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A dissipative boundary condition in Electromagnetism: existence, uniqueness and asymptotic stability (**)

1 - Introduction

Let us consider an electromagnetic system, occupying a spatial region Ω and regard its boundary as a conductor. A boundary condition with memory has been introduced in [**1**], in order to prove the existence, uniqueness and asymptotic stability of the solution of Maxwell's equations. The magnetic field on the boundary is expressed through a functional on the history of the electric field.

For time-harmonic dependence, the tangential electric and magnetic fields are linked by the relation

$$\boldsymbol{E}(\boldsymbol{x}, \omega) = \eta(\boldsymbol{x}, \omega)\, \boldsymbol{H}(\boldsymbol{x}, \omega) \times \boldsymbol{n}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial\Omega, \tag{1.1}$$

where ω represents the angular frequence, $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$ and η is a suitable scalar which describes the conductivity feature of the boundary, usually given by

$$\eta(\omega) = \sqrt{\frac{\mu(\omega)}{\varepsilon(\omega) + i\sigma(\omega)/\omega}}\,.$$

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Letting $\omega \to \infty$, the previous equation gives Graffi's condition

$$\lim_{\omega \to \infty} \eta(\omega) = \sqrt{\frac{\mu}{\varepsilon}} .$$

For good conductors we can neglect $\omega\varepsilon$ in comparison with σ, thus obtaining Schelkunoff's relation [**5**]

$$\eta(\omega) = (1+i) \sqrt{\frac{\mu\omega}{2\sigma}} . \tag{1.2}$$

In [**1**] Fabrizio and Morro provide a generalization of the condition (1.1) for fields with arbitrary temporal dependence, introducing a hereditary model described by the boundary condition

$$\boldsymbol{E}(\boldsymbol{x}, t) = \lambda_0(\boldsymbol{x}) \boldsymbol{H}(\boldsymbol{x}, t) \times \boldsymbol{n}(\boldsymbol{x}) + \int_0^\infty \lambda(\boldsymbol{x}, s) \boldsymbol{H}^t(\boldsymbol{x}, s) \times \boldsymbol{n}(\boldsymbol{x}) \, ds , \qquad \boldsymbol{x} \in \partial\Omega . \tag{1.3}$$

Under the only hypothesis that $\lambda(\boldsymbol{x}, \cdot) \in L^1(\mathbb{R}^+)$, which ensures the property of fading memory, they prove that, when time-harmonic fields are considered, (1.3) reduces to (1.1) with

$$\eta(\boldsymbol{x}, \omega) = \lambda_0(\boldsymbol{x}) + \int_0^\infty \lambda(\boldsymbol{x}, s) \, e^{i\omega s} \, ds .$$

Thus $\eta(\boldsymbol{x}, \omega)$ satisfies Graffi's condition

$$\lim_{\omega \to \infty} \eta(\boldsymbol{x}, \omega) = \lambda_0(\boldsymbol{x}) .$$

In this paper we consider a boundary condition with memory like (1.3) in which the kernel $\lambda(\boldsymbol{x}, s)$ is such that $\eta(\omega)$ satisfies Schelkunoff's condition (1.2), namely

$$\lim_{\omega \to \infty} \eta(\omega) = \infty .$$

Such different assumptions on the memory kernel cause some difficulties in dealing with the asymptotic behaviour of the solution.

Moreover we assume that the boundary condition agrees with the dissipation principle of electromagnetic energy. Thus, in analogy with [**1**], we give the thermodynamic restrictions on the memory kernel which characterize a dissipative boundary.

The main part of the paper is devoted to the differential problem obtained from Maxwell's equations. In 4 we introduce the functional spaces which allow to define the weak solution of the problem. By introducing the concept of boundary free energy, we prove an existence and uniqueness theorem for the weak solution in the bounded domain $Q = \Omega \times (0, T)$.

Finally, in last Section, we prove a theorem of existence and uniqueness by letting $T \to \infty$ and we show that the weak solution is asymptotically stable if the sources satisfy suitable hypotheses of decay.

2 - Boundary condition

The behaviour of an electromagnetic system in a connected region $\Omega \subset \mathbb{R}^3$ is described by Maxwell's equations

$$\frac{\partial \boldsymbol{D}}{\partial t} = \nabla \times \boldsymbol{H} - \boldsymbol{J}\,, \qquad \nabla \cdot \boldsymbol{D} = \varrho\,, \tag{2.1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}\,, \qquad \nabla \cdot \boldsymbol{B} = 0\,. \tag{2.2}$$

The material occupying Ω is supposed to be linear, namely described by the constitutive equations

$$\boldsymbol{D}(\boldsymbol{x}, t) = \varepsilon(\boldsymbol{x})\, \boldsymbol{E}(\boldsymbol{x}, t)\,,$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = \mu(\boldsymbol{x})\, \boldsymbol{H}(\boldsymbol{x}, t)\,,$$

where $0 < \varepsilon_m \leqslant \varepsilon(\boldsymbol{x}) \leqslant \varepsilon_M$, $0 < \mu_m \leqslant \mu(\boldsymbol{x}) \leqslant \mu_M$. Moreover, for formal simplicity, we let the charge density ϱ vanish. The electric density of current $\boldsymbol{J}$ is given by the sum of a term due to the conduction feature of the material and a term due to external sources, namely

$$\boldsymbol{J} = \sigma \boldsymbol{E} + \boldsymbol{J}_f\,,$$

where $\boldsymbol{J}_f$ is regarded as a known function of $\boldsymbol{x}$ and t.

For greater generality, we introduce another source, called magnetic current, $\boldsymbol{I}_f$ and consider it as an assigned function. These hypotheses allow to rewrite the system (2.1)-(2.2) in the form

$$\varepsilon \dot{\boldsymbol{E}} = \nabla \times \boldsymbol{H} - \sigma \boldsymbol{E} - \boldsymbol{J}_f \qquad \nabla \cdot \varepsilon \boldsymbol{E} = 0 \tag{2.3}$$

$$\mu \dot{\boldsymbol{H}} = -\nabla \times \boldsymbol{E} + \boldsymbol{I}_f\,, \qquad \nabla \cdot \mu \boldsymbol{H} = 0\,. \tag{2.4}$$

The previous equations must be completed by suitable initial conditions:

$$\boldsymbol{E}(\boldsymbol{x},\,0)=\boldsymbol{E}_0(\boldsymbol{x})\,,$$

$$\boldsymbol{H}(\boldsymbol{x},\,0)=\boldsymbol{H}_0(\boldsymbol{x})\,.$$

Moreover, denoting by

$$\widehat{\boldsymbol{E}}^t(\boldsymbol{x},\,s)=\int\limits_{t-s}^{t}\boldsymbol{E}(\boldsymbol{x},\,\xi)\,\mathrm{d}\xi$$

the «integral history» of the electric field $\boldsymbol{E}$, we assume a boundary condition

