Riv. Mat. Univ. Parma (6) **1** (1998), 39-52

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CR-structures on $SO_g(M)$ (**)

1 - Introduction

The geometry of an orientable Riemannian manifold (M, g) may be studied in the terms of its linear bundle or, more precisely, via the bundle of the positive orthogonal frames $P := SO_g(M)$.

Remind that P is a parallelisable manifold. Thus, we may consider the canonical parallelism submitted to the Levi-Civita connection of (M, g). with respect to this connection, we have the splitting

$$T_u P = H_u P \oplus F_u P ,$$

where $H_u P$ (resp. $F_u P$) is identificated with $\mathbb{R}^m$ (resp. $\mathfrak{so}(m)$), cf. (3).

Since the Lie-algebra $\mathfrak{so}(m)$ is endowed with suitable CR-structures (the $\mathfrak{u}$-normal ones, as defined in (2)), we introduce an almost-CR-structure $(CP, \mathbb{J})$ on P submitted to the above splitting. Let us recall the classical definition of a CR-structure:

Definition 1.1. *A* CR-*structure on a given smooth manifold P is a pair (CP, J) such that*

1. *CP is an even-dimensional subbundle of TP*;
2. *the linear map $J: CP \to CP$ is such that $J^2 = -Id$*;
3. for any X, Y sections of CP, $[JX, Y] + [X, JY]$ is a section of CP;

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(**) Received March 21, 1998. AMS classification 32 C 16, 22 E 10, 53 C 20.

4. *the Nijenhuis tensor*

$$N_J(X, Y) := [JX, JY] - [X, Y] - J([JX, Y] + [X, JY])$$

vanishes identically.

The fourth condition is said to be the *condition of integrability* of (CP, J). We speak of an *almost*-CR-*structure* (CP, J) when the tensor N_J does not vanish identically: i.e. when (CP, J) is not integrable. A natural question is the study of its integrability in dependence on the differential geometry of the manifold, [**AHR**], [**WE**].

Thus we are interested in the integrability of our almost-CR-structures on the principal bundle $P = SO_g(M)$. In particular, we look for conditions involving the curvature of the Riemannian manifold (M, g)). The main result obtained, Theorem 4.4, assures that

Theorem 1.2 *The almost-*CR-*structures, as constructed in Sections* 3 *and* 4, *are integrable if and only if the manifold* (M, g) *has constant sectional curvature.*

Notice that the construction of the almost-CR-structures defined in Section 3 and 4 depends on the algebraic properties of the Lie-algebra $\mathfrak{so}(m)$ of the structure group $SO(m)$. Thus, Section 2 is devoted to the description of $\mathfrak{so}(m)$, giving the classical splittings

$$\mathfrak{so}(m+1) = \mathfrak{so}(m) \oplus \mathbb{R}^m,$$

$$\mathfrak{so}(2n) = \mathfrak{u}(n) \oplus \mathfrak{s}(n).$$

Via these decompositions, we divide our study in the four cases $m = 4n$, $4n+1$, $4n+2$, $4n+3$.

In Section 2, the nonexistence of Lie-CR-structures [1] is proved for $\mathfrak{so}(m)$. Moreover, the $\mathfrak{u}$-normal CR-structures are introduced, together with a characterization of them (Proposition 2.5).

The first part of Section 3 recalls the main notations on the principal bundle $P = SO_g(M)$. Furthermore, after the exposition of a result of Pacini on the $4n$-di-

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(1) Such special CR-structures are defined on Lie-groups as the ones with respect to which both the right and the left translations are CR-maps. In the terms of the Lie-algebra $\mathfrak{g}$, they are determined by an ideal $\mathfrak{p} \subseteq \mathfrak{g}$ and a map $J\colon \mathfrak{p} \to \mathfrak{p}$ such that $J^2 = -Id$ and $J\,\mathrm{ad}_X = \mathrm{ad}_X J$, $\forall X \in \mathfrak{g}$.

mensional case, proved in [**PA**], the even-dimensional case is completed with the study of $m = 4n + 2$.

Finally Section 4 provides an algebraic condition on the curvature equivalent to the integrability of J in the odd-case. Moreover, we conclude with an example of manifols whose curvature satisfies such a condition.

