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A note on the matrix representations of the Lie algebras L_r^s for quantized Hamiltonians where $rs = 0$ (**)

1 - Introduction

The Lie-algebraic decomposition formulas of Baker, Campbell, Hausdorff and Zassenhaus [**9**] and their matrix representations, were used by Steinberg in his method to solve certain types of linear partial differential equations [8]. The required matrix representations must be faithful and of low-dimension. Recently, the method has been used to derive solutions of the Schrödinger's wave equations for the Hamiltonian model of coupled quantized harmonic oscillators of the form [7], $H = K_0 + \lambda(K_+ + K_-)$, $\lambda \in \mathbb{R}^*$ (the set of nonzero real numbers) is the coupling parameter, and the Lie algebras L_r^s generated by the three operators K_0, $K_\pm$; $r, s \in \mathbb{R}$ with $[K_+, K_-] = sK_0$, $[K_0, K_\pm] = \pm rK_\pm$.

All considered matrix representations should satisfy the physical requirements namely, $K_- = K_+^\dagger$ († is used for Hermitian conjugation), K_0 is real and diagonal and $(K_+ + K_-)$ is real.

For $r = 1$, $s = 2$ the model corresponds to two-level optical atom model, while for $r = 1$, $s = -2$ it corresponds to light amplifier model. When $rs \neq 0$, it was proved in [**5**], that L_r^s has faithful matrix representations only if $rs > 0$. When $rs = 0$, we show that L_0^0 has three faithful matrix representations of dimension 3, as the least dimension. But L_0^s, $s \neq 0$ and L_r^0, $r \neq 0$ have none.

Unless otherwise stated, **0** is the zero matrix of appropriate size, I_k is the identity matrix of order k, $N = \{1, 2, \dots, n\}$, while $X = [x_{ij}]$, $Y = [y_{ij}]$ and

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$Z = [\delta_{ij} z_{ij}]$ are $n \times n$ representation matrices for K_+, K_- and K_0 respectively. Thus, $z_{ij} \in \mathbb{R}$, $y_{ij} = x_{ij}^\dagger = \overline{x}_{ji}$, where δ_{ij} is the Kronecker delta; $i, j \in N$. Obviously for a faithful representation

$$\lambda Z + \mu X + \nu Y = \mathbf{0} \tag{1}$$

is only satisfied by $\lambda = \mu = \nu = 0$.

The defining relations of L_r^s are: $[K_+, K_-] = sK_0$, $[K_0, K_-] = -rK_-$, or

$$[K_+, K_-] = sK_0, \qquad [K_0, K_+] = rK_+ . \tag{2}$$

Hence from (2)

$$[X, Y] = sZ, [Z, X] = rX \tag{3}$$

and for $i, j \in N$, we have

$$x_{ij}(z_{ii} - z_{jj} - r) = 0 , \tag{4}$$

$$s\delta_{ij} z_{ij} = \sum_{l=1}^{n} (x_{il} x_{lj}^\dagger - x_{il}^\dagger x_{lj}) , \tag{5}$$

$$sz_{ii} = \sum_{l=1}^{n} (|x_{il}|^2 - |x_{li}|^2) . \tag{6}$$

Lemma 1. *For any $p, q \in N$, let $\sigma = (pq)$ be a permutation on N that is applied to the rows as well as to the columns of X, Y and Z, then the resulting matrices X', Y' and Z', respectively are also representation matrices for K_+, K_- and K_0, respectively.*

Proof. Let P be the elementary matrix, obtained by applying σ to the rows of I_n. Since $P = P^{-1} = P^T$, $X' = P^{-1}XP$, $Y' = P^{-1}YP$ and $Z' = P^{-1}ZP$. Also, the physical properties are satisfied by X', Y' and Z'. ■

Remark 1. Using Lemma 1, the matrix Z can be rearranged as $Z = \operatorname{diag}(\alpha_1 I_{m_1}, \alpha_2 I_{m_2}, \ldots, \alpha_t I_{m_t})$, with different α_i's, $i = 1, 2, \ldots t$; $t \in N$. If Z is singular, we take $\alpha_t = 0$.

2 - Faithful matrix representations for L_0^s, $s \neq 0$

Lemma 2. *The matrices X, Y and Z consist of diagonal blocks of equal corresponding sizes.*

Proof. If z_{ii} and z_{jj} are from different diagonal blocks, then from (4) as $r = 0$ we have $x_{ij} = x_{ji} = 0$. ■

Theorem 3. *L_0^s, $s \neq 0$, has no faithful matrix representations.*

Proof. From (1), $Z \neq \mathbf{0}$. So, let $\alpha_1 I_{m_1} = \alpha I_m \neq \mathbf{0}$. From Lemma 2 the first diagonal block of X has order m. Using (6), we have

$$\alpha m s = \sum_{i=1}^{m} s z_{ii} = \sum_{i=1}^{m} \sum_{l=1}^{n} (|x_{il}|^2 - |x_{li}|^2) = \sum_{i=1}^{m} \sum_{l=1}^{m} (|x_{il}|^2 - |x_{li}|^2) = 0\,,$$

but $\alpha m s \neq 0$. ■

3 - Faithful matrix representations for L_r^0, $r \neq 0$

Lemma 4. *X and Y are real matrices whose diagonal elements are all zeros.*

Proof. The following results are immediate from (4) when $r \neq 0$.