Definition 2.1. *The boundary condition*

$$\Sigma\subset L^2(0,\,T,\,H^1(\Omega))\times L^2(0,\,T,\,H^1(\Omega))$$

is a set of fields $(\boldsymbol{E},\boldsymbol{H})$ *such that*

$$\boldsymbol{H}(\boldsymbol{x},\,t)\times\boldsymbol{n}(\boldsymbol{x})=\int\limits_0^\infty\lambda'(\boldsymbol{x},\,s)\,\widehat{\boldsymbol{E}}^t(\boldsymbol{x},\,s)\,\mathrm{d}s\qquad \boldsymbol{x}\in\partial\Omega \tag{2.5}$$

where $\widehat{\boldsymbol{E}}^t(\boldsymbol{x},\,\cdot)\in L^\infty(0,\,\infty)$, $\lambda'\in C(\Omega;\,L^1(0,\,\infty))$ *and* $s\lambda'(\boldsymbol{x},\,\cdot)\notin L^1(0,\,\infty)$. *The dual boundary condition of* Σ, *henceforth denoted by* $\Sigma^*\subset L^2(0,\,T,\,H^1(\Omega))\times L^2(0,\,T,\,H^1(\Omega))$ *is a set of fields* $(\boldsymbol{e},\boldsymbol{h})\in\Sigma^*$ *such that*

$$\int\limits_0^T\int\limits_{\partial\Omega}[\boldsymbol{e}(\boldsymbol{x},\,t)\times\boldsymbol{H}(\boldsymbol{x},\,t)\cdot\boldsymbol{n}(\boldsymbol{x})+\boldsymbol{E}(\boldsymbol{x},\,t)\times\boldsymbol{h}(\boldsymbol{x},\,t)\cdot\boldsymbol{n}(\boldsymbol{x})]\,\mathrm{d}a\,\mathrm{d}t=0$$

for each $\boldsymbol{E},\boldsymbol{H}\in\Sigma$.

Note that in the definition (2.5) the initial integral history $\widehat{\boldsymbol{E}}^0$ must be regarded as known and for simplicity it will be assumed $\widehat{\boldsymbol{E}}^0=0$.

If we consider time-harmonic fields in which the dependence on the angular frequence ω is given by $e^{-i\omega t}$, the constitutive equation (2.5) is expected to be

$$\boldsymbol{E}(\boldsymbol{x},\,\omega)=\eta(\boldsymbol{x},\,\omega)\,\boldsymbol{H}(\boldsymbol{x},\,\omega)\times\boldsymbol{n}(\boldsymbol{x})\,. \tag{2.6}$$

Indeed by substituting the representations

$$\boldsymbol{E}(\boldsymbol{x}, \omega) = \boldsymbol{E}_0(\boldsymbol{x})\, e^{-i\omega t},$$

$$\boldsymbol{H}(\boldsymbol{x}, \omega) = \boldsymbol{H}_0(\boldsymbol{x})\, e^{-i\omega t},$$

in (2.5) we obtain the identity

$$\boldsymbol{H}(\boldsymbol{x}, \omega) \times \boldsymbol{n}(\boldsymbol{x}) = \frac{\boldsymbol{E}(\boldsymbol{x}, \omega)}{i\omega} \int_0^\infty \lambda'(\boldsymbol{x}, s)(e^{i\omega s} - 1)\, ds\,. \tag{2.7}$$

In this way, comparison between (2.6) and (2.7) yields

$$\eta^{-1}(\boldsymbol{x}, \omega) = \frac{1}{i\omega} \int_0^\infty \lambda'(\boldsymbol{x}, s)(e^{i\omega s} - 1)\, ds\,.$$

According to the hypotheses about the kernel λ', it follows that η approaches zero as $\omega \to 0$. This is coherent with the static boundary condition

$$\boldsymbol{E}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x}) = 0\,.$$

Concerning the dissipativity of the boundary, we give the following

Definition 2.2. *A boundary condition is called locally dissipative if*

$$\int_0^d \boldsymbol{E}(\boldsymbol{x}, t) \cdot \boldsymbol{H}(\boldsymbol{x}, t) \times \boldsymbol{n}(\boldsymbol{x})\, dt \geq 0 \tag{2.8}$$

holds for every cycle of period d and $\boldsymbol{x} \in \partial\Omega$.

The following theorem ensures a necessary and sufficient condition for the dissipativity of the boundary.

Theorem 2.1. *The boundary condition* (2.5) *is locally dissipative if and only if*

$$\int_0^\infty \lambda'(\boldsymbol{x}, s) \sin \omega s\, ds \geq 0\,, \qquad \forall \boldsymbol{x} \in \Omega, \quad \omega \in \mathbb{R}^+. \tag{2.9}$$

Proof. In order to prove that the condition (2.9) is necessary, it suffices to consider the periodic field $\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{E}_0(\boldsymbol{x}) \cos \omega t$ with period $d = 2\pi/\omega$. In this way, we obtain

$$0 \leqslant \int_0^d \boldsymbol{E}(\boldsymbol{x}, t) \cdot \boldsymbol{H}(\boldsymbol{x}, t) \times \boldsymbol{n}(\boldsymbol{x}) \, dt$$

$$= \boldsymbol{E}_0^2(\boldsymbol{x}) \int_0^d \cos \omega t \int_0^\infty \lambda'(\boldsymbol{x}, s) \int_{t-s}^t \cos \omega \xi \, d\xi \, ds \, dt .$$

By integrating with respect to ξ and t, from the previous inequality it follows

$$\frac{\boldsymbol{E}_0^2(\boldsymbol{x})}{\omega^2} \int_0^\infty \lambda'(\boldsymbol{x}, s) \sin \omega s \, ds \geqslant 0 .$$

To show the sufficiency, we expand a generic periodic field in a Fourier series

$$\boldsymbol{E}(t) = \sum_{k=1}^\infty A_k \cos k\omega t + B_k \sin k\omega t .$$

Substitution in (2.7) yields the identity

$$\int_0^d \boldsymbol{E}(t) \cdot \boldsymbol{H}(t) \times \boldsymbol{n} \, dt = \frac{\pi}{\omega} \sum_{k=1}^\infty \frac{A_k^2 + B_k^2}{k} \int_0^\infty \lambda'(t) \sin k\omega t \, dt .$$

The inequality (2.9) implies that each term of the series is positive. Hence (2.8) holds. ∎

3 - Boundary free energy

In this section we define a functional, henceforth called boundary free energy, $\widehat{\psi}: \mathcal{D} \to \mathbb{R}^+$, where $\mathcal{D}$ is a subset of the history space, such that $L^\infty(0, \infty, L^2(\partial\Omega)) \subset \mathcal{D}$. The boundary free energy assigns to each history $\widehat{\boldsymbol{E}}^t$ the positive value $\psi(t) := \widehat{\psi}(\widehat{\boldsymbol{E}}^t)$ such that for each $t \in (0, T)$

$$\dot{\psi}(t) \leqslant \boldsymbol{E}(t) \cdot \boldsymbol{H}(t) \times \boldsymbol{n} , \tag{3.1}$$

where $\boldsymbol{H}(\boldsymbol{x}) \times \boldsymbol{n}$ is expressed in terms of $\widehat{\boldsymbol{E}}^t$ by (2.5).