Manifolds and maps will be $\mathcal{C}^\infty$. The Lie-algebra of a Lie-group G is denoted with $\mathfrak{g}$. The direct sum of linear spaces is given by $\oplus$, while the one of Lie-algebras by $\odot$: thus, $\mathfrak{g} \odot \mathfrak{h}$ means that $[X, Y]$ vanishes for any X in $\mathfrak{g}$ and Y in $\mathfrak{h}$. Moreover, we often omit the point on which a map or a vector is defined: so, if x is in $\boldsymbol{R}^m$, Bx denotes the element $B(u)x$ of $H_u P$.

2. - Levi-flat CR-structures on $\mathfrak{so}(m)$

The structure group $SO(m)$ of the principal bundle $P = SO_g(M)$ may be regarded as its vertical fiber. Moreover, the Levi-Civita connection of M determines the horizontal distribution HP and we have that $H_u P \simeq \mathbb{R}^m$, $F_u P \simeq \mathfrak{so}(m)$.

The algebraic properties of $\mathfrak{so}(m) := \{M \in \mathbf{gl}(m, \boldsymbol{R}) / M^t + M = 0\}$ shall suggest the definition of a suitable family of almost-CR-structures on P. First of all we recall two classical splittings of $\mathfrak{so}(m)$. Hence, we investigate the existence of CR-structures on $\mathfrak{so}(m)$ submitted to these splittings. Thus, we start with the study of this semisimple Lie algebra.

The characterizing condition $M^t + M = 0$ implies that the generic element is of the form

$$M = \begin{pmatrix} M_1 & v \\ -v^t & 0 \end{pmatrix},$$

with M_1 in $\mathfrak{so}(m-1)$ and v in $\mathbb{R}^{m-1}$. Consequentely, the dimension of $\mathfrak{so}(m)$ is $m(m-1)/2$ and $\mathfrak{so}(m)$ splits as

$$\mathfrak{so}(m) = \mathfrak{so}(m-1) \oplus \mathbb{R}^{m-1}. \tag{1}$$

The Lie-product defines the following relations involving the previous decomposition (1)

$$\begin{aligned} &[M, N]_{\mathfrak{so}(m)} = [M, N]_{\mathfrak{so}(m-1)}, && \forall M, N \in \mathfrak{so}(m-1) \\ &[M, v] = Mv, && \forall M \in \mathfrak{so}(m-1),\ \ v \in \mathbb{R}^{m-1} \\ &[u, v] = vu^t - uv^t, && \forall u,\ \ v \in \mathbb{R}^{m-1}. \end{aligned}$$

Notice that the splitting (1) and the involutive map

$$\alpha\begin{pmatrix} M_1 & v \\ -v^t & 0 \end{pmatrix} = \begin{pmatrix} M_1 & -v \\ v^t & 0 \end{pmatrix}.$$

produce the orthogonal symmetric Lie-algebra $(\mathfrak{so}(m), \alpha)$, which corresponds to the globally symmetric space

$$S^{m-1} = SO(m)/SO(m-1).$$

Whenever m is even (: $m = 2n$), consider the element of $\mathfrak{so}(2n)$

$$J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Then, the generic matrix M in $\mathfrak{so}(2n)$ may be written as $M = U + S$ where $U = M - J_0 M J_0/2$ and $S = M + J_0 M J_0/2$.

Consequentely, $\mathfrak{so}(2n)$ splits as

(2) $$\mathfrak{so}(2n) = \mathfrak{u}(n) \oplus \mathfrak{s}(n),$$

with

$$\mathfrak{u}(n) = \{U \in \mathfrak{so}(2n)/UJ_0 = J_0 U\},$$

$$\mathfrak{s}(n) = \{S \in \mathfrak{so}(2n)/SJ_0 = -J_0 S\}.$$

Alternatively, the subspaces $\mathfrak{u}(n)$ and $\mathfrak{s}(n)$ may be seen as

$$\mathfrak{u}(n) = \left\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \Big/ A \in \mathfrak{so}(n),\ B = B^t\right\},$$

$$\mathfrak{s}(n) = \left\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \Big/ A,\ B \in \mathfrak{so}(n)\right\}.$$

Moreover, $\mathfrak{u}(n)$ is a real subalgebra of $\mathfrak{so}(2n)$, $[\mathfrak{u}(n), \mathfrak{s}(n)] \subseteq \mathfrak{s}(n)$, and $[\mathfrak{s}(n), \mathfrak{s}(n)] \subseteq \mathfrak{u}(n)$.

