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A note on regular rings, III (**)

Dedicated to Professor C. FAITH on his 70th birthday

Introduction

Von Neumann regular rings were introduced by John Von Neumann some sixty years ago (cf. [**5**]). Strongly regular rings were first studied by R. F. Arens, I. Kaplansky in 1948. Since several years, von Neumann regular, V-rings and associated rings are extensively studied (cf. for example, [**1**], [**2**], [**5**], [**7**]-[**10**], [**22**], [**23**]). Throughout, A denotes an associative ring with identity and A-modules are unital. J, Z will stand respectively for the Jacobson radical and the left singular ideal of A. A left (right) ideal of A is called reduced if it contains no non-zero nilpotent element. An ideal of A will always mean a two-sided ideal. Following E. H. Feller, A is called left duo if every left ideal of A is an ideal.

Recall that (1) A is left (resp. right) quasi-duo if every maximal left (resp. right) ideal of A is an ideal of A; (2) A is fully (resp. (*a*) fully left; (*b*) fully right) idempotent if every ideal (resp. (*a*) left ideal; (*b*) right ideal) of A is idempotent. Fully idempotent left and right quasi-duo rings need not be regular [**22**]. As before, we write "A is ELT" (resp. MELT) if every essential (resp. maximal essential, if it exists) left ideal of A is an ideal of A. Similarly, ERT and MERT rings are defined on the right side. MERT rings effectively generalize ERT rings, right quasi-duo rings and semi-simple Artinian rings. Rings whose right ideals one quasi-injective are examples of ERT rings (Jain-Mohamed-Singh).

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Various conditions for ERT (MERT) rings to be von Neumann regular have been given (cf. [**1**], [**14**], [**15**], [**16**]).

Currently, A is called (1) a left (resp. right) SF-ring if every simple left (resp. right) A-module is flat; (2) a left (resp. right) V-ring if every simple left (resp. right) A-module is injective (definition given by C. Faith). A right quasi-duo ring A is strongly regular in each of the following cases: (1) A is left SF; (2) A is right SF; (3) A is a left V-ring; (4) A is a right V-ring (cf. for example, [**1**], [**17**], [**18**]).

In 1974, we introduced p-injective modules (especially, simple p-injective modules) to study von Neumann regular rings and V-rings.

Recall that a left A-module M is p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to A (cf. [**10**], p. 340). Rings whose cyclic singular left modules are p-injective need not be regular (as shown in the following example).

Example. Set $A = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$, where K is a field. Any cyclic singular left A-module is isomorphic to $C = A/M$, where $M = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$ and ${}_AC$ is p-injective. But the Jacobson radical of A is $J = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ and therefore A cannot be von Neumann regular.

$I = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$ is an injective maximal left ideal of A. Since $J^2 = 0$ and A is (left and right) hereditary Artinian, then all singular left (and right) A-modules are injective [4, Proposition 3.5]. Also, all non-singular left (and right) A-modules are projective while the maximal left and right quotient rings of A coincide. For simple left (or right);

A-modules, p-injectivity coincides with injectivity (but not with flatness).

We propose a new characterization of regular rings.

Theorem 1. *The following conditions are equivalent*:

1) *A is von Neumann regular*;

2) *Every cyclic singular left A-module is either p-injective or flat and every principal right ideal of A is the right annihilator of an element of A.*

Proof. If A is von Neumann regular, then every left (right) A-module is p-injective and flat. Thus (1) implies(2).

Assume (2). Let $b \in A$. If $bA = r(c)$, $c \in A$, K a complement left ideal of A such that $L = l(b) \oplus K$ is an essential left ideal of A, first suppose that $L = A$. Since ${}_A A$ is p-injective (in as much as every principal right ideal of A is a right annihilator), then ${}_A Ab(\approx A/l(b))$ is p-injective which implies that ${}_A A/Ab$ is flat. Then A/Ab, being a finitely related flat left A-module, is projective which implies that ${}_A Ab$ is a direct summand of ${}_A A$. Next consider the case when $L \neq A$. Then ${}_A A/L$ is cyclic singular.