The definition of the functional ψ is not unique: here we exhibit two possible choices, namely

$$\psi_1(t) = \frac{1}{2}\int_0^{\infty} \lambda'(s)\,|\widehat{\boldsymbol{E}}^t(s)|^2\,\mathrm{d}s\,,$$

$$\psi_2(t) = -\frac{1}{2}\int_0^{\infty}\int_0^{\infty} \lambda_{12}(|u_1 - u_2|)\,\widehat{\boldsymbol{E}}^t(u_1)\cdot\widehat{\boldsymbol{E}}^t(u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2,$$

where $\lambda_{12}(|u_1 - u_2|) := (\partial^2\lambda/\partial u_1\,\partial u_2)(|u_1 - u_2|)$.

The former needs some more hypotheses about the kernel λ', the latter requires only the assumptions given in Section 2.

It's clear that the positiveness of λ' is a sufficient condition in order that ψ_1 may be positive valued. To verify the estimate (3.1), we assume that the second derivative $\lambda''(s)$ exists and satisfies $\lambda''(s) \leqslant 0$ for each $s \in \mathbb{R}^+$.

Consider the identity

$$\dot{\psi}_1(t) = \boldsymbol{E}(t)\cdot\int_0^{\infty} \lambda'(s)\,\widehat{\boldsymbol{E}}^t(s)\,\mathrm{d}s - \int_0^{\infty} \lambda'(s)\,\widehat{\boldsymbol{E}}^t(s)\cdot\boldsymbol{E}(t-s)\,\mathrm{d}s\,.$$

Keeping (2.5) into account and integrating by parts, we obtain

$$\dot{\psi}_1(t) \leqslant \boldsymbol{E}(t)\cdot\boldsymbol{H}(t)\times\boldsymbol{n}\,.$$

Concerning ψ_2, we observe that, owing to the identity

$$\lambda_{12}(|u_1 - u_2|) = -\lambda''(|u_1 - u_2|) - 2\lambda'(|u_1 - u_2|)\,\delta(u_1 - u_2)$$

we can write ψ_2 in the equivalent form

$$\psi_2(t) = 2\int_0^{\infty} \widehat{\boldsymbol{E}}^t(u_1)\cdot\int_0^{u_1} [\lambda''(u_1 - u_2) + \lambda'(0)\,\delta(u_1 - u_2)]\,\widehat{\boldsymbol{E}}^t(u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2.$$

Therefore the application of Parseval-Plancherel theorem gives

$$\psi_2(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} [\lambda'_c(\omega) + \lambda'(0)][\,|\widehat{\boldsymbol{E}}^t_c(\omega)|^2 + |\widehat{\boldsymbol{E}}^t_s(\omega)|^2]\,\mathrm{d}\omega\,,$$

where the subscripts c, s mean respectively the cosine and the sine Fourier transform. By an integration by parts, we can prove the identity

$$\lambda_c''(\omega) + \lambda'(0) = \omega \lambda_s'(\omega).$$

Hence, owing to the condition (2.9), i.e. owing to the dissipativity of the boundary, ψ_2 is positive valued. Moreover, a direct computation shows the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_2(\widehat{\boldsymbol{E}}^t) = \frac{2}{\pi}\int_{-\infty}^{+\infty} \lambda_s'(\omega)\, \widehat{\boldsymbol{E}}_s^t(\omega)\cdot \boldsymbol{E}(t)\, \mathrm{d}\omega = \boldsymbol{E}(t)\cdot \boldsymbol{H}(t)\times \boldsymbol{n}\ .$$

4 - Existence and uniqueness

In order to state a theorem of existence and uniqueness, first we shall consider the problem (2.3)-(2.5) in a weak formulation. To this purpose, we introduce the functional spaces

$$\mathcal{K}(\Omega) = \{\boldsymbol{E},\, \boldsymbol{H} \in H^1(\Omega)\colon\ \nabla\cdot \varepsilon \boldsymbol{E} = 0,\quad \nabla\cdot \mu \boldsymbol{H} = 0\},$$

$$\mathcal{H}(Q) = \left\{\boldsymbol{E}, \boldsymbol{H} \in L^2(0,T; \mathcal{K}(\Omega))\colon \boldsymbol{H}(\boldsymbol{x},t)\times \boldsymbol{n}(\boldsymbol{x}) = \int_0^\infty \lambda'(\boldsymbol{x},s)\,\widehat{\boldsymbol{E}}^t(\boldsymbol{x},s)\,\mathrm{d}s \ \text{ on } \ \partial\Omega\times(0,T)\right\},$$

$$\mathcal{H}^*(Q) = \left\{\boldsymbol{e}, \boldsymbol{h} \in L^2(0,T; \mathcal{K}(\Omega))\colon \boldsymbol{h}(\boldsymbol{x},t)\times \boldsymbol{n}(\boldsymbol{x}) = \int_0^\infty \lambda'(\boldsymbol{x},s)\,\widehat{\boldsymbol{e}}^t(\boldsymbol{x},-s)\,\mathrm{d}s \ \text{ on } \ \partial\Omega\times(0,T)\right\},$$

$$\mathcal{V}(Q) = \mathcal{H}(Q)\cap H^1(0,\, T;\, L^2(\Omega)),$$

$$\mathcal{W}(Q) = \mathcal{H}^*(Q)\cap H^1(0,\, T;\, L^2(\Omega)),$$

$$\mathcal{J}(\Omega) = \left\{\boldsymbol{e}\in L^2(\Omega)\colon \int_\Omega \nabla\phi\cdot \boldsymbol{e}\,\mathrm{d}v = 0,\quad \forall\phi\in C_0^\infty(\Omega)\right\},$$

$$\mathcal{R}(\Omega) = \{\boldsymbol{e}\in\mathcal{J}(\Omega)\colon\ \nabla\times \boldsymbol{e}\in\mathcal{J}(\Omega)\}.$$

By assuming $\widehat{\boldsymbol{E}}^0 = 0$ and $\widehat{\boldsymbol{e}}^T = 0$, one can prove that the boundary condition given in the definition of the space $\mathcal{H}^*(Q)$ is the dual boundary condition of (2.5) in the sense of definition 2.1.