Furthermore, the Lie algebra $\mathfrak{u}(n)$ is compact, its center is $\mathbb{R}J_0$ and its derived algebra is

$$\mathfrak{su}(n) := \left\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathfrak{u}(n)/\operatorname{tr} B = 0\right\}.$$

Via the splittings (1) and (2), we obtain a suitable decomposition of each $\mathfrak{so}(m)$:

$$\mathfrak{so}(4n) = \mathfrak{u}(2n) \oplus \mathfrak{s}(2n) \ni X = U + S\,,$$

$$\mathfrak{so}(4n+1) = \mathfrak{u}(2n) \oplus \mathfrak{s}(2n) \oplus \mathbb{R}^{4n} \ni X = U + S + v\,,$$

$$\mathfrak{so}(4n+2) = \mathbb{R}J_0 \oplus \mathfrak{s}u(2n+1) \oplus \mathfrak{s}(2n+1) \ni X = xJ_0 + U + S\,,$$

$$\mathfrak{so}(4n+3) = \mathbb{R}J_0 \oplus \mathfrak{s}u(2n+1) \oplus \mathfrak{s}(2n+1) \oplus \mathbb{R}^{4n+2} \ni X = xJ_0 + U + S + v\,.$$

Remark 2.1. *Finally, the equation* $[u, v] = vu^t - uv^t$ *implies that*

$$[\mathbb{R}^{2n}, \mathbb{R}^{2n}] \subseteq \mathfrak{so}(2n)\,.$$

Moreover, the rank of $[u, v]$ *is* 0*, when* u *and* v *are linearly dependent, or* 2*, when they are independent. Thus,* $[u, v]$ *is different from* J_0*. In particular, the subspace* $[\mathbb{R}^{2n}, \mathbb{R}^{2n}]$ *is contained in* $\mathfrak{su}(n) \oplus \mathfrak{s}(n)$*.*

The datum of a CR-structure $(\mathfrak{p}, J)$ on a real Lie-algebra $\mathfrak{g}_0$ is totally equivalent to a left invariant CR-structure on the corresponding Lie group G_0. Furthermore, $(\mathfrak{p}, J)$ is *Levi-flat* in the case that $\mathfrak{p}$ is a subalgebra of $\mathfrak{g}_0$; while it is a *Lie-CR-structure* (LCR-structure) when $\mathfrak{p}$ is an ideal and $\mathrm{ad}_x J = J\,\mathrm{ad}_x$, for any x in $\mathfrak{g}_0$.

Moreover, we speak of an (almost-)complex structure J when $\mathfrak{p}$ coincides with $\mathfrak{g}_0$. When the further condition $\mathrm{ad}_x J = J\,\mathrm{ad}_x$ is satisfied, J is said to be *biinvariant*. A description of these structures is given by Snow in the reductive case, [**SN**].

A known result of Morimoto assures that any even-dimensional reductive Lie-algebra admits infinitely many complex structures, [**MO**]. Thus, both $\mathfrak{u}(2n)$ and $\mathfrak{u}_0(2n+1)$ admit complex structures. Hence, the set of Levi-flat CR-structures LfCR $(\mathfrak{so}(m))$ is not empty, even if neither $\mathfrak{u}(2n)$ nor $\mathfrak{su}(2n+1)$ admit biinvariant complex structures. In fact, let J be a biinvariant one and g a biinvariant metric on $\mathfrak{u}(2n)$. A direct computation shows that $\mathfrak{u}(2n)$ should be abelian, which is false. Furthermore, they do not admit LCR-structures, [**GO**]. The same fact holds for $\mathfrak{su}(2n+1)$.

In the following, we shall select a family of Levi-flat CR-structures compatible with the splittings given in the present Section.

