)\,, \qquad u(t_0) = u_0\,, \tag{1.2}$$

with t_0 on $\mathbb{R}^+$ and u_0 on $[0,\, U)$. Moreover, by $I = [t_0,\, \omega)$, $t_0 < \omega \leqslant \infty$, we denote the right maximal interval of existence of the solution $u(t) = u(t,\, t_0,\, u_0)$.

Assume that $u(t) > 0$ on an interval $J \subset I$. This implies that $f(u(t)) > 0$ on J.

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Hence from (1.1) we obtain, by separating variables,

$$\int_{u(t')}^{u(t'')} \frac{ds}{f(s)} = \int_{t'}^{t''} a(s)\, ds, \qquad \forall t', t'' \in J. \tag{1.3}$$

Then define

$$A(t) = \exp\left\{-\int_0^t a(s)\, ds\right\}, \qquad \forall t \in \mathbb{R}^+.$$

Moreover, if $\int_0^u \frac{ds}{f(s)} < \infty$ for every $u \in (0, U)$, define

$$F(u) = \exp\left\{\int_0^u \frac{ds}{f(s)}\right\}, \qquad \forall u \in [0, U), \tag{1.4}$$

otherwise, put

$$F(u) = \exp\left\{\int_{\overline{u}}^u \frac{ds}{f(s)}\right\}, \qquad \forall u \in (0, U), \tag{1.5}$$

where $\overline{u}$ is a suitable constant on $(0, U)$.

From (1.3) we obtain so that

$$A(t'')\, F(u(t'')) = A(t')\, F(u(t')), \qquad \forall t', t'' \in J. \tag{1.6}$$

Remark 1.1. If $\int_0^u \frac{ds}{f(s)} = \infty$ for every $u \in (0, U)$, it is easy to show that there exists a unique solution $u(t)$ for the Cauchy problem (1.2) (see also [**6**]). This implies that $u(t) > 0$ on I, provided that $u_0 > 0$. Thus from (1.6) we get

$$F(u(t)) = \frac{A(t_0)\, F(u_0)}{A(t)}, \qquad \forall t \in I, \tag{1.7}$$

where $F(u)$ is given by (1.5).

In Section 2 we investigate the case where $\int_0^u ds/f(s) = \infty$, for every

$u \in (0, U)$. Such assumption enable us to give necessary and sufficient stability conditions (Theorems 2.1, 2.2, 2.3) that, although not completely original, improve and complete classical results (e.g. [**5**], Chap. 3). It is worth noting that all results of Section 2 continue to hold when $f(u)$ is not defined on the origin (see Example 2.3).

In Section 3 we assume that

$$a(t)\, f(0) \geq 0\,, \qquad \forall t \in \mathbb{R}^+, \tag{1.8}$$

together with $\int_0^u \frac{ds}{f(s)} < \infty$ for every $u \in (0, U)$. Under such assumptions we show that the origin of (1.1) is stable if and only if $a(t) \leq 0$ on $\mathbb{R}^+$ (Theorem 3.1). Otherwise we prove that the origin can be eventually uniformly stable (Theorem 3.2).

Remark 1.2. Condition (1.8) ensures that if a solution $u(t)$ is defined on the right maximal interval of existence $I = [t_0, \omega)$ with $\omega < \infty$, then necessarily (see Nagumo's theorem [**1**], Chap. 4) $u(t) \to U$ as $t \to \omega^-$. It is worth noting that (1.8) automatically holds in the important case where $f(0) = 0$.

2 - Stability criteria

Throughout this section, and without further mention, we shall suppose that

$$\int_0^u \frac{ds}{f(s)} = \infty\,, \qquad \forall u \in (0, U).$$

Then $F(u)$, defined by (1.5), is a strictly increasing function mapping the interval $(0, U)$ homeomorphically onto the interval $(0, F(U))$, where $F(U) = \lim_{u \to U} F(u)$. Thus $F(u)$ may be considered a function belonging to class K in the sense of Hahn [**4**].

