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Maximum modulus, maximum term and approximation error of an entire harmonic function in R^3 (**)

Introduction

The harmonic functions in R^3 are the solutions of the Laplace equation

$$\frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2} + \frac{\partial^2 H}{\partial x_3^2} = 0 . \tag{0.1}$$

A harmonic functions regular about the origin can be expanded as

$$H \equiv H(r, \theta, \phi) = \sum_{n=0}^{\infty} r^n \sum_{m=0}^{n} (a_{nm}^{(1)} \cos m\phi + a_{nm}^{(2)} \sin m\phi)\, P_n^m(\cos\theta) , \tag{0.2}$$

where $x_1 = r\cos\theta$, $x_2 = r\sin\theta\cos\phi$, $x_3 = r\sin\theta\sin\phi$ and $P_n^m(t)$ are associated Legendre's functions of the first kind of degree m and order n. A harmonic polynomial of degree k is a polynomial of degree k in x_1, x_2 and x_3 which satisfies (0.1).

A harmonic function is said to be regular in $D_R = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < R^2,\ 0 < R \leq \infty\}$ if the series (0.2) converges uniformly on every compact subset of D_R. The harmonic function is called entire if it is regular in D_∞. The order

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ϱ and type T of an entire harmonic function are defined as

$$\limsup_{r\to\infty} \frac{\log\log M(r, H)}{\log r} = \varrho(H) \equiv \varrho\,, \qquad 0 \leqslant \varrho \leqslant \infty\,,$$

and, for functions having order $\varrho(0 < \varrho < \infty)$,

$$\limsup_{r\to\infty} \frac{\log M(r, H)}{r^{\varrho}} = T(H) \equiv T, \qquad 0 \leqslant t \leqslant T\,,$$

where $M(r, H) = \max_{x_1^2 + x_2^2 + x_3^2 = r^2} |H(x_1, x_2, x_3)|$.

Let H_R, $0 < R < \infty$ denote the class of all harmonic functions H regular in D_R and continuous on D_R, the closure of D_R. For $H \in H_R$, let $E_n(H, R)$, the error in approximating the function H by harmonic polynomials in uniform norm, be defined by

$$E_n(H, R) = \inf_{g\in\pi_n} \|H - g\|_R,$$

where π_n consists of all polynomials of degree at most n and

$$\|H - g\|_R = \max_{x_1^2 + x_2^2 + x_3^2 \in \overline{D}_R} |H(x_1, x_2, x_3) - g(x_1, x_2, x_3)|\,.$$

One way of characterizing the growth of an entire harmonic function in terms of approximation error is to relate $E_n(H, R)$ with order ϱ and T. Various authors ([**3**], [**4**], [**5**], [**6**]) established a relation between the growth parameters and appoximation error of an entire harmonic function, but as compared to the direct estimates of $E_n(H, R)$ or $M(r, H)$ these are still rather crude.

The purpose of this paper is to set up more precise interrelation between $M(r, H)$, $\mu(r, H)$ and $E_n(H, R)$ for entire harmonic functions of relatively slow growth, in terms of direct estimates for these quantities. Here, we call an entire harmonic function slowly increasing if $M(r, H)$ increases essentially, not faster than

$$\exp\,(c(\beta - 1))\left[\frac{\log(r/r'R)}{2\beta}\right]^{\beta/(\beta-1)} \tag{0.3}$$

for $\beta = 2$ and an arbitrary $c > 0$ (critical value $\beta = 2$ has been found to be significant). For rapidly increasing functions there are direct estimates of $M(r, H)$, $\mu(r, H)$ and $E_n(H, r)$. If H increases like (0.3) with $1 < \beta < 2$, for example Proposition 2 and Corollary 1 apply, and for still more rapidly increasing harmonic functions of classical order $\varrho > 0$, Proposition 1 and Theorem 2 apply. Though in the later case a necessary and sufficient characterization of the growth of H in terms of $E_n(H, R)$ is possible, the results are still sharper than the limit relations mentioned above.