First of all, let us prove the

Proposition 2.2. *The semisimple Lie-algebra* $\mathfrak{so}(m)$ *does not admit* LCR-*structures.*

Proof. In the even-dimensional case $m = 2n$, let $(\mathfrak{p}, \Phi)$ be in $\mathrm{LCR}(\mathfrak{so}(2n))$. Since

$$[J_0, \Phi M] = \Phi[J_0, M] = 0 , \qquad \forall M \in \mathfrak{p} \cap \mathfrak{u} ,$$

we have that Φ maps $\mathfrak{p} \cap \mathfrak{u}$ into itself. Thus, $\mathfrak{p} \cap \mathfrak{u}$ must vanish, otherwise $(\mathfrak{p} \cap \mathfrak{u}, \Phi|_{\mathfrak{p} \cap \mathfrak{u}})$ should be an LCR-structure of $\mathfrak{u}$.

Take now an element M in $\mathfrak{p} \cap \mathfrak{s}$, then

$$2J_0 M = [J_0, M]$$

is in $\mathfrak{p} \cap \mathfrak{s}$. In particular, this implies that $\mathfrak{p}$ is contained in $\mathfrak{s}$. In fact, let $M = U + S$ be the generic element of $\mathfrak{p}$, then

$$-4S = [J_0, 2J_0 S] = [J_0, [J_0, M]]$$

is in $\mathfrak{p}$ and U vanishes. Finally, since $\mathfrak{p}$ is an ideal contained in $\mathfrak{p}$, it is abelian. Hence, $\mathfrak{p}$ vanishes, too.

Let us conclude with the odd-dimensional case: $m = 2n + 1$. For any $(\mathfrak{p}, \Phi) \in \mathrm{LCR}(\mathfrak{so}(2n+1))$, with the same argument than in the even-case, we prove that $\mathfrak{p} \cap \mathfrak{u}$ vanishes.

Then consider the element $M = U + S + v \in \mathfrak{p}$ and compute

$$[J_0, [J_0, M]] = [J_0, 2J_0 S + J_0 v] = -4S - v$$

hence

$$[J_0, M - 4S - v] = [J_0, U - 3S] = -6J_0 S$$

is in $\mathfrak{p}$. Consequentely, v and U are in $\mathfrak{p}$, too. In conclusion U vanishes and $M = S + v$.

In order to conclude the proof of Proposition 2.2, we need the following technical

Lemma 2.3. *In correspondence of a nonvanishing element* v *of* $\mathbb{R}^{2n}$, *there exists a proper subspace* E *in* $\mathbb{R}^{2n}$ *such that* $[v, e]$ *is in* $\mathfrak{u}(n)$, *for all* e *in* E. *Moreover,* $[v, e]$ *is nonvanishing, for almost every* $e \in E$. ◆

Now, observe that $[M, e] = Se + [v, e]$ is an element of $\mathfrak{p}$; hence $[v, e]$ is in $\mathfrak{s}(n)$ and, by Lemma 2.3, v vanishes. Thus, $\mathfrak{p}$ is contained in $\mathfrak{s}$ and it is an abelian ideal, which is false. ■

Even if there are no LCR-structures, Morimoto's result assures that there are CR-structures on $\mathfrak{so}(m)$ according with the

Definition 2.4. *An almost complex structure Φ on $\mathfrak{so}(4n)$ (respectively on $\mathfrak{so}(4n+1)$) is said to be* $\mathfrak{u}$*-normal if there exists a complex structure j on $\mathfrak{u}(2n)$ such that $\Phi X = jU + J_0 \circ S$ (resp. $\Phi X = jU + J_0 \circ S + J_0 v$).*

Obviously, the definition extends even to the other cases: an almost-CR-structure Φ on $\mathfrak{so}(4n+2)$ (resp. on $\mathfrak{so}(4n+3)$) is $\mathfrak{u}$*-normal if there exists a complex structure j on $\mathfrak{su}(2n+1)$ such that $\Phi X = jU + J_0 \circ S$ (risp. $\Phi X = jU + J_0 \circ S + J_0 v$).*

Notice that a CR-structure $\mathfrak{u}$-normal on $\mathfrak{so}(4n+2)$ is Levi-flat (as the restriction to $\mathfrak{u}_0 \oplus \mathfrak{s}$ of a CR-structure $\mathfrak{u}$-normal on $\mathfrak{so}(4n+3)$).

Let us conclude the Section with an useful characterization of the $\mathfrak{u}$-normality, whose proof consists in an algebraic computation.