Definition 4.1. *A pair* $\boldsymbol{E}, \boldsymbol{H}\in\mathcal{V}(Q)$ *is called a strong solution of the problem* (2.3)-(2.5) *with sources* $\boldsymbol{I}_f, \boldsymbol{J}_f\in L^2(0,\, T,\, \mathcal{J}(\Omega))$ *and initial data* $\boldsymbol{E}_0, \boldsymbol{H}_0$

$\in \mathcal{K}(\Omega)$ *if*

$$\varepsilon \dot{\boldsymbol{E}} = \nabla \times \boldsymbol{H} - \sigma \boldsymbol{E} - \boldsymbol{J}_f, \qquad \text{a.e. in } Q,$$

$$\mu \dot{\boldsymbol{H}} = - \nabla \times \boldsymbol{E} + \boldsymbol{I}_f, \qquad \text{a.e. in } Q,$$

$$\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(\boldsymbol{x}), \qquad \boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0(\boldsymbol{x}), \quad \text{in } L^2(\Omega).$$

Definition 4.2. *A pair* $(\varepsilon \boldsymbol{E}, \mu \boldsymbol{H}) \in L^2(0, T, \mathcal{J}(\Omega))$ *is called a weak solution of the problem* (2.3)-(2.5) *with sources* $\boldsymbol{I}_f, \boldsymbol{J}_f \in L^2(0, T, \mathcal{J}(\Omega))$ *and initial data* $\varepsilon \boldsymbol{E}_0, \mu \boldsymbol{H}_0 \in \mathcal{J}(\Omega)$ *if*

$$\int_0^T \int_\Omega (\varepsilon \boldsymbol{E} \cdot \dot{\boldsymbol{e}} + \boldsymbol{H} \cdot \nabla \times \boldsymbol{e} + \mu \boldsymbol{H} \cdot \dot{\boldsymbol{h}} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{h} - \sigma \boldsymbol{E} \cdot \boldsymbol{e} - \boldsymbol{J}_f \cdot \boldsymbol{e} + \boldsymbol{I}_f \cdot \boldsymbol{h}) \, \mathrm{d}v \, \mathrm{d}t = - \int_\Omega (\varepsilon \boldsymbol{E}_0 \cdot \boldsymbol{e}_0 + \mu \boldsymbol{H}_0 \cdot \boldsymbol{h}_0) \, \mathrm{d}v \tag{4.1}$$

for each $\boldsymbol{e}, \boldsymbol{h} \in \mathcal{W}(Q)$ *such that*

$$\begin{cases} \boldsymbol{e}(\boldsymbol{x}, 0) = \boldsymbol{e}_0(\boldsymbol{x}), & \boldsymbol{h}(\boldsymbol{x}, 0) = \boldsymbol{h}_0(\boldsymbol{x}), \\ \boldsymbol{e}(\boldsymbol{x}, T) = \boldsymbol{h}(\boldsymbol{x}, T) = 0 \,. \end{cases} \tag{4.2}$$

In view of definition 4.2 it is clear that every strong solution of the problem (2.3)-(2.5) is also a weak solution.

The previous results allow us to prove the uniqueness theorem for the weak solution.

Theorem 4.1. *The weak solution of the problem* (2.3)-(2.5) *is unique.*

Proof. Let $(\boldsymbol{E}_1, \boldsymbol{H}_1)$, $(\boldsymbol{E}_2, \boldsymbol{H}_2)$ be two solutions with the same sources $\boldsymbol{I}_f$, $\boldsymbol{J}_f$ and initial conditions $\boldsymbol{E}_0(\boldsymbol{x})$, $\boldsymbol{H}_0(\boldsymbol{x})$.

If we denote by $\boldsymbol{E} := \boldsymbol{E}_1 - \boldsymbol{E}_2$ $\boldsymbol{H} := \boldsymbol{H}_1 - \boldsymbol{H}_2$, the linearity of equations (2.3), (2.4) and of the boundary condition (2.5) implies that the pair $(\boldsymbol{E}, \boldsymbol{H})$ is a weak solution of the same problem with sources $\boldsymbol{I}_f = \boldsymbol{J}_f = 0$ and histories $\widehat{\boldsymbol{E}}^0 = \widehat{\boldsymbol{H}}^0 = 0$.

Accordingly, by (4.1) the identity

$$\int_0^T \int_\Omega (\boldsymbol{H} \cdot \nabla \times \boldsymbol{e} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{h} + \varepsilon \boldsymbol{E} \cdot \dot{\boldsymbol{e}} + \mu \boldsymbol{H} \cdot \dot{\boldsymbol{h}} - \sigma \boldsymbol{E} \cdot \boldsymbol{e}) \, \mathrm{d}v \, \mathrm{d}t = 0 \tag{4.3}$$

holds for each $\boldsymbol{e}, \boldsymbol{h} \in \mathcal{W}(Q)$ such that $\boldsymbol{e}(\boldsymbol{x}, T) = \boldsymbol{h}(\boldsymbol{x}, T) = 0$. We shall prove that $\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{H}(\boldsymbol{x}, t) = 0$ identically.

The arbitrariness of $\boldsymbol{e}, \boldsymbol{h}$ in the equation (4.3) yields

$$\int_0^T\int_\Omega (\mu \boldsymbol{H}\cdot\dot{\boldsymbol{h}} - \boldsymbol{E}\cdot\nabla\times\boldsymbol{h})\,\mathrm{d}v\,\mathrm{d}t = 0\,, \tag{4.4}$$

$$\int_0^T\int_\Omega (\varepsilon \boldsymbol{E}\cdot\dot{\boldsymbol{e}} + \boldsymbol{H}\cdot\nabla\times\boldsymbol{e} - \sigma\boldsymbol{E}\cdot\boldsymbol{e})\,\mathrm{d}v\,\mathrm{d}t = 0\,. \tag{4.5}$$

We make the identifications

$$\widetilde{\boldsymbol{E}}(\boldsymbol{x},\tau) = \int_0^\tau \boldsymbol{E}(\xi)\,\mathrm{d}\xi\,,$$

$$\widetilde{\boldsymbol{H}}(\boldsymbol{x},\tau) = \int_0^\tau \boldsymbol{H}(\xi)\,\mathrm{d}\xi$$

and choose

$$\boldsymbol{e}(\boldsymbol{x},\tau) = \tilde{\boldsymbol{e}}(\boldsymbol{x})(T-\tau)\,,$$

$$\boldsymbol{h}(\boldsymbol{x},\tau) = \tilde{\boldsymbol{h}}(\boldsymbol{x})(T-\tau)\,,$$

with $\tilde{\boldsymbol{e}}, \tilde{\boldsymbol{h}} \in C_0^\infty(\Omega)$. Then, applying (4.4) and (4.5) at $t = T$ and differentiating with respect to t, we obtain

$$\int_\Omega (\mu\boldsymbol{H}\cdot\tilde{\boldsymbol{h}} + \widetilde{\boldsymbol{E}}\cdot\nabla\times\tilde{\boldsymbol{h}})\,\mathrm{d}v = -\int_{\partial\Omega}\widetilde{\boldsymbol{E}}\cdot\tilde{\boldsymbol{h}}\times\boldsymbol{n}\,\mathrm{d}a = 0\,, \tag{4.6}$$

$$\int_\Omega (\widetilde{\boldsymbol{H}}\cdot\nabla\times\tilde{\boldsymbol{e}} - \varepsilon\boldsymbol{E}\cdot\tilde{\boldsymbol{e}} - \sigma\widetilde{\boldsymbol{E}}\cdot\tilde{\boldsymbol{e}})\,\mathrm{d}v = \int_{\partial\Omega}\tilde{\boldsymbol{e}}\cdot\boldsymbol{H}\times\boldsymbol{n}\,\mathrm{d}a = 0\,. \tag{4.7}$$