If ${}_A A/L$ is flat, since $c \in l(b) \subseteq L$, then $c = cd$ for some $d \in L$ [6], Theorem 3.57. Now $1 - d \in r(c)$ which implies that $1 = d + bu$ for some $u \in A$. Since $d = v + k$, $v \in l(b)$, $k \in K$, $1 = v + k + bu$ and $c = cv + ck + cbu$ implies that $c - cv = ck \in l(b) \cap K = 0$, whence $c = cv$. Therefore $1 - v \in r(c) = bA$ which yields $1 = v + bw$, $w \in A$. Thus $b = bwb$. Finally suppose that ${}_A A/L$ is p-injective. If f: $Ab \to A/L$ is the map defined by $f(ab) = a + L$ for all $a \in A$, there exist $z \in A$ such that $1 + L = f(b) = bz + L$ which yields $1 - bz = t + h$, $t \in l(b)$, $h \in K$. Now $c = ct + ch$ and $ch = c - ct \in l(b) \cap K = 0$ implies that $1 - t \in r(c) = bA$. If $1 - t = by$, $y \in A$, then $b = byb$. In any case, ${}_A Ab$ is a direct summand of A which proves that A is von Neumann regular and thus (2) implies (1).

Simple flat left A-modules need not be p-injective and the converse is not true either (cf. above example).

Write «A is left SPF» if every simple left A-module is either p-injective or flat.

Proposition 2. *If A is a left* SPF *ring which is either left or right quasi-duo, then A/J is strongly regular.*

Proof. We know that $B = A/J$ is a reduced ring (cf. the proof of «(2) implies (3)» in [**14**], Theorem 2.1. Since B is a left SPF ring then B, being reduced, is fully left (and right) idempotent. Now B is strongly regular (in as much as B is either left or right quasi-duo) by [**16**], Proposition 4(8).

Remark 1. If A is a left SPF ring, then A is left quasi-duo iff A is right quasi-duo (such rings need not be regular as shown by the example given above). Consequently, if A is a ring which is either left or right quasi-duo, then A is left SPF iff A is right SPF.

Proposition 3. *If A is left* SPF, *then $J = 0$ iff J is a reduced ideal of A.*

Proof. Suppose that J is a non-zero reduced ideal of A. If $0 \neq b \in J$, set $L = Ab + l(b)$. If we suppose that $L = A$, then $1 = ab + c$, $a \in A$, $c \in l(b)$, which implies that $b = ab^2$. Since $b \in J$, $b - bab \in J$ and $(b - bab)^2 = 0$ yields $b = bab$.

Therefore $b = be$, where $e = ab$ is idempotent. Since J cannot contain a non-zero idempotent, then $b = 0$. This proves that $L \neq A$. Let M be a maximal left ideal of A containing L. If ${}_AA/M$ is p-injective, define the map $f\colon Ab \to A/M$ by $f(ab) = a + M$ for all $a \in A$. Then $f(b) = bc + M$, for some $c \in A$ and therefore $1 + M = bc + M$ which implies that $1 - bc \in M$, whence $1 \in M$ (because $bc \in J \subseteq M$), contradicting $A \neq M$. If A/M is flat, since $b \in M$, then $b = bd$ for some $d \in M$ [6], Theorem 3.57. Now $1 - d \in r(b) = l(b) \subseteq M$ which implies that $1 \in M$, again a contradiction! This proves that if J is reduced, then $J = 0$.

Remark 2. If we consider the left singular ideal Z of A, without any hypothesis on A, we know that $Z = 0$ iff Z is a reduced ideal of A.

ELT rings whose simple left modules are either injective or projective need not be regular (cf. above example). If A is semi-prime, then it is well-known that every essential right ideal of A which is an ideal of A must be left essential. The converse is obviously not true.