1 - Auxiliary results

Here, we mention two results which have been utilized in the text:

Lemma 1. *Let $H \in H_R$ be entire and $r' > 1$. Then, for all $r > 2r'R$ and sufficiently large values of n, we have*

$$E_n(H, R) \leqslant KM(r, H)\left(\frac{r'R}{r}\right)^{n+1},$$

where K is a constant.

The *proof* of the lemma is left to the reader.

Lemma 2. *Let $H \in H_R$. Then, for $R_* < R$ and $n \geqslant 1$, there exists an entire function $h(z)$ such that*

$$h(z) = \sum_{n=1}^{\infty} (2n+1)^2 E_{n-1}(H, R)\left(\frac{z}{R_*}\right)^n$$

and

$$M(r, H) \leqslant |a_{00}^{(1)}| + K_0 M(r, h),$$

where $M(r, h) = \max_{|z|=r} |h(z)|$.

The *proof* of this lemma follows form [3], Lemma 4.

2 - Main results

I - Maximum modulus and approximation error

We first prove two propositions for a class of entire harmonic functions which include all H of order $\varrho > 0$. Then, we restrict ourselves to entire harmonic functions of slow growth in order to obtain a characterization theorem of desired precision.

We denote by $C^2[x, \infty)$ the class of twice continuously differentiable functions on $[x, \infty)$ and for any $\alpha \in C^2[x, \infty)$ with $\alpha'' > 0$, set

$$A(r) = \exp\{\log r(\alpha')^{-1}(\log r) - \alpha((\alpha')^{-1})(\log r))\}. \tag{2.1}$$

Proposition 1. *Let $H \in H_R$. Then H has an analytic continuation as an entire harmonic function and let $\alpha \in C^2[x, \infty)$ be such that $\alpha'(x) \to \infty$ and $\alpha''(x) > 0$ for each $x \geqslant x_1$. If $M(r, H) = O(A(r/r'R))$, $r \to \infty$, then we have*

$$E_n(H, R) = O\left(\frac{1}{\psi(n+1)}\right), \qquad n \to \infty, \tag{2.2}$$

where $\psi(x) = \exp \alpha(x)$, $x \geqslant x_1$.

Proof. By Lemma 1, we have for $r' > 1$ and $r > 2r'R$,

$$E_n(H, R) \leqslant KM(r, H)\left(\frac{r'R}{r}\right)^{n+1}.$$

For $r/r'R = \exp \alpha'(n+1)$, above inequality corresponds to (2.2).

We define Γ to be the class of functions $\alpha \in C^2[x, \infty)$ for some $x_1 \geqslant 0$ for which there exists a function ω such that $x - \omega(x) \geqslant x_1$, $x \geqslant x_1$.

$$\lim_{x\to\infty} \alpha'(x) = \infty, \qquad \lim_{x\to\infty} \omega^2(x)\,\alpha''(x) = \infty, \qquad \lim_{x\to\infty} \alpha''(x) = 0$$

and $\alpha''(x + \delta\omega(x)) \simeq \alpha''(x)$ as $x \to \infty$ on δ such that $|\delta| < 1$.

Proposition 2. *Let $H \in H_R$ have analytic continuation as an entire harmonic function satisfying* (2.2) *for some ψ such that $\alpha(x) = \log \psi(x) \in \Gamma$. Then*

$$M\left(\frac{r}{r'R}\right) = O\left(\left[\alpha''\left((\alpha')^{-1}\left(\frac{\log r}{r'R}\right)\right)\right]^{-1/2} A\left(\frac{r}{r'R}\right)\right), \qquad r \to \infty. \tag{2.3}$$

Proof. Using Lemma 2, we have

$$M(r, H) \leqslant |a_{00}^{(1)}| + K_0 \sum_{n=1}^{\infty} (2n+1)^2 E_{n-1}(H, R)\left(\frac{r}{R_*}\right)^n.$$