Proposition 2.5. *An* $\mathfrak{u}$*-normal almost-CR-structure Φ is integrable. Vice versa, if Φ is integrable, then Φ is* $\mathfrak{u}$*-normal if and only if*

1. $\Phi \mathfrak{u} \subseteq \mathfrak{u}$;
2. $\Phi \mathfrak{s} \subseteq \mathfrak{s}$;
3. $[J_0, A] = \Phi(A) + J_0 \Phi(A) J_0$, $\forall A \in \mathfrak{so}(m)$;

4_1. $\Phi \mathbb{R}^{4n} \subseteq \mathbb{R}^{4n}$, in the case $m = 4n+1$;

4_2. $\Phi \mathbb{R}^{4n+2} \subseteq \mathbb{R}^{4n+2}$, in the case $m = 4n+3$. ■

3. - The geometrical situation

Let (M, g) be an m-dimensional orientable Riemannian manifold. Consider the principal bundle $P = SO_g(M)$ of the positive orthonormal frames on M. Let $\pi: P \to M$ denote the smooth canonical projection.

The action of $SO(m)$ on P induces an injection $\sigma: \mathfrak{so}(m) \to \mathcal{X}(P): x \mapsto x^*$. If x is in $\mathfrak{so}(m)$, and u in P, then $x_u^* = \sigma(x)(u)$ is the tangent vector at $t = 0$ to the curve $\gamma(t) = u \exp(tx)$. Obviously, $\sigma(x)(u)$ is an element of the vertical subspace of $T_u P$, $F_u P = T_u \pi^{-1} \pi(u)) = \operatorname{Ker} \pi_*(u)$. Thus, the map

$$\tau_u: \mathfrak{so}(m) \to F_u P: x \mapsto x^*(u)$$

is an isomorphism.

Take now the Levi-Civita connection Γ induced by g on $P = SO_g(M)$. By definition, Γ consists in the datum of an $SO(m)$-invariant subbundle HP of TP such that $TP = HP \oplus FP$. In particular, for every $u \in P$,

$$T_u P = H_u P \oplus F_u P \ni X = X^h + X^v$$

and

$$H_{ua} P = (R_a)_* H_u, \; a \in SO(m).$$

Let $\omega(u)$: $T_u P \to \mathfrak{so}(m)$: $X \mapsto \tau_u^{-1}(X^v)$ be the connection 1-form defined by Γ. Then,

1. $\omega(u) X$ vanishes if and only if X is contained in $H_u P$;
2. $\omega x^* = x$, $\forall x \in \mathfrak{so}(m)$;
3. $\omega(u) \circ \tau(u) = \theta(u)$, where θ is the canonical 1-form on $SO(m)$, determined setting $\theta x = x$, for any x of $\mathfrak{so}(m)$; hence $\theta(u)$ is the map $H_u P \to \mathbb{R}^m$: $X \mapsto u^{-1}(\pi_*(X))$, where the element $u \in SO_g(M)$ may be seen as a positive isometry

$$u: (\mathbb{R}^m, g_{\text{euc}}) \to (T_{\pi(u)} M, g_{\pi(u)}).$$

Finally, consider the map B: $\mathbb{R}^m \to HP$ defined choosing $B(u)\,\xi$ as the unique element of $H_u P$ such that

$$\pi_*(B(u)\,\xi) = u\xi .$$

Then,

$$\theta(u)\, B(\xi) = \xi .$$

Since $SO_g(M)$ is parallelisable, consider a bases (e_i) of $\mathbb{R}^m$ and one (ε_j) of $\mathfrak{so}(m)$. Then $\mathcal{B}(u) := (B(u)\, e_i, \, \varepsilon_j^*)$ is the canonical parallelism of $SO_g(M)$ determined by Γ. In these terms, we shall construct a CR-structure on $SO_g(M)$ whose integrability is determined by the Riemannian geometry of (M, g).

In order to do this, let us return to the splitting

$$T_u P = H_u P \oplus F_u P \tag{3}$$

where $F_u P$ is isomorphic to $\mathfrak{so}(m)$ via τ_u and $H_u P$ is isomorphic to $\mathbb{R}^m$ via $\theta(u)$. Suppose J_1 (resp. J_2) be a CR-structure on HP (resp. FP) and define the almost-CR-structure

$$\mathbb{J} := J_1 \oplus J_2 .$$

Notice that in the present paper we are interested in giving geometrical conditions on (M, g) which are equivalent to the fact that $\mathbb{J}$ is a CR-structure. The results are obtained in correspondence of a suitable choice of J_1 and J_2.