Theorem 4. *Suppose that every essential right ideal of A is an essential left ideal. If every simple left A-module is either p-injective or projective, then A is von Neumann regular.*

Proof. Suppose that A is not semi-prime. Then, there exist $0 \neq b \in A$ such that $(Ab)^2 = 0$. Let K be a complement right ideal of A such that $R = AbA \oplus K$ is an essential right ideal of A. Then $Kb \subseteq K \cap AbA = 0$ implies that $R \subseteq l(b)$. Since, by hypothesis, R is an essential left ideal of A, then $b \in Z$. But $Ab \subseteq J$ which implies that $0 \neq b \in J \cap Z$. Since every simple left A-module is either p-injective or projective, then $Z \cap J = 0$ by [**12**], Proposition 3. This contradiction proves that A must be semi-prime. In that case, A is fully left idempotent by [**12**], Proposition 6 and since A is ERT, then A is von Neumann regular by [**13**], Propositon 9.

The proof of Theorem 4 yields the next two remarks

Remark 3. If A is right continuous (in the sense of Y. Utumi [**7**]) such that $J \cap Z = 0$ and every essential right ideal of A is an essential left ideal, then A is regular.

Remark 4. If A contains an injective maximal right ideal, $Z \cap J = 0$ and every essential right ideal is an essential left ideal, then A is a left and right self-injective regular, left and right V-ring of bounded index (cf. [**15**], Lemma 1).

Question 1. Does Theorem 4 hold if «simple left» is replaced by «simple right»?

Question 2. Does Theorem 4 hold if «projective» is replaced by «flat»?

As in previous papers, we often pay particular attention to cyclic and simple modules. Such modules play an important role in ring theory (cf. [3]). Certain rings are completely characterized by cyclic or simple modules. For example, the following well-known results emphasize that fact: (1) A is semi-simple Artinian iff every cyclic left A-module is injective (B. Osofsky); (2) If A is commutative, then A is von Neumann regular iff A is a V-ring (I. Kaplansky).

The example preceding Theorem 1 shows that rings containing an injective maximal left ideal need not be self-injective (otherwise, the ring quoted there would have been semi-simple Artinian!). Call A a left (resp. right) MI-ring if A contains an injective maximal left (resp. right)ideal.

Theorem 5. *Let A be a left MI-ring with an injective maximal left ideal M which is not an ideal of A. If Ma is a projective left A-module for every $a \in A$, then A is left self-injective regular.*

Proof. $A = M \oplus U$, where U is a minimal projective left ideal of A. Now $MU \neq 0$ (otherwise, $MU = 0$ implies that $U = AU = MAU = MU = 0$, a contradiction!). Then $MU = U = Au$, $u = u^2 \in A$ and $Mv \neq 0$ for some $v \in U$. The map g: $M \to U$ defined by $g(m) = mv$ yields $M/\ker g \approx U$ and since U is projective, $M \approx \ker g \oplus (M/\ker g)$. Therefore U is injective which implies that $A = M \oplus U$ is left self-injective. Suppose that $Z \neq 0$. If $0 \neq z \in Z$, then $Mz \neq 0$ (otherwise, $M = l(z)$ is an essential left ideal of A, contradicting M a direct summand of A).

If $M = Ae$, $e = e^2 \in A$, then $Aez = Mz$ is a projective left A-module which implies that $l(ez)$ is a direct summand of A. Since $ez \in Z$, then $ez = 0$ which contradicts $Mz \neq 0$. This proves that $Z = 0$ and A is therefore left self-injective regular.

Theorem 6. *Let A contain an injective maximal left ideal M such that $r(M)$ contains no non-zero nilpotent ideal of A. Then A is left self-injective.*

Proof. $A = M \oplus V$, where $M = Ae$, $e = e^2 \in A$, $V = Av$, $v = 1 - e$.
If $MV = 0$, then $M = MA$ is an ideal of A and $VA \subseteq r(M) = vA$ which implies

that $AvA = vA$. Now $MV = 0$ implies that $(1 - v)Av = 0$ and therefore $av = vav$ for all $a \in A$. Also, $VM = AvM \subseteq AvA = vA = r(M)$ and $(VM)^2 = 0$ implies that $VM = 0$ by hypothesis. Therefore $vA(1 - v) = 0$ which yields $va = vav$ for all $a \in A$. Thus $av = va$ for all $a \in A$ which proves that v is a central idempotent. Then $V = vA$ and $M = eA$ which yields $A_A = M_A \oplus V_A$. Now A/M_A is projective which implies that ${}_A A/M$ is injective. Thus ${}_A V$ is injective and $A = M \oplus V$ is left self-injective. Now suppose that $MV \neq 0$. Since ${}_A V$ is projective, the proof of Theorem 5 shows that ${}_A V$ is again injective. In any case, $A = M \oplus V$ is left self-injective.