The validity of (4.6) and (4.7) for each $\tilde{\boldsymbol{e}}, \tilde{\boldsymbol{h}} \in C_0^\infty(\Omega)$ implies that both $\nabla\times\widetilde{\boldsymbol{E}}$ and $\nabla\times\widetilde{\boldsymbol{H}}$ exist and belong to $L^2(0, T, L^2(\Omega))$. Moreover they satisfy

$$\nabla\times\widetilde{\boldsymbol{E}} = -\mu\boldsymbol{H}\,, \tag{4.8}$$

$$\nabla\times\widetilde{\boldsymbol{H}} = -\varepsilon\boldsymbol{E} + \sigma\widetilde{\boldsymbol{E}}\,. \tag{4.9}$$

Multiplying (4.8) by $\tilde{\boldsymbol{H}}$ and (4.9) by $\tilde{\boldsymbol{E}}$ and subtracting, we obtain

$$\int_{\partial\Omega} \tilde{\boldsymbol{E}}\cdot\tilde{\boldsymbol{H}}\times\boldsymbol{n}\,\mathrm{d}a = -\int_{\Omega}(\mu\boldsymbol{H}\cdot\tilde{\boldsymbol{H}} + \varepsilon\boldsymbol{E}\cdot\tilde{\boldsymbol{E}} + \sigma\tilde{\boldsymbol{E}}^2)\,\mathrm{d}v\,. \tag{4.10}$$

Notice that the system (4.8), (4.9) can be written in the equivalent form

$$\mu\dot{\tilde{\boldsymbol{H}}} = -\nabla\times\tilde{\boldsymbol{E}}\,,$$

$$\varepsilon\dot{\tilde{\boldsymbol{E}}} = \nabla\times\tilde{\boldsymbol{H}} - \sigma\tilde{\boldsymbol{E}}\,.$$

Accordingly, $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{H}}$ are regular solutions of Maxwell's equations. Therefore, by the definition of the boundary free energy given in Section 3, we obtain the estimate

$$\dot{\psi}(t) \leqslant \tilde{\boldsymbol{E}}(t)\cdot\tilde{\boldsymbol{H}}(t)\times\boldsymbol{n}\,.$$

Hence, by integrating (4.10) in the interval $(0, t)$ we prove the inequality

$$\int_{\partial\Omega}[\psi(\boldsymbol{x}, t) - \psi(\boldsymbol{x}, 0)]\,da \leqslant -\frac{1}{2}\int_{\Omega}[\mu\tilde{\boldsymbol{H}}^2(\boldsymbol{x}, t) + \varepsilon\tilde{\boldsymbol{E}}^2(\boldsymbol{x}, t)]\,\mathrm{d}v\,. \tag{4.11}$$

Since $\widehat{\boldsymbol{E}}^0 = 0$, the boundary free energy vanishes at $t = 0$.

The positive-definiteness of ψ and the inequality (4.11) imply in particular $\tilde{\boldsymbol{E}}(\boldsymbol{x}, t) = \tilde{\boldsymbol{H}}(\boldsymbol{x}, t) = 0$ for each $\boldsymbol{x}\in\Omega$, $t\in(0, T)$. Hence $\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{H}(\boldsymbol{x}, t) = 0$. ■

We can prove an existence theorem of the problem (2.3)-(2.5). Let

$$A\colon \mathcal{V}(Q)\times\mathcal{V}(Q)\to L^2(0, T, \mathcal{J}(\Omega))\times L^2(0, T, \mathcal{J}(\Omega))\times\mathcal{H}(\Omega)\times\mathcal{H}(\Omega)$$

be the operator defined by

$$A(\boldsymbol{E}, \boldsymbol{H}) = (\nabla\times\boldsymbol{H} - \varepsilon\dot{\boldsymbol{E}} - \sigma\boldsymbol{E},\ \mu\dot{\boldsymbol{H}} + \nabla\times\boldsymbol{E},\ \varepsilon\boldsymbol{E}_0,\ \mu\boldsymbol{H}_0)\,.$$

First, we state the following

Lemma 4.1. *The range $\mathcal{R}(A)$ of the operator A is dense in the Hilbert space $\mathcal{H}(Q) := L^2(0, T, \mathcal{J}(\Omega))\times L^2(0, T, \mathcal{J}(\Omega))\times\mathcal{J}(\Omega)\times\mathcal{J}(\Omega)$.*

Proof. Suppose by contradiction that $\mathcal{R}(A)$ is not dense in $\mathcal{H}(Q)$, then there exists a non-zero element $\Phi = (\boldsymbol{e}, \boldsymbol{h}, \boldsymbol{e}_0, \boldsymbol{h}_0)\in\mathcal{H}(Q)$, which is orthogonal to $\overline{\mathcal{R}(A)}$.

Hence each pair $(\boldsymbol{E}, \boldsymbol{H}) \in \mathcal{V}(Q) \times \mathcal{V}(Q)$ satisfies the relation

$$\int_0^T \int_\Omega [(\nabla \times \boldsymbol{H} - \varepsilon \dot{\boldsymbol{E}} - \sigma \boldsymbol{E}) \cdot \boldsymbol{e} + (\mu \dot{\boldsymbol{H}} + \nabla \times \boldsymbol{E}) \cdot \boldsymbol{h}] \, \mathrm{d}v \, \mathrm{d}t + \int_\Omega (\varepsilon \boldsymbol{E}_0 \cdot \boldsymbol{e}_0 + \mu \boldsymbol{H}_0 \cdot \boldsymbol{h}_0) \, \mathrm{d}v = 0 \,. \tag{4.12}$$

By choosing first $\boldsymbol{H}(\boldsymbol{x}, t) = \boldsymbol{E}_0(\boldsymbol{x}) = 0$, then $\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{H}_0(\boldsymbol{x}) = 0$ we obtain

$$\int_0^T \int_\Omega [\nabla \times \boldsymbol{E} \cdot \boldsymbol{h} - (\varepsilon \dot{\boldsymbol{E}} + \sigma \boldsymbol{E}) \cdot \boldsymbol{e}] \, \mathrm{d}v \, \mathrm{d}t = 0 \,,$$

$$\int_0^T \int_\Omega (\nabla \times \boldsymbol{H} \cdot \boldsymbol{e} + \mu \dot{\boldsymbol{H}} \cdot \boldsymbol{h}) \, \mathrm{d}v \, \mathrm{d}t = 0 \,.$$

These equations mean that $(\boldsymbol{e}, \boldsymbol{h})$ is a weak solution of the problem obtained by the former after the temporal inversion $\tau = T - t$. This backward problem has sources $\boldsymbol{I}_f = \boldsymbol{J}_f = 0$ and initial data $\boldsymbol{e}(T) = \boldsymbol{h}(T) = 0$.

Proceeding like in the proof of the theorem 4.1, one can prove an uniqueness theorem also in this case. Hence, $\boldsymbol{e}(\boldsymbol{x}, t) = \boldsymbol{h}(\boldsymbol{x}, t) = 0$.

Substitution in (4.12) yields $\boldsymbol{e}_0(\boldsymbol{x}) = \boldsymbol{h}_0(\boldsymbol{x}) = 0$ which contradicts the hypothesis. ■

Now we can prove the existence theorem.