Now, we proceed to define both J_1 and J_2. Take a maximal even-dimensional subspace E_{2n} of $\mathbb{R}^m$: E coincides with $\mathbb{R}^m$ when $m = 2n$ and it is an hyperplane when $m = 2n + 1$. Let $(e_1 \ldots e_{2n})$ be a bases of E and set

$$J_0 e_i = e_{n+i}, \qquad J_0 e_{n+i} = -e_i, \qquad i \leq n .$$

Consider the subbundle $EP \subsetneq HP$ such that $E_u P := B(u)\, S = \mathrm{Span}\,(B(u) e_i)$. Of course, when m is even, EP coincides with HP.

Then, define $J_1(u)$: $E_u P \to E_u P$ as

$$J_1(u) := B(u) \circ J_0 \circ \theta(u) .$$

It is clear that $J_1^2 = -id$.

In an analogous way, let Φ be a $\mathfrak{u}$-normal CR-structure on $\mathfrak{so}(m)$. Notice that Φ is really a complex structure for $m = 4n$, $4n + 1$, while it is a CR-structure of codimension one for $m = 4n + 2$, $4n + 3$. More precisely Φ is defined either on $\mathfrak{su}(2n) \oplus \mathfrak{s}(2n)$ or on $\mathfrak{su}(2n) \oplus \mathfrak{s}(2n) \oplus \mathbb{R}^{4n+2}$, in these last cases. Take, now, the subbundle KP, where $K_u P := \tau_u \mathfrak{k}$ and $\mathfrak{k}$ is the subspace on which Φ is defined. Finally, the CR-structure J_2 is given setting

$$J_2(u) := \tau_u \circ \Phi \circ \tau_u^{-1} .$$

Proposition 3.1. *The pair (KP, J_2) defines a* CR-*structure, that is an almost* CR-*structure such that N_{J_2} vanishes identically.*

In fact, N_{J_2} coincides with N_Φ.

Remark 3.2. *Since we want to focus our attention on the dependence on J_0 and on Φ we denote J_2 as Φ^*, J_1 as J_0 and $\mathbb{J}$ as $\mathbb{J}_\Phi$:*

$$\mathbb{J}_\Phi = J_0 \oplus \Phi^* .$$

In conclusion of the present Section, let us study the even-dimensional cases. The results on $4n$-dimensional manifolds have been obtained by Pacini, in [**PA**].

Theorem 3.3. *When $n > 1$ and $m = 4n$, the complex structure $\mathbb{J}_\Phi$ is integrable if and only if M has costant sectional curvature.*

Since the proof is the same in both the even-dimensional cases: $m = 4n$,

$4n+2$, it is developed for $m=4n+2$ (the computations for $m=4n$ are exposed in detail in [**PA**]); the following lemmas are useful preliminaries to Theorem 3.6.

Lemma 3.4. *For all* $a \in \mathbf{gl}(4n)$ *and* $x \in \mathbb{R}^{4n}$,

$$[a^*, B[u]\,x] = B[u](ax)\,,$$

$$[B[u]x, B[u]\,y] \qquad \textit{is vertical}. \qquad \blacklozenge$$

Lemma 3.5. *Since* Φ *is* $\mathfrak{u}$*-normal, then* $N_{\mathbb{J}_\Phi}$ *vanishes on the mixed pairs,* (Bx, a^*)*. Also the vice versa is true.* $\blacklozenge$

Let Φ be an $\mathfrak{u}$-normal CR-structure on $\mathfrak{so}(4n+2)$. Then Φ is defined on the ideal $\mathfrak{k} = \mathfrak{u}_0 \oplus \mathfrak{s}$. The corresponding structure

$$\mathbb{J} := J_0^* \oplus \Phi^*$$

on $SO_g(M)$ is given on the subbundle $HP \oplus KP$. Since $[B(u)x, B(u)\,y]$ is contained in $F_u P$, $HP \oplus KP$ is involutive if and only if $[B(u)\,x, B(u)\,y] \neq J_0^*$, $\forall x, y \in \mathbb{R}^{4n+2}$. Otherwise, such a condition is satisfied, as a consequence of Remark 2.1. Moreover the condition

$$[x, y] - [\mathbb{J}_\Phi x, \mathbb{J}_\Phi y] \in HP \oplus KP\,, \tag{4}$$

is always satisfied, $\forall x, y \in HP \oplus KP$. In order to see if J_Φ is a CR-structure we have to verify that

$$N_{\mathbb{J}_\Phi} \equiv 0\,. \tag{5}$$

Such a condition is satisfied by any pair of vertical elements. Furthermore, it is true even for mixed pairs, since Φ is $\mathfrak{u}$-normal (Lemma 3.5).