Recall that A is biregular if, for every $a \in A$, AaA is generated by a central idempotent. It is well-known that left self-injective regular rings need not be biregular.

Corollary 7. *Let A be a semi-prime ring such that for each $a \in A$, AaA is a left annihilator. If A is left MI, then A is left self-injective biregular.*

Proof. A is left self-injective by Theorem 6 (in as much as in a semi-prime ring A, given an idempotent u, Au is a minimal left ideal iff uA is a minimal right ideal). Now, for any $a \in A$, $AaA = l(T)$, where T is an ideal of A. Since A is semi-prime, $AaA \cap l(AaA) = 0$ and $l(T) = r(T)$. Since A is left self-injective, by Ikeda-Nakayama's theorem [**3**], p. 189, $A = r(AaA \cap l(AaA)) = r(AaA) + r(l(AaA))$. Since $AaA = l(T) = r(T)$, $A = r(AaA) + AaA$ which yields $A = r(AaA) \oplus AaA$. Then $AaA = uA$, $u = u^2 \in A$. Since A is semi-prime, u is central in A. This proves that A is biregular. Corollary 7 slightly improves [**20**], Proposition 5.

Theorem 8. *Let A be a semi-prime* ELT *left MI-ring. Then A is a left and right self-injective biregular left and right V-ring of bounded index.*

Proof. By Theorem 6, A is left self-injective. If $Z \neq 0$, there exist $0 \neq z \in Z$ such that $z^2 = 0$ (Remark 2). Then $l(z)$ is an ideal of A which implies that $AzA \subseteq l(z)$, whence $(Az)^2 = 0$. Since A is semi-prime, $Az = 0$, which contradicts $z \neq 0$. This proves that $Z = 0$ and A is therefore left self-injective regular. By [**15**], Lemma 1.1, A is a right self-injective left and right V-ring of bounded index. In that case, A is biregular by [**5**], Theorem 7.20, Corollary 8.24 and Corollary 9.15 (in as much as any prime factor ring of A, being ELT regular, is primitive with non-zero socle).

It is well-known that A is left pseudo-Frobeniusean iff A is left self-injective with finitely generated essential left socle (cf. for example, [**3**], [**10**]). In that case, every left ideal of A is a left annihilator.

Proposition 9. *Let A be a left and right MI-ring with finitely generated essential left and right socles. If every minimal right ideal of A is a right annihilator, then A is left and right pseudo-Frobeniusean.*

Proof. Let M be an injective maximal left ideal of A. Then $A = M \oplus V$, where $M = Ae$, $e = e^2 \in A$, $V = Av$, $v = 1 - e$. Since A has an essential right socle, then vA contains a minimal right ideal R. Now $M = l(vA) = l(R)$ which implies that ${}_VA \subseteq r(l(vA)) = r(l(R)) = R$ (because R is a right annihilator). Therefore $r(M) = vA = R$. By Theorem 6, A is left self-injective which implies that A is left pseudo-Frobeniusean. In that case, every left ideal of A is a left annihilator. Then the above argument shows that A is also right self-injective and consequently, A is right pseudo-Frobeniusean.

F. Dischinger and W. Müller have given an example of a left pseudo-Frobeniusean ring which is not right pseudo-Frobeniusean (cf. [**10**], p. 580).

Question 3. Does Proposition 9 hold if the following hypothesis is dropped: every minimal right ideal of A is a right annihilator? Note that in a von Neumann regular ring A, the following properties are equivalent: (a) A is ELT; (b) A is ERT; (c) A is MELT; (d) A is MERT. A result of Zhang Jule [**22**], Theorem 1 asserts that MELT fully idempotent rings need not be regular (thus answering in the negative a question raised in [**16**]).