Theorem 4.2. *If* $(\boldsymbol{J}_f, \boldsymbol{I}_f, \varepsilon \boldsymbol{E}_0, \mu \boldsymbol{H}_0) \in \mathcal{K}(Q)$, *the problem* (2.3)-(2.5) *has a weak solution.*

Proof. Since $\mathcal{R}(A)$ is dense in $\mathcal{K}(Q)$, there exists a sequence $(\boldsymbol{J}_f^{(n)}, \boldsymbol{I}_f^{(n)}, \varepsilon \boldsymbol{E}_0^{(n)}, \mu \boldsymbol{H}_0^{(n)}) \in \mathcal{R}(A)$ which converges to $(\boldsymbol{J}_f, \boldsymbol{I}_f, \varepsilon \boldsymbol{E}_0, \mu \boldsymbol{H}_0)$ in $\mathcal{K}(Q)$.

Accordingly, for each n there exists a pair $(\boldsymbol{E}^{(n)}, \boldsymbol{H}^{(n)}) \in \mathcal{V}(Q)$ such that

$$-\varepsilon \dot{\boldsymbol{E}}^{(n)} + \nabla \times \boldsymbol{H}^{(n)} - \sigma \boldsymbol{E}^{(n)} = \boldsymbol{J}_f^{(n)},$$

$$\mu \dot{\boldsymbol{H}}^{(n)} + \nabla \times \boldsymbol{E}^{(n)} = \boldsymbol{I}_f^{(n)}.$$

The solution $(\boldsymbol{E}^{(n)} - \boldsymbol{E}^{(m)}, \boldsymbol{H}^{(n)} - \boldsymbol{H}^{(m)})$ satisfies the estimate (see [**6**])

$$\varepsilon_m \|\boldsymbol{E}^{(n)} - \boldsymbol{E}^{(m)}\|^2_{L^2(Q)} + \mu_m \|\boldsymbol{H}^{(n)} - \boldsymbol{H}^{(m)}\|^2_{L^2(Q)} \leqslant c\{\|\boldsymbol{J}_f^{(n)} - \boldsymbol{J}_f^{(m)}\|^2_{L^2(Q)}$$

$$+ \|\boldsymbol{I}_f^{(n)} - \boldsymbol{I}_f^{(m)}\|^2_{L^2(Q)} + \varepsilon_M \|\boldsymbol{E}_0^{(n)} - \boldsymbol{E}_0^{(m)}\|^2_{L^2(\Omega)} + \mu_M \|\boldsymbol{H}_0^{(n)} - \boldsymbol{H}_0^{(m)}\|^2_{L^2(\Omega)}\} .$$

It follows that $\{\boldsymbol{E}^{(n)}\}, \{\boldsymbol{H}^{(n)}\}$ are Cauchy sequences in $L^2(Q)$, hence there exist $\boldsymbol{E}, \boldsymbol{H} \in L^2(Q)$ such that

$$\lim_{n \to \infty} \boldsymbol{E}^{(n)} = \boldsymbol{E} , \qquad \lim_{n \to \infty} \boldsymbol{H}^{(n)} = \boldsymbol{H} .$$

It remains to show that $(\boldsymbol{E}, \boldsymbol{H})$ is a weak solution. Consider $\boldsymbol{e}, \boldsymbol{h} \in \mathcal{W}(Q)$ such that $\boldsymbol{e}(T) = \boldsymbol{h}(T) = 0$. They satisfy the identities

$$\int_0^T \int_\Omega \boldsymbol{H}^{(n)} \cdot \nabla \times \boldsymbol{e} \, dv \, dt + \int_0^T \int_\Omega \boldsymbol{E}^{(n)} \cdot \dot{\boldsymbol{e}} \, dv \, dt = \int_0^T \int_\Omega \nabla \times \boldsymbol{H}^{(n)} \cdot \boldsymbol{e} \, dv \, dt$$

$$- \int_0^T \int_\Omega \dot{\boldsymbol{E}}^{(n)} \cdot \boldsymbol{e} \, dv \, dt - \int_{\boldsymbol{E}_0}^{(n)} \cdot \boldsymbol{e}_0 \, dv ,$$

$$\int_0^T \int_\Omega \boldsymbol{E}^{(n)} \cdot \nabla \times \boldsymbol{h} \, dv \, dt + \int_0^T \int_\Omega \boldsymbol{H}^{(n)} \cdot \dot{\boldsymbol{h}} \, dv \, dt = \int_0^T \int_\Omega \nabla \times \boldsymbol{E}^{(n)} \cdot \boldsymbol{h} \, dv \, dt$$

$$- \int_0^T \int_\Omega \dot{\boldsymbol{H}}^{(n)} \cdot \boldsymbol{h} \, dv \, dt - \int_\Omega \boldsymbol{H}_0^{(n)} \cdot \boldsymbol{h}_0 \, dv .$$

As $n \to \infty$, the sum of the previous equations gives the relation (4.1). ∎

5. – Asymptotic stability

In the previous section we stated an existence and uniqueness theorem in a domain $Q = \Omega \times (0, T)$. Now we examine the asymptotic behaviour of the weak solutions, by letting $T \to \infty$.

First we have to extend the definition (4.2) to the unbounded case. The pair $(\varepsilon \boldsymbol{E}, \mu \boldsymbol{H}) \in L^2(0, \infty, \mathcal{J}(\Omega))$ will be called a weak solution of the problem (2.3)-

(2.5) if it satisfies the integral equation

$$\int_0^\infty \int_\Omega (\varepsilon \boldsymbol{E} \cdot \dot{\boldsymbol{e}} + \boldsymbol{H} \cdot \nabla \times \boldsymbol{e} + \mu \boldsymbol{H} \cdot \dot{\boldsymbol{h}} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{h} - \sigma \boldsymbol{E} \cdot \boldsymbol{e} - \boldsymbol{J}_f \cdot \boldsymbol{e} + \boldsymbol{I}_f \cdot \boldsymbol{h}) \, \mathrm{d}v \, \mathrm{d}t = - \int_\Omega (\varepsilon \boldsymbol{E}_0 \cdot \boldsymbol{e}_0 + \mu \boldsymbol{H}_0 \cdot \boldsymbol{h}_0) \, \mathrm{d}v \, . \tag{5.1}$$

In what follows, we denote by f_F the temporal Fourier transform of f, considered as a causal function. Moreover we denote by $\mathcal{J}^*(\Omega)$ and $\mathcal{R}^*(\Omega)$ the space of complex functions, whose real and imaginary part belong to $\mathcal{J}(\Omega)$ and $\mathcal{R}(\Omega)$ respectively.

The Parseval-Plancherel theorem allows to rewrite (5.1) in the equivalent form

$$\int_{-\infty}^{+\infty} \int_\Omega [\varepsilon \boldsymbol{E}_F^* \cdot (i\omega \boldsymbol{e}_F - \boldsymbol{e}_0) + \mu \boldsymbol{H}_F^* \cdot (i\omega \boldsymbol{h}_F - \boldsymbol{h}_0) + \boldsymbol{H}_F^* \cdot \nabla \times \boldsymbol{e}_F - \boldsymbol{E}_F^* \cdot \nabla \times \boldsymbol{h}_F - \sigma \boldsymbol{E}_F^* \cdot \boldsymbol{e}_F - \boldsymbol{J}_{fF}^* \cdot \boldsymbol{e}_F + \boldsymbol{I}_{fF}^* \cdot \boldsymbol{h}_f] \, \mathrm{d}v \, \mathrm{d}t + 2\pi \int_\Omega (\varepsilon \boldsymbol{E}_0 \cdot \boldsymbol{e}_0 + \mu \boldsymbol{H}_0 \cdot \boldsymbol{h}_0) \, \mathrm{d}v \, , \tag{5.2}$$

where the symbol * means the complex conjugate.