(7) $$\text{If } x_{ij} \neq 0, \quad \text{then } z_{ii} - z_{jj} = r\,.$$

(8) $$\text{If } x_{ij} \neq 0, \quad \text{then } x_{ji} = 0\,.$$

(9) $$x_{ii} = 0, \quad \text{for } i = 1, 2, \ldots, n\,.$$

Since $X + Y$ is real, then for $i, j \in N$, $x_{ij} + \overline{x}_{ji}$ is real, but from (8) x_{ij} and x_{ji} cannot both be nonzero. Then x_{ij} must be real. ■

We note from (6) and Lemma 4, as $s = 0$, that

(10) $$\sum_{l=1}^{n} x_{il}^2 = \sum_{l=1}^{n} x_{li}^2.$$

Lemma 5. *If X has m zero rows (or columns), where $0 \leqslant m < n$, then L_r^0 has a representation of dimension $(n - m)$.*

Proof. If the i-th row of X is a zero row, then from (10) the i-th column of X must be a zero column, and vice versa. Use Lemma 1 so that, $X = \operatorname{diag}(X', \mathbf{0})$, $Y = \operatorname{diag}(Y', \mathbf{0})$ and $Z = \operatorname{diag}(Z', Z'')$, where, $X' = [x'_{ij}]$, $Y' = X'^{\dagger}$ and $Z' = [\delta_{ij} z'_{ij}]$ are $(n-m) \times (n-m)$ matrices, and X' has no zero rows or columns. For $i, j = 1, 2, \ldots, (n-m)$; $x'_{ij} = x_{ij}$ and from (3) and (10) we have $r x_{ij} = \sum_{l=1}^{n} (\delta_{il} z_{il} x_{lj} - x_{il} \delta_{lj} z_{lj}) = \sum_{l=1}^{n-m} (\delta_{il} z'_{il} x'_{lj} - x'_{il} \delta_{lj} z'_{lj}) = r x'_{ij}$ and $\sum_{l=1}^{n-m} x'^2_{il} = \sum_{l=1}^{n-m} x'^2_{li}$.

Hence, $[Z', X'] = rX'$ and $[X', Y'] = \mathbf{0}$. Therefore, X', Y' and Z' are representation matrices for K_+, K_- and K_0, respectively. ■

Remark 2. We use Lemma 5 to eliminate all zero rows and zero columns of X. Thus, if $X \neq \mathbf{0}$ then it can be considered that X has no zero row or zero column.

Theorem 6. *The nontrivial matrix representations of L_r^0 are not faithful and of the form $X = Y = \mathbf{0}$ and Z is a nonzero real diagonal matrix.*

Proof. *Case 1*: If $n = 1$, then $X = Y = x_{11} = 0$, from Lemma 4.

Case 2: If $n = 2$, then from (10) $x_{12}^2 = x_{21}^2$ and from (8) either x_{12} or x_{21} is zero. Thus $X = Y = \mathbf{0}$.

Case 3: For $n \geqslant 3$, suppose that $X \neq \mathbf{0}$ and has k nonzero elements. Then from (8-9) $n \leqslant k \leqslant (n^2 - n)/2$. From Remark 2, for each $i \in N$ choose an $x_{ij} \neq 0$. Then from (7), we have $z_{ii} - z_{jj} = r$ or equivalently, $c_{ii} - c_{jj} = 1$, where $c_{ll} = z_{ll}/r$; $l = i, j$. After this, the proof follows the same lines as that of Theorem 3.1 of [4], where it was proved that the system of equations $c_{ii} - c_{jj} = 1$, is inconsistent.

In all cases, for a nontrivial representation Z is a nonzero real diagonal matrix. ■

4 - Faithful matrix representations for L_0^0

Lemma 7. *L_0^0 has no faithful matrix representations of dimension* 2.

Proof. Suppose that $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $Y = X^\dagger$ and $Z = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$ are representation matrices for K_+, K_- and K_0, respectively, where $a, b, c, d \in \mathbb{C}$; $u, v \in \mathbb{R}$. Since $X + Y$ is real and from (3) as $r = s = 0$, we have:

$$b + \bar{c} = \text{real}\,, \tag{11}$$

$$|b|^2 = |c|^2 \quad \text{and} \quad \bar{c}(a - d) = b(\bar{a} - \bar{d})\,, \tag{12}$$

$$b(v - u) = 0\,. \tag{13}$$

From (11) and (12), we have either $c = b$ or $c = -\bar{b}$.