Theorem 2.1. *The origin of* (1.1) *is stable if and only if there exists a constant* $c > 0$ *such that*

$$A(t) \geq c\,, \qquad \forall t \in \mathbb{R}^+. \tag{i}$$

Proof. Let us suppose that condition (i) holds and let t_0 on $\mathbb{R}^+$ and ε on

$(0, U)$ be given. Then, since $c \leqslant A(t_0)$, a constant $\delta = \delta(t_0, \varepsilon)$ on $(0, \varepsilon]$ exists such that $F(\delta) = (c/A(t_0))\, F(\varepsilon)$. Fix arbitrarily u_0 on $(0, \delta)$. By (1.7) we have

$$F(u(t)) < \frac{A(t_0)\, F(\delta)}{A(t)} = \frac{c}{A(t)}\, F(\varepsilon) \qquad \forall t \in I\,,$$

where $u(t) = u(t, t_0, u_0)$. Hence, in view of condition (i),

$$F(u(t)) < F(\varepsilon) \qquad \forall t \in I\,.$$

This implies that $u(t) < \varepsilon$ on I. Hence $I = [t_0, \infty)$, and the origin of (1,1) is stable.

Conversely, suppose the origin of (1.1) be stable. Take ε on $(0, U)$, and denote by δ a constant on $(0, \varepsilon]$ such that $u(t, 0, \delta) \leqslant \varepsilon$ for every $t \geqslant 0$. Since $A(t_0) = A(0) = 1$, from (1.7) we obtain so that

$$A(t) \geqslant \frac{F(\delta)}{F(\varepsilon)} \qquad \forall t \in \mathbb{R}^+\,,$$

and thus condition (i) holds by $c = F(\delta)/F(\varepsilon)$. This completes the proof.

Theorem 2.2. *The origin of* (1.1) *is uniformly stable if and only if there exists a constant* $k \geqslant 1$ *such that*

(i) $$A(t') \leqslant kA(t''), \qquad \forall t' \geqslant 0, \quad \forall t'' \geqslant t'.$$

Proof. Let us suppose that condition (i) holds, and let ε on $(0, U)$ be given. Then a constant $\delta = \delta(\varepsilon)$ on $(0, \varepsilon]$ exists such that $F(\delta) = F(\varepsilon)/k$. By (1.7) we have, for every $t_0 \geqslant 0$ and u_0 on $(0, \delta)$,

$$F(u(t)) < \frac{A(t_0) F(\delta)}{A(t)} = \frac{A(t_0)}{kA(t)} F(\varepsilon) \qquad \forall t \in I\,.$$

On the other hand, condition (i) ensures that $A(t_0) \leqslant kA(t)$ for every $t \geqslant t_0$. Thus

$$F(u(t)) < F(\varepsilon)\,, \qquad \forall t \in I\,.$$

i.e. $u(t) < \varepsilon$ on I and $I = [t_0, \infty)$. Since δ not depends on t_0, the origin of (1.1) is uniformly stable.

Conversely, assume the origin be uniformly stable. Fix ε on $(0, U)$. Then there exists a constant $\delta = \delta(\varepsilon)$ on $(0, \varepsilon]$ such that $u(t, t_0, \delta) \leqslant \varepsilon$ for every $t_0 \geqslant 0$ and

$t \geq t_0$. Then from (1.7) there follows that

$$A(t_0) \leq \frac{F(\varepsilon)}{F(\delta)} A(t), \qquad \forall t_0 \geq 0, \quad \forall t \geq t_0,$$

and so condition (i) is satisfied by putting $k = F(\varepsilon)/F(\delta)$. This completes the proof of Theorem 2.2.

Corollary 2.1. *Suppose that there exist two constants $c_1 > 0$ and c_2 such that*

(i) $$c_1 \leq A(t) \leq c_2, \qquad \forall t \in \mathbb{R}^+ .$$

Then the orign of (1,1) *is uniformly stable.*

Proof. It is clear that condition (i) of Theorem 2.2 is satisfied by $k = c_2/c_1$.

Theorem 2.3. *The origin of* (1.1) *is equi-asymptotically stable if and only if*

(i) $$A(t) \to \infty \quad as \quad t \to \infty .$$

Proof. We recall that the origin is called equi-asymptotically stable if it is stable and equi-attractive [7].

Assume that condition (i) holds. Then Theorem 2.1 ensures that the origin of (1.1) is stable. So it is enough to prove that it is equi-attractive, i.e.