Finally, we have that the Nijenhuis tensor vanishes on the pairs of horizontal elements if and only if $K(u) = \lambda Id$. In fact, there is the

Theorem 3.6. *Let* (M, g) *be a Riemannian orientable* m*-dimensional manifold, with* $m = 4n+2$*. Then, the structure* $(HP \oplus KP, \mathbb{J})$ *is integrable if and only if* (M, g) *has constant sectional curvature.*

Proof. First of all define

$$K(u)\colon \bigwedge^2 \mathbb{R}^m \to \mathfrak{so}(m)\colon x \wedge y \mapsto \Omega(u)(Bx, By)$$

where Ω is the 2-form of curvature. Then,

$$\omega N_{\mathbb{J}}(Bx, By) = -2K((x + iJ_0 x) \wedge (y + iJ_0 y))$$

and hence $N_{\mathbb{J}}$ vanishes identically on the horizontal vectors if and only if $\operatorname{Ker} K(u) \supseteq \overset{0,2}{\underset{J_0}{\wedge}}(\mathbb{R}^m)$.

– Let us suppose $\mathbb{J}$ integrable. Since $\operatorname{Im} \overset{0,2}{\underset{J_0}{\wedge}}(\mathbb{R}^m) = \operatorname{Re} \overset{0,2}{\underset{J_0}{\wedge}}(\mathbb{R}^m)$, we have that

$$[\Phi, \operatorname{ad}(g^{-1})]\, K(u)\left(\operatorname{Re} \overset{0,2}{\underset{J_0}{\wedge}}(\mathbb{R}^m)\right) = \{0\},$$

for all u in P and g in $U(2n+1)$.

Moreover, $\operatorname{Re} \overset{0,2}{\underset{J_0}{\wedge}}(\mathbb{R}^{4n})$ corresponds to $\mathfrak{s}(2n)$ in the canonical identification of $\overset{2}{\wedge}(\mathbb{R}^{4n})$ with $\mathfrak{so}(4n)$; thus, $\forall u \in P$, $\forall a \in SO(4n)$,

$$K(u)\operatorname{ad}(a)\, \mathfrak{s}(2n+1) \subseteq \operatorname{ad}(a)\, \mathfrak{s}(2n+1).$$

Furthermore, $SO(m) \to \operatorname{Aut} \mathfrak{so}(m)$: $a \mapsto \operatorname{ad}(a)$ is an irreducible representation. Hence, since $\operatorname{Span}\{\operatorname{ad}(a)\mathfrak{s}(2n) : a \in SO(4n)\}$ is $SO(4n)$-invariant, it is

$$\operatorname{Span}\{\operatorname{ad}(a)\mathfrak{s}(2n) : a \in SO(4n)\} = \mathfrak{so}(4n);$$

then take the orthogonal projection $p(a)$: $\mathfrak{so}(4n) \to \operatorname{ad}(a)\mathfrak{s}(2n)$.

Finally, notice that $K(u)$ is symmetric. Then $\operatorname{ad}(a)\mathfrak{s}(2n+1)$ is $K(u)$-invariant if and only if $[p(a), K(u)] = 0$. By Schur's lemma, $K(u) = \lambda Id + \mu H$, where $H^2 = -Id$. Since $K(u)$ is symmetric and H is not diagonalizable, μ vanishes. So, we deduce that $K(u) = \lambda Id$, that means that (M, g) has constant sectional curvature.