Remark. If we apply the Fourier transform to (2.3)-(2.5), we obtain the system

$$\nabla \times \boldsymbol{H}_F = (i\omega\varepsilon + \sigma) \boldsymbol{E}_F + \boldsymbol{J}_{fF} - \varepsilon \boldsymbol{E}_0, \tag{5.3}$$

$$\nabla \times \boldsymbol{E}_F = - i\omega\mu \boldsymbol{H}_F + \boldsymbol{I}_{fF} - \mu \boldsymbol{H}_0, \tag{5.4}$$

$$\boldsymbol{H}_F \times \boldsymbol{n} = - \lambda_F \boldsymbol{E}_F. \tag{5.5}$$

A straightforward check shows that the solutions of (5.3)-(5.5) satisfy (5.2).

Lemma 5.1. *For each* $\omega \in \mathbb{R}$, *if* $\boldsymbol{J}_{fF}(\omega)$, $\boldsymbol{I}_{fF}(\omega) \in \mathcal{J}^*(\Omega)$, $\varepsilon \boldsymbol{E}_0$, $\mu \boldsymbol{H}_0 \in \mathcal{J}^*(\Omega) \cap \mathcal{R}^*(\Omega)$, $\lambda_c(\omega) < 0$, *the problem* (5.3)-(5.5) *has a unique solution* $\boldsymbol{E}_F(\omega)$, $\boldsymbol{H}_F(\omega) \in \mathcal{R}^*(\Omega)$. *Moreover,* $\boldsymbol{E}_F(\boldsymbol{x}, \omega)$, $\boldsymbol{H}_F(\boldsymbol{x}, \omega)$ *depend continuously on* ω.

Proof. Consider the homogeneous problem associated with (5.3)-(5.4)

$$\nabla \times \boldsymbol{H}_F = (i\omega\varepsilon + \sigma) \boldsymbol{E}_F, \tag{5.6}$$

$$\nabla \times \boldsymbol{E}_F = - i\omega\mu \boldsymbol{H}_F, \tag{5.7}$$

$$\boldsymbol{H}_F \times \boldsymbol{n} = - \lambda_F \boldsymbol{E}_F. \tag{5.8}$$

According to the complex Poynting theorem, we obtain from (5.6)-(5.8) the identity

$$\int_{\partial\Omega} \lambda_F |\boldsymbol{E}_F \times \boldsymbol{n}|^2 \,\mathrm{d}a = \int_{\Omega} [\sigma |\boldsymbol{E}_F|^2 + i\omega(\varepsilon |\boldsymbol{E}_F|^2 - \mu |\boldsymbol{H}_F|^2)] \,\mathrm{d}v \,. \tag{5.9}$$

Hence, by comparing the real part, (5.9) gives

$$\int_{\partial\Omega} \lambda_c |\boldsymbol{E}_F \times \boldsymbol{n}|^2 \,\mathrm{d}a = \int_{\Omega} \sigma |\boldsymbol{E}_F|^2 \,\mathrm{d}v \,.$$

Since λ_c is strictly negative, $\boldsymbol{E}_F = 0$. Substitution in (5.9) yields $\boldsymbol{H}_F = 0$. This proves the uniqueness. In order to prove the existence of the solution, we observe that, according to the uniqueness, the problem (5.6)-(5.8) admits no eigensolutions, except the trivial one. The Theorem 3.12 of [7] ensures the existence of the solution $(\boldsymbol{E}_F(\omega), \boldsymbol{H}_F(\omega))$. Moreover, in view of lemma 44.1 of [8], the pair $(\boldsymbol{E}_F(\omega), \boldsymbol{H}_F(\omega))$ is proved to be analytic with respect to ω and to satisfy the estimate

$$\int_{\Omega} [\,|\boldsymbol{E}_F(\omega)|^2 + |\boldsymbol{H}_F(\omega)|^2] \,\mathrm{d}v \leq C(\omega) \int_{\Omega} [\,|\boldsymbol{J}_{fF}(\omega)|^2 + |\boldsymbol{I}_{fF}(\omega)|^2] \,\mathrm{d}v \,. \tag{5.10}$$ ■

Let $\mathcal{B}^*(\Omega)$ denote the following set

$$\mathcal{B}^*(\Omega) = \{(\boldsymbol{E}_F, \boldsymbol{H}_F) \in \mathcal{R}^*(\Omega) \times \mathcal{R}^*(\Omega) \colon \boldsymbol{H}_F \times \boldsymbol{n} = \lambda_F \boldsymbol{E}_F \ \text{ on } \partial\Omega\} \,.$$

In view of lemma 5.1, the differential operator $A_\omega \colon \mathcal{B}^*(\Omega) \to \mathfrak{J}^*(\Omega) \times \mathfrak{J}^*(\Omega)$ induced by the system (5.3)-(5.5) is surjective. Therefore, there exist the tensor Green's functions $\Pi_1(\boldsymbol{x}, \boldsymbol{x}', \omega)$, $\Pi_2(\boldsymbol{x}, \boldsymbol{x}', \omega)$ which satisfy in the weak form the system

$$\nabla \times \Pi_2(\boldsymbol{x}, \boldsymbol{x}', \omega) + (\sigma + i\omega\varepsilon)\, \Pi_1(\boldsymbol{x}, \boldsymbol{x}', \omega) = \delta(\boldsymbol{x} - \boldsymbol{x}')\, \boldsymbol{I} \,, \tag{5.11}$$

$$\nabla \times \Pi_1(\boldsymbol{x}, \boldsymbol{x}', \omega) - i\omega\mu \Pi_2(\boldsymbol{x}, \boldsymbol{x}', \omega) = 0 \,, \tag{5.12}$$

$$\Pi_2(\boldsymbol{x}, \boldsymbol{x}', \omega) \times \boldsymbol{n}(\boldsymbol{x}) = \lambda_F(\boldsymbol{x}, \omega)\, \Pi_1(\boldsymbol{x}, \boldsymbol{x}', \omega) \,, \tag{5.13}$$

where $\boldsymbol{I}$ is the identity tensor.