(2.1) $$\forall t_0 \geq 0, \quad \exists r \in (0, U), \quad \forall \varepsilon \in (0, U), \quad \exists T > 0, \quad \forall u_0 \in (0, r), \quad \forall t \geq t_0 + T$$

$$u(t, t_0, u_0) < \varepsilon .$$

Fix arbitrarily $t_0 \geq 0$. Since the origin is stable, there exists a constant $r = r(t_0)$ on $(0, U)$ such that all solutions $u(t) = u(t, t_0, u_0)$ with $u_0 < r$ exist globally in the future. Then if $u_0 \in (0, r)$ we have, in view of (1.7),

(2.2) $$F(u(t)) < \frac{A(t_0)\, F(r)}{A(t)} \qquad \forall t \geq t_0.$$

Moreover, however we take ε on $(0, U)$ condition (i) ensures that there is a constant $T = T(t_0, r, \varepsilon)$ for which

$$A(t) \geqslant \frac{A(t_0)\, F(r)}{F(\varepsilon)}, \qquad \forall t \geqslant t_0 + T .$$

Thus, taking into account (2.2), we obtain that $F(u(t)) < F(\varepsilon)$ on $[t_0 + T, \infty)$. This prove that condition (i) is sufficient for equi-asymptotic stability.

Conversely, assume the origin of (1.1) equi-asymptotically stable. Thus, by the definition (2.1) of equi-attractivity, all solutions starting from $u_0 \in (0, r)$ tend to 0 uniformly with respect to u_0. On the other hand (1.7) implies

$$A(t) = \frac{A(t_0)\, F(u_0)}{F(u(t))}, \qquad \forall t \geqslant t_0,$$

and so $A(t) \to \infty$ as $t \to \infty$. This completes the proof.

Example 2.1. Let us consider the scalar equation

$$\dot{u} = e^{t}(\sin 2t - \cos^2 t)\log(1+u).$$

Since $A(t) = \exp\{e^t \cos^2 t - 1\}$, we can apply Theorem 2.1, but we cannot apply Theorem 2.2 or Theorem 2.3.

Example 2.2. Let us consider the equation

$$\dot{u} = (e^{-t} - e^{t}\cos^2 t)\, u .$$

Since $\int_{t'}^{t''} a(s)\, ds \leqslant 1$ for every $t' \geqslant 0$ and $t'' \geqslant t'$, then condition (i) of Theorem 2.2 holds by choosing $k = e$. On the other hand $A(t)$ is not bounded, and so we cannot apply Corollary 2.1.

Example 2.3. Let us consider the equation

$$\dot{u} = e^{t}(1 - 5\sin^2 t)\, e^{-1/u}.$$

By applying Theorem 2.3 we recognize that the origin is equi-asymptotically stable. Moreover, Theorem 2.2 ensures that the stability is not uniform.

3 - Eventual stability

It will be supposed in this section that

$$a(t)\, f(0) \geqslant 0, \qquad \forall t \in \mathbb{R}^+,$$

and

$$\int_0^u \frac{ds}{f(s)} < \infty, \qquad \forall u \in (0, U).$$

Then $F(u)$, defined by (1.4), is now a strictly increasing function mapping the interval $[0, U)$ homeomorphically onto the interval $[1, F(U))$.

Theorem 3.1. *The origin of* (1.1) *is stable if and only if* $a(t) \leqslant 0$ *on* $\mathbb{R}^+$.

Proof. It is clear that when $a(t) \leqslant 0$ on $\mathbb{R}^+$ then the origin of (1.1) is stable. Thus we limit ourselves to show that when there is $\bar{t} > 0$ for which $a(\bar{t}) > 0$ then the origin of (1.1) is unstable.

In fact, take $J = [t_1, t_2]$ such that $\bar{t} \in J$ and $a(t) > 0$ on J. Moreover, fix t_0 on (t_1, t_2) and choose ε on $(0, U)$ for which

$$F(\varepsilon) \leqslant \frac{A(t_0)}{A(t_2)}. \tag{3.1}$$

Notice that such ε exists since $A(t_2) < A(t_0)$. Suppose for contradiction that there is u_0 on $(0, U)$ such that $u(t, t_0, u_0) = u(t) < \varepsilon$ on $[t_0, t_2]$. On the other hand $u(t) > 0$ on $[t_0, t_2]$, since it is here increasing. Then we can apply (1.6), putting $t' = t_0$ and $t'' = t_2$. Taking into account that $F(u(t_2)) < F(\varepsilon)$, we obtain so that

$$A(t_0)\, F(u_0) < A(t_2)\, F(\varepsilon)$$

which contradicts (3.1) since $F(u_0) > 1$. This completes the proof.

Theorem 3.2. *The origin of* (1.1) *is eventually uniformly stable if and only if*

(i) $\quad \forall \sigma > 1, \quad \exists T \geqslant 0, \quad \forall t' \geqslant T, \quad \forall t'' \geqslant t' \quad A(t') < \sigma A(t'')$.