Since $(2n+1)^{2/n} \to 1$ as $n \to \infty$ and $r' > 1$, it follows that $(2n+1)^2 \leqslant (r')^n$ for

$n \geqslant n_0(r')$. Thus, in vew of (2.2),

$$
\begin{aligned}
M(r, H) &\leqslant |a_{00}^{(1)}| + K_0 \sum_{n=1}^{\infty} E_{n-1}(H, R)\left(\frac{rr'}{R_*}\right)^n \\
&\leqslant O\left(|a_{00}^{(1)}| + K_0 \sum_{n=0}^{\infty}\left(\frac{rr'}{R_*}\right)^{n+1} \psi(n+1)\right).
\end{aligned} \tag{2.4}
$$

Define $h(x, t) = xt - \alpha(t)$. The right side estimate of (2.4) is evaluated as

$$
\sum_{n=1}^{\infty} \frac{1}{\psi(n+1)}\left(\frac{rr'}{R_*}\right)^{n+1} = \sum_{n=0}^{\infty} \exp\left[h\left(\log\left(\frac{rr'}{R_*}\right)\right), n+1\right], \qquad r \to \infty .
$$

Since $\alpha \in \Gamma$, the asymptotic relation

$$
\sum_{n=0}^{\infty}\left\{h\left(\log\left(\frac{rr'}{R_*}\right), n+1\right)\right\} \simeq \sqrt{2\pi}\left[\alpha''\left((\alpha')^{-1}\left(\log\left(\frac{r}{r'R}\right)\right)\right)\right]^{-1/2} A\left(\frac{r}{r'R}\right), \qquad r \to \infty ,
$$

follows easily by using [**1**] Theorem 28.3. This concludes (2.3).

Remark. The conclusions of Propositions 1 and 2 are best possible in the sense that O can not be replaced by o in (2.2) and (2.3), respectively. In case ψ grows at least as rapidly as $\exp(x^t)$, $t \geqslant 2$, this will be a consequence of Theorem 1 below. For Proposition 2 and general ψ with $\alpha \in \Gamma$ this is also clear from (2.4) by choosing H with $E_n(H, R) = 1/\psi(n+1)$.

Now, we consider the case of slowly increasing entire harmonic function expressed in terms of a particular $\psi(x)$ of the form $\psi(x) = \exp(cx^{\beta})$, $c > 0$, it means that the following theorem will cover the case $\beta \geqslant 2$ whereas Propositions 1 and 2 cover the cases $\beta > 1$ and $1 < \beta < 2$, respectively. Let $\overline{\Gamma}$ denote the class of functions $\alpha \in C^2[x, \infty)$, for some $x_1 \geqslant 0$ with $\lim_{x \to \infty} \alpha''(x) = \infty$, $\alpha''(x)$ exist for $x \geqslant x_1$ and $\lim_{x \to \infty} \alpha'''(x)(\alpha''(x))^{-3/2} = 0$.

Theorem 1. *Let $H \in H_R$ have an analytic continuation as an harmonic function for some $\alpha \in \overline{\Gamma}$ and let $A(r)$ be defined by* (2.1). *The following statements are equivalent*

(i) $M(r, H) = O(A(r/r'R))$, $r \to \infty$.
(ii) $E_n(H, R) = O(1/\psi(n+1))$, $n \to \infty$.

Proof. The (i)$\Rightarrow$(ii) is conteined in Proposition 1. For the converse, consider first the case when $\lim_{x \to \infty} \alpha''(x) = c > 0$ with $c \to \infty$. Then, the *proof* follows as in Proposition 2. If $\lim_{x\to\infty} \alpha''(x) = \infty$, it follows again that

$$(2.5)\quad M(r, H) = O\left[|a_{00}^{(1)}| + K_0 \sum_{n=0}^{\infty} \left(\frac{rr'}{R_*} \right)^{n+1} \frac{1}{\psi(n+1)} \right], \qquad r \to \infty \ \textit{and}\ R_* < R .$$