It can be shown, using (12-13), when $c = b = 0$ or $c = -\bar{b} = 0$ that (1) has a nontrivial solution in λ, μ and ν. Hence the representation is not faithful. Also, if $c = b \neq 0$ or $c = -\bar{b} \neq 0$, then from (13), as $Z \neq \mathbf{0}$, we have $u = v = l$, for some $l \in \mathbb{R}^*$. It can be shown, using (12), that (1) has a nontrivial solution. ■

Theorem 8. *L_0^0 has three types of faithful matrix representations of dimension* 3, *as the least dimension, namely*:

type 1: $Z = uI_3,\ u \neq 0,\ X = \mathrm{diag}(a, b, c),\ \Delta_1 \neq 0,$

type 2: $Z = \mathrm{diag}(u, u, v),\ X = \mathrm{diag}(a, b, c),\ \Delta_2 \neq 0,$

type 3: $Z = \mathrm{diag}(u, v, w),\ X = \mathrm{diag}(a, b, c),\ \Delta_3 \neq 0,$

where,

$$\Delta_1 = \begin{vmatrix} 1 & a & \bar{a} \\ 1 & b & \bar{b} \\ 1 & c & \bar{c} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} u & a & \bar{a} \\ u & b & \bar{b} \\ v & c & \bar{c} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} u & a & \bar{a} \\ v & b & \bar{b} \\ w & c & \bar{c} \end{vmatrix};$$

for $a, b, c \in \mathbb{C}$ and different $u, v, w \in \mathbb{R}$. $Y = X^\dagger = \overline{X}$.

Proof. Since $r = 0$, in view of (4), Lemma 2 is also applicable to L_0^0. Thus we have the following three cases.

Case 1: Let $Z = uI_3 \neq \mathbf{0}$ and let A, $B = A^\dagger$ and $C = Z$, be 3×3 representation matrices for K_+, K_- and K_0 respectively. Since $AA^\dagger = A^\dagger A$, A is a normal matrix. Hence, $A = U^\dagger DU$ for some unitary matrix U and a complex diagonal matrix D. Since $A + A^\dagger$ is real, $U^\dagger(D + \overline{D})\,U = R$ for some real matrix R. But $U^\dagger(D + \overline{D})\,U$ is a Hermitian matrix, therefore, R is symmetric and consequently, has real eigenvalues. Also, $D + \overline{D} = URU^\dagger$ is a real diagonal matrix. Hence, $U^\dagger$ can be chosen as a real matrix since its columns consist of the eigen vectors of R. Therefore, $U = O$ for some orthogonal matrix O. Hence, $A = O^{-1}DO$, $B = O^{-1}\overline{D}O$ and $C = O^{-1}uI_3O$. Obviously, this representation is conjugate to the representation $X = D = \mathrm{diag}(a, b, c)$, $Y = \overline{D}$ and $Z = uI_3$ which in view of (1) is faithful if $\Delta_1 \neq 0$.

Case 2: Let $Z = \mathrm{diag}(u, u, v)$, let A, $B = A^\dagger$ and $C = Z$, be 3×3 representation matrices for K_+, K_- and K_0 respectively. From Lemma 2, $A = \mathrm{diag}(A', c)$, A' is a 2×2 matrix. Since $AA^\dagger = A^\dagger A$, $A'A'^\dagger = A'^\dagger A'$, and as $A + A^\dagger$ is real, $A' + A'^\dagger$ is also real. With a similar argument for A' instead of A as in Case 1, we find that $A' = O'^{-1}D'O'$ for some orthogonal matrix O' and diagonal matrix $D' = \mathrm{diag}(a, b)$. Let $O = \mathrm{diag}(O', 1)$, and $D = \mathrm{diag}(D', c)$, then we have $A = O^{-1}DO$, $B = O^{-1}\overline{D}O$ and $C = O^{-1}ZO$, since O is an orthogonal matrix. Clearly, this representation is conjugate to the representation $X = D = \mathrm{diag}(a, b, c)$, $Y = \overline{D}$ and $Z = \mathrm{diag}(u, u, v)$ which in view of (1) is faithful if $\Delta_2 \neq 0$.

Case 3: If $Z = \mathrm{diag}(u, v, w)$, then from Lemma 2, X is a complex diagonal matrix. So, let $X = \mathrm{diag}(a, b, c)$ which from (1), is faithful if $\Delta_3 \neq 0$. ■

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Abstract

Consider the Lie algebras L_r^s: $[K_+, K_-] = sK_0$, $[K_0, K_\pm] = \pm rK_\pm$; $r, s \in \mathbb{R}$, K_0 *is a Hermitian operator and* $K_- = K_+^\dagger$. *In* [4], [5] *the faithful matrix representations of* L_r^s *and* ${}^cL_r^s$ *were discussed for* $rs \neq 0$. *In this note we consider the case* $rs = 0$. *We prove that* L_0^0 *has three types of faithful* 3*-dimensional representations, as the least dimension, while* L_0^s, $s \neq 0$ *and* L_r^0, $r \neq 0$ *have none.*

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