Proof. The origin is called eventually uniformly stable if [8]

(3.2) $\quad \forall \varepsilon \in (0, U), \quad \exists T \geqslant 0, \quad \exists \delta \in (0, \varepsilon), \quad \forall t_0 \geqslant T, \quad \forall u_0 < \delta, \quad \forall t \geqslant t_0$

$$u(t, t_0, u_0) < \varepsilon,$$

where $u(t, t_0, u_0)$ denotes the maximal solution of (1.1) satisfying $u(t_0) = u_0$.

Fix arbitrarrily ε on $(0, U)$. Condition (i) ensures that there are two constants $T \geqslant 0$ and $\delta \in (0, \varepsilon)$ such that

$$A(t') < \frac{F(\varepsilon)}{F(\delta)} A(t''), \qquad \forall t' \geqslant T, \quad \forall t'' \geqslant t'. \tag{3.3}$$

Take $t_0 \geqslant T$ and u_0 on $(0, \delta)$, and put $u(t) = u(t, t_0, u_0)$. Suppose for contradiction that there exists $\bar{t} > t_0$ for which $u(\bar{t}) = \varepsilon$. Thus we can choose t' on $(t_0, \bar{t})$ such that $u(t') = \delta$ and $\delta \leqslant u(t) \leqslant \varepsilon$ on $[t', \bar{t}]$. Hence from (1.6) one has, putting $t'' = \bar{t}$,

$$A(t')\, F(\delta) = A(\bar{t})\, F(\varepsilon)$$

which contradicts (3.3) since $\bar{t} > t' \geqslant T$. So condition (i) ensures eventual uniform stability of the origin.

Assume now the origin not eventually uniformly stable, namely

$$\exists \varepsilon \in (0, U),\ \forall T \geqslant 0,\ \forall \delta \in (0, \varepsilon),\ \exists t_0 \geqslant T,\ \exists u_0 \in (0, \delta),\ \exists \bar{t} > t_0\colon u(\bar{t}, t_0, u_0) = \varepsilon. \tag{3.4}$$

Take ε satisfying (3.4) and take arbitrarily a constant σ on the interval $(1, F(\varepsilon))$. Moreover, fix arbitrarily $T \geqslant 0$ and δ on $(0, \varepsilon)$. Without loss of generality constant δ can be chosen so smal l that $F(\delta) \leqslant F(\varepsilon)/\sigma$. Assumption (3.4) ensures that there exist $t_0 \geqslant T$, $t' > t_0$ and $\bar{t} > t'$ such that $u(t') = \delta$, $u(\bar{t}) = \varepsilon$ and $\delta \leqslant u(t) \leqslant \varepsilon$ on $[t', \bar{t}]$. From (1.6) there follows that

$$A(t') = \frac{F(\varepsilon)}{F(\delta)} A(\bar{t}) \geqslant \sigma A(\bar{t}).$$

Hence, $\exists \sigma > 1$, $\forall T \geqslant 0$, $\exists t' \geqslant T$, $\exists \bar{t} > t'$: $A(t') \geqslant \sigma A(\bar{t})$. We have so showed that condition (i) is untrue. This completes the proof of Theorem 3.2.

Easily we show that the following sufficient conditions for eventual uniform stability are corollaries of Theorem 3.2.

Corollary 3.1. *If there exists a constant $C > 0$ such that $A(t) \to C$ as $t \to \infty$, then the origin of* (1.1) *is eventually uniformly stable.*

Corollary 3.2. *Assume that the continuous function $a_+(t) = \max\{a(t), 0\}$, for every $t \in \mathbb{R}^+$, belongs to class $L^1(\mathbb{R}^+)$. Then the origin of* (1.1) *is eventually uniformly stable.*

Remark 3.1. When $a_+(t)$ belongs to $L^1(\mathbb{R}^+)$ also condition (i) of Theorem 2.2 holds. Consequently, by applying Theorem 2.2 we show that: *if* $\int_0^u (ds/f(s))) = \infty$ *for every* $u > 0$, *and the continuous function* $a_+(t)$ *belongs to class* $L^1(\mathbb{R}^+)$, *then the origin of* (1.1) *is uniformly stable.*

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Summary

Stability criteria of the origin $u = 0$ *of the scalar differential equation* $\dot{u} = a(t)\, f(u)$ *are provided. The present paper is one of a series on the stability of scalar comparison equation. The above mentioned equation is the most usual comparison equation that we obtain by applying the comparison method to Liapunov's direct method. Many applications to the stability of the origin of Lagrange equations of holonomic mechanical systems will be provided in next papers.*

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