We have to show that the right side is $O(A(r/r'R))$ as $r \to \infty$. We set $\eta(x, \xi) = x\xi - \alpha(\xi)$, using a result of Sirovich [7], p. 96-98 and Evgrafov [2], p. 18

$$(2.6)\qquad \int_0^{\infty} e^{\eta(x, \xi)} \mathrm{d}\xi = e^{\eta(x, \xi_0(x))} \left\{ \frac{2\pi}{-n_{\xi\xi}(x, \xi_0(x))} \right\}^{1/2}, \qquad x \to \infty .$$

Here $\eta_{\xi\xi}$ denotes the second derivative with respect to ξ and $\xi_0(x) = (\alpha')^{-1}(x)$. The hypothesis of [5], p. 98, case 2, are satisfied since, for each $x > x_1$, $\eta(x, \xi)$, a function of ξ has a global maximum at $\xi_0 = \xi_0(x) = (\alpha')^{-1}(x)$, i.e., $\eta_{\xi}(x, \xi_0(x)) = 0$. Moreover, $\eta_{\xi}(x, \xi_0(x)) = -\alpha''((\alpha')^{-1}(x)) \neq 0$ and we have $(\alpha')^{-1}(x) \to \infty$, as $x \to \infty$, it follows by the definition of $\overline{\Gamma}$ that

$$-\lim_{x\to\infty} \eta_{\xi\xi}(x, \xi_0(x)) = \infty ,$$

as well as

$$-\lim_{x\to\infty} \eta_{\xi\xi}(x, \xi_0(x))\, \eta(x, \xi_0(x))^{3/2} = \infty .$$

Now, we set $x_r = \xi_0(\log(r/r'R))$ and $K_r = [x_r]$, where $[x_r]$ denotes the integral parts of x_r. Using (2.5), we have

$$\sum_{n=0}^{\infty} \left(\frac{rr'}{R_*} \right)^{n+1} \frac{1}{\psi(n+1)} = \sum_{n=0}^{\infty} \exp\left\{ \eta\left(\log\left(\frac{rr'}{R_*} \right), n+1 \right) \right\}$$

$$\simeq \sum_{n=0}^{\infty} \exp\left\{ \eta\left(\log\left(\frac{r}{r'R^*} \right), n+1 \right) \right\}, \qquad r \to \infty$$

$$\leqslant \int_0^{k_r} \exp\left\{ \eta\left(\log\left(\frac{r}{r'R} \right), \xi \right) \right\} \mathrm{d}\xi + \exp\left\{ \eta\left(\log\left(\frac{r}{r'R} \right), K_r \right) \right\}$$

$$+ \exp\left\{ \eta\left(\log\left(\frac{r}{r'R} \right), K_r + 1 \right) \right\} + \int_{K_{r+1}}^{\infty} \exp\left\{ \eta\left(\log\left(\frac{r}{r'R} \right), \xi \right) \right\} \mathrm{d}\xi$$

$$\leqslant 2 \exp\left\{ \eta\left(\log\left(\frac{r}{r'R} \right), x_r \right) \right\} + \int_0^{\infty} \exp\left\{ \eta\left(\log\left(\frac{r}{r'R} \right), \xi \right) \right\} \mathrm{d}\xi .$$

As a recourse of definition of η, (2.6) and (2.1), we obtain

$$M(r, H) = O\left(|a_{00}^{(1)}| + K_0 A\left(\frac{r}{r'R}\right)\left\{2+\sqrt{2}\pi\alpha''\left((\alpha')^{-1}\left(\log\left(\frac{r}{r'R}\right)\right)^{-1/2}\right)\right\}\right), \quad r \to \infty$$

holds if and only if

$$E_n(H, R) = O\left(\frac{1}{\exp(c(n+1))^{\beta}}\right), \qquad n \to \infty .$$

The asymptotic relation (2.6) is given by this for ψ, in the cases $\beta \in N$ and $\beta > 2$, respectively.