Proposition 10. *Let A be a* MELT *fully idempotent ring. If every prime factor ring of A contains an injective maximal left ideal, then A is unit-regular.*

Proof. Let B be a prime factor ring of A. Since B contains an injective maximal left ideal M, then $r(M)$ is a minimal non-nilpotent right ideal of B and by Theorem 6, B is left self-injective. Since B has non-zero socle, then B is left non-singular which implies that B is regular. Now B, being an ELT left self-injective regular ring, is a right self-injective left and right V-ring by Theorem 8. Thus B contains simple faithful injective projective left and right modules and by a result of D. Handelman, B is simple Artinian. A is therefore unit-regular by [**5**], Corollary 1.18 and Theorem 6.10.

Question 4. Does Proposition 10 hold if «prime factor ring» is weakened to «primitive factor ring»?

We now turn to a new characterization of strongly regular rings

Theorem 11. *The following conditions are equivalent*:

1) *A is strongly regular*;
2) *A is a reduced ring such that every prime factor ring is left MI.*

Proof. If A is strongly regular, then every prime factor ring is a division ring. Thus (1) implies (2). Assume (2). Let B be a prime factor ring of A. Then by Theorem 6, B is left self-injective and since B has non-zero socle, then B is regular. By [**2**], Corollary 4.5, A is regular. Thus (2) implies (1).

Properly semi-prime rings (noted PSP) are defined and studied in [**2**]. Since the ring A considered here has an identity, if A is left PSP, then A must be semi-prime [**2**], p. 844.

Proposition 12. *The following conditions are equivalent for a* MELT *ring A*:

1) *A is biregular and unit-regular*
2) *A is a left* PSP *ring whose prime factors are left MI.*

Proof. Assume (1). Let B be a prime factor ring of A. Since B is biregular, then B is simple. Since B is MELT, then every maximal left ideal of B is a direct summand of B which implies that B is Artinian. Thus (1) implies (2).

(2) implies (1) by [**2**], Theorem 4.2 and Proposition 10.

Finally, we give a «test module» for A to be primitive left self-injective regular with non-zero socle.

Theorem 13. *The following conditions are equivalent*:

1) *A is primitive left self-injective regular with non-zero socle*
2) *A is a prime ring with an injective maximal left ideal*
3) *There exist an injective maximal left ideal of A which contains no non-zero ideal of A.*

Proof. (1) implies (2) evidently.

Since a non-zero ideal in a prime ring is an essential left ideal, then (2) implies (3).

Assume (3). Let M be an injective maximal left ideal of A which contains no non-zero ideal of A. Then A/M is a simple, faithful left A-module which implies that A is primitive with non-zero socle. This yields $Z = 0$ (because Z cannot contain a non-zero idempotent). Since A is left self-injective by Theorem 6, then (3) implies (1).

Remark 5. Theorem 13 shows that left MI-rings (even when primitive) need not be right MI. (Otherwise, any primitive left self-injective regular ring with non-zero socle would be necessarily simple Artinian!).

A nice corollary follows.

Corollary 18. *The following conditions are equivalent*:

1) *A is simple Artinian*;

2) *A is a prime ring which is both left and right MI*;

3) *There exist an injective maximal left ideal and an injective maximal right ideal of A which contain no non-zero ideal of A.*

In a recent paper [**11**], it is proved that a right quasi-duo ring which is either fully right or fully left idempotent must be regular. This is also true for MERT rings.

Our last remark is motivated by [**2**], Theorem 4.2.

Remark 6. If A is a left PSP ring whose simple left modules are either p-injective or projective, then A is biregular.

We conclude by noting that interesting conditions are given recently by J.Y. Kim-J.K. Park (Math. Japonica 45 (1997), 311-313) for von Neumann regular rings to be semi-simple Artinian.

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Summary

In this sequel to [**20**] *and* [**21**], *new characterizations of von Neumann regular rings and strongly regular rings are given. Conditions for certain rings to be regular, self-injective regular and biregular are considered. A «test module» is given for a ring to be primitive left self-injective regular with non-zero socle* (*Theorem 13*).

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