The equations (5.3)-(5.5) and (5.11)-(5.13) give the expression of the electric field

$$E_F(x, \omega) = -\int_\Omega \{\Pi_1(x, x', \omega)[J_{fF}(x', \omega) - \varepsilon E_0(x')] - \Pi_2(x, x', \omega)[I_{fF}(x', \omega) - \mu H_0(x')]\}\, dv' . \tag{5.14}$$

Analogously, if we consider the Green's functions Ξ_1, Ξ_2, defined as the solution of the system

$$\nabla \times \Xi_2(x, x', \omega) + (\sigma + i\omega\varepsilon)\, \Xi_1(x, x', \omega) = 0 , \tag{5.15}$$

$$\nabla \times \Xi_1(x, x', \omega) - i\omega\mu \Xi_2(x, x', \omega) = \delta(x - x')\, I , \tag{5.16}$$

$$\Xi_2(x, x', \omega) \times n(x) = \lambda_F(x, \omega)\, \Xi_1(x, x', \omega) , \tag{5.17}$$

we obtain the relation

$$H_F(x, \omega) = \int_\Omega \{\Xi_1(x, x', \omega)[J_{fF}(x', \omega) - \varepsilon E_0(x')] - \Xi_2(x, x', \omega)[I_{fF}(x', \omega) - \mu H_0(x')]\}\, dv' . \tag{5.18}$$

The following lemma shows the asymptotic behaviour of Green's functions.

Lemma 5.2. *Green's functions satisfy the asymptotic condition*

$$\lim_{\omega \to \infty} i\omega \int_\Omega \int_\Omega [\varepsilon \Pi_1(x, x', \omega)\, f_1(x) \cdot \phi(x', \omega) + \mu \Pi_2(x, x', \omega)\, f_2(x) \cdot \phi(x', \omega)]\, dv\, dv' = \lim_{\omega \to \infty} \int_\Omega \phi(x, \omega) \cdot f_1(x)\, dv \tag{5.19}$$

for each $f_1, f_2 \in \mathcal{R}^*(\Omega)$.

Proof. Keeping (5.11)-(5.13) into account we obtain the integral relation

$$\int_\Omega \int_\Omega [i\omega\mu \phi(x', \omega) \cdot \Pi_2(x, x', \omega)\, f_2(x) + (\sigma + i\omega\varepsilon)\, \phi(x, \omega) \cdot \Pi_1(x, x', \omega)\, f_1(x) - \phi(x', \omega) \cdot \Pi_1(x, x', \omega)\, \nabla \times f_2(x) + \phi(x', \omega) \cdot \Pi_2(x, x', \omega)\, \nabla \times f_1(x)]\, dv\, dv'$$

$$= \int_\Omega \phi(x, \omega) \cdot f_1(x)\, dv .$$

Hence, letting $\omega \to \infty$, since $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ are independent of ω, we obtain (5.19). ■

In a similar way, we can prove the asymptotic conditions for Ξ_1 and Ξ_2

$$\lim_{\omega\to\infty} i\omega \int_\Omega\int_\Omega [\varepsilon\Xi_1(\boldsymbol{x},\boldsymbol{x}',\omega)\,\boldsymbol{f}_1(\boldsymbol{x})\cdot\phi(\boldsymbol{x}',\omega)+\mu\Xi_2(\boldsymbol{x},\boldsymbol{x}',\omega)\,\boldsymbol{f}_2(\boldsymbol{x})\cdot\phi(\boldsymbol{x}',\omega)]\,\mathrm{d}v\,\mathrm{d}v' \tag{5.20}$$
$$= -\lim_{\omega\to\infty}\int_\Omega \phi(\boldsymbol{x},\omega)\cdot\boldsymbol{f}_2(\boldsymbol{x})\,\mathrm{d}v\,.$$

As a consequence of lemmas (5.1)-(5.2) we obtain the following

Theorem 5.1. *Let* $\boldsymbol{I}_f$, $\boldsymbol{J}_f \in H^1(0,\infty;\mathcal{J}(\Omega))\cap L^1(0,\infty;\mathcal{J}(\Omega))$, $\boldsymbol{I}_f(\boldsymbol{x},0) = \boldsymbol{J}_f(\boldsymbol{x},0)=0$, $\boldsymbol{E}_0$, $\boldsymbol{H}_0 \in \mathcal{R}(\Omega)\cap\mathcal{J}(\Omega)$. *Then, there exists a unique solution* $\boldsymbol{E}$, $\boldsymbol{H} \in L^2(0,\infty;\mathcal{R}(\Omega))$ *in the sense of definition* (5.1).

Proof. According to lemma (5.2), from (5.14) we obtain the condition

$$\lim_{\omega\to\infty}\int_\Omega [i\omega\boldsymbol{E}_F(\boldsymbol{x},\omega)-\boldsymbol{E}_0(\boldsymbol{x})]\cdot[i\omega\boldsymbol{e}_F(\boldsymbol{x},\omega)-\boldsymbol{e}_0(\boldsymbol{x})]\,\mathrm{d}v \tag{5.21}$$
$$\sim \lim_{\omega\to\infty} i\omega\int_\Omega \boldsymbol{J}_{fF}(\boldsymbol{x},\omega)\cdot[i\omega\boldsymbol{e}_F(\boldsymbol{x},\omega)-\boldsymbol{e}_0(\boldsymbol{x})]\,\mathrm{d}v\,.$$

Since $\boldsymbol{J}_f \in H^1(0,\infty;L^2(\Omega))\cap L^1(0,\infty;L^2(\Omega))$ the previous equation implies

$$\lim_{\omega\to\infty}\int_\Omega \boldsymbol{E}_F(\boldsymbol{x},\omega)\cdot[i\omega\boldsymbol{e}_F(\boldsymbol{x},\omega)-\boldsymbol{e}_0(\boldsymbol{x})]\,\mathrm{d}v$$

$$\sim \lim_{\omega\to\infty}\frac{1}{i\omega}\int_\Omega \boldsymbol{E}_0(\boldsymbol{x})\cdot[i\omega\boldsymbol{e}_F(\boldsymbol{x},\omega)-\boldsymbol{e}_0(\boldsymbol{x})]\,\mathrm{d}v\,.$$

Therefore $\boldsymbol{E}_F \in L^2(0,\infty;L^2(\Omega))$.

In the same way, we prove that $\boldsymbol{H}_F \in L^2(0,\infty;L^2(\Omega))$.

In view of (5.21)

$$i\omega\boldsymbol{E}_F - \boldsymbol{E}_0 \in L^2(0,\infty;L^2(\Omega))\,.$$

Analogously,

$$i\omega\boldsymbol{H}_F - \boldsymbol{H}_0 \in L^2(0,\infty;L^2(\Omega))\,.$$

The Paley-Wiener theorem allows to conclude that $\boldsymbol{E}_F$, $\boldsymbol{H}_F$ are the Fourier transforms of casual functions. Their inverse Fourier transforms $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy the relation (5.1). Moreover, according to (5.2), $\boldsymbol{E}_F(\omega)$, $\boldsymbol{H}_F(\omega) \in \mathcal{R}(\Omega)$. Thus $(\boldsymbol{E}, \boldsymbol{H})$ is a weak solution of the problem (2.3)-(2.5). ■

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Abstract

In this paper, we study the differential problem of Maxwell's equations with a memory boundary condition in the Schelkunoff's hypothesis [1]. *Such a condition describes an electromagnetic solid with a conducting and dissipative boundary. We define a boundary free energy in order to prove a theorem of existence and uniqueness for the weak solution. Finally, we prove an asympotic stability theorem.*

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