II - Maximum term and approximation error

Consider the function $h(z) = \sum_{n=1}^{\infty} (2n+1)^2 E_{n-1}(H, R)(z/R_*)^n$. Since $h(z)$ is an entire harmonic function and we know that h and H have same maximum term, we denote it by $\mu(r, H)$. A satisfactory characterization of $\mu(r, H)$ in terms of approximation error holds for large class of entire functions, including those of order $\varrho > 0$ and type $T \geq 0$.

Theorem 2. *Let $H \in H_R$ have an analytic continuation as an entire harmonic function with maximum term $\mu(r, H)$ and let α, ψ and $A(r)$ be given as usual. Then condition* (2.2) *is equivalent to* $\mu(r, H) = O(A(r/r'R))$, $r \to \infty$.

Proof. Let (2.2) be satisfied, so that for each $r > 0$, $n \geq n_0$,

$$(2.7) \qquad (2n+3)^2 E_n(H,R)\left(\frac{r}{r'R_*}\right)^{n+1} \leq M\left(\frac{r}{r'R_*}\right)^{n+1}\frac{1}{\psi(n+1)} \qquad \text{for } r' > 1 .$$

For fixed r, the maximum over x of the function $(r/r'R_*)^{n+1}/\psi(n+1)$ is attained at $x = (\alpha')^{-1}(\log(r/r'R_*))$ and has value $A(r/r'R_*)$, provided $(\alpha')^{-1}(\log(r/r'R_*)) > x_1$. Therefore,

$$\max_{n \geq n_0}(2n+3)^2 E_n(H, R)\left(\frac{r}{r'R_*}\right)^{n+1} \leq M \max_{n \geq n_0}\left(\frac{r}{r'R_*}\right)^{n+1}\frac{1}{\psi(n+1)} = MA\left(\frac{r}{r'R_*}\right),$$

for each $r > r_0$. Choosing $r_1 > r_0$ large enough, so that

$$E_{n_0}(H, R)\left(\frac{r}{r'R_*}\right)^{n+1} \geq \max\left\{E_n(H, R)\left(\frac{r}{r'R_*}\right)^{n+1}; \ 0 \leq n < n_0\right\}.$$

For each $r > r_1$, it also follows that $\mu(r, H) \leqslant MA(r/r' R_*)$, $r > r_1$. Conversely, (2.7) implies that

$$E_n(H, R) \leqslant \frac{MA(r/r' R_*)}{(r/r' R_*)^{n+1}}, \qquad r > r_0, \tag{2.8}$$

where r_0 and M are constant. Choosing $r/r' R_* = \exp \alpha'(n+1)$ for some $n \in N$ in (2.8), we have for n large enough, since $\alpha'(x) \to \infty$ as $x \to \infty$, and $R_* < R$ is arbitrary, this implies (2.2).

III - Maximum modulus and maximum term

Corollary 1. *Let $H \in H_R$ have an analytic continuation as an entire harmonic function satisfying $\mu(r, H) = O(A(r/r' R))$, $r \to \infty$ with $A(r)$ defined by* (2.1), *for $\alpha \in \Gamma$. Then*

$$M(r, H) = O\left((\alpha''(\alpha'))^{-1}\left(\log\left(\frac{r}{r' R}\right)\right)^{-1/2} A\left(\frac{r}{r' R}\right)\right), \qquad r \to \infty.$$

This is direct consequence of Theorem 2 and Proposition 2.

Corollary 2. *Let $H \in H_R$ be given as in Theorem* 1. *The following statements are equivalent*

(i) $\mu(r, H) = O(A(r/r' R))$, $r \to \infty$.
(ii) $M(r, H) = O(A(r/r' R))$, $r \to \infty$.

Theorems 1 and 2 combine to establish above equivalence and it is concerned with functions of slow growth only.

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Abstract

Generally, the growth of an entire harmonic function is measured in terms of order and type. Here, we have established the relations between maximum modulus, maximum term and approximation error in the form of direct estimates.

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