– Vice versa, let $K(u) = \lambda Id$. Then an easy computation shows that

$$K(u)(\alpha + i\beta) = 0, \qquad \forall \alpha, \beta \in \mathfrak{s}(2n+1).$$

and hence, $\operatorname{Ker} K(u) \supseteq \overset{0,2}{\underset{J_0}{\wedge}}(\mathbb{R}^{4n})$. Thus, N_J vanishes identically. ■

Theorem 3.6 implies that the integrability of $(HP \oplus KP, \mathbb{J})$ does not depend on the choice of the $\mathfrak{u}$-normal structure Φ. Thus, we may take $jU := J_0 \circ U$ and $\Phi M = J_0 \circ M$. The same fact will be true in the odd cases.

4. - The odd cases

Whenever $m = 4n+1$, the vertical fiber $F_u P$ has dimension $2n(4n+1)$, while $H_u P$ is odd-dimensional. Thus, in correspondence of the hyperplane $E_h = \mathrm{Span}_{i \neq h}(e_i) \subseteq \mathbb{R}^{4n+1}$, define $E_u^h P$ as $B(u)\, E_h$; then $E^h P \oplus FP$ is an even-dimensional subbundle. On such a subbundle set

$$\mathbb{J}_h = J_0 \oplus \Phi^* .$$

Remark that, given x, y vertical, the element $[x, y] - [\mathbb{J}_h x, \mathbb{J}_h y]$ is in FP and x, y satisfy $N_{\mathbb{J}_h}(x, y) = 0$ (cf. Proposition 3.1). Thus, the study of the integrability of $\mathbb{J}_h$ reduces to the horizontal and mixed pairs.

First of all, let us prove that

$$[x, y] - [\mathbb{J}_h x, \mathbb{J}_h y] \in E^h P \oplus FP, \tag{6}$$

for all nonvertical x, y in $E^h P \oplus FP$.

Lemma 4.1. *For all* $a \in \mathfrak{so}(m)$ *and* $x \in S_u P$, $B_u(\Phi(a)\, x) + B_u(a J_0 x) \in E_u P$.

Set $a = \begin{pmatrix} U+S & v \\ -v^t & 0 \end{pmatrix}$ and $\Phi(a) = \begin{pmatrix} jU + J_0 S & J_0 v \\ v^t J_0 & 0 \end{pmatrix}$. Then, the proof consists in computing

$$\Phi(a) \begin{pmatrix} x \\ 0 \end{pmatrix} + a \begin{pmatrix} J_0 x \\ 0 \end{pmatrix} = \begin{pmatrix} (jU + J_0 \circ S)\, x + (U+S)\, J_0 x \\ v^t J_0 x - v^t J_0 x \end{pmatrix}. \qquad \blacksquare$$

Thanks to Lemma 3.4, Lemma 4.1 means that the condition (6) is satisfied for mixed pairs. Moreover, by an analogous computation, the $\mathfrak{u}$-normality of Φ implies that $N_{\mathbb{J}_h}(a^*, B(u)\, x)$ is always zero.

In the following, we shall consider just pairs of horizontal elements. Since, $\theta[Bx, By]$ vanishes, for all x, y, $[Bx, By]$ is an element of FP. Hence, the condition (6) is satisfied even in the horizontal case.

The map $K(u)\colon \bigwedge^2 \mathbb{R}^m \to \mathfrak{so}(m)\colon x \wedge y \mapsto \Omega(u)(Bx, By)$ is useful to characterize the integrability of $\mathbb{J}_h$. In fact $N_{\mathbb{J}_h}$ vanishes identically if and only if

$$K(u)\,((x + iJ_0 x) \wedge (y + iJ_0 y)) \equiv 0 .$$

Thus, $\mathbb{J}_h$ is integrable if and only if $\mathrm{Ker}\, K(u) \supseteq \bigwedge_{J_0}^{0,2} E_h$. The same argument of Theorem 3.6 assures that

Proposition 4.2. *The almost-CR-structure $\mathbb{J}_h$ is integrable if and only if $K[u] = \lambda Id$ on $\bigwedge_{J_0}^{0,2} E_h$.*

A geometrical result about this situation is given in the terms of all the J_h. In fact, we have the

Theorem 4.3. *In the above hypothesis, (M, g) is a Riemannian manifold with constant sectional curvature if and only if all the almost CR-structures $\mathbb{J}_h$ $(h = 1, \ldots, 4n+1)$ are integrable.*

The other odd-case, corresponding to $m = 4n + 3$, has the same characterization. In fact, consider the subbundle $EP \oplus KP$, where E_u is the image via $B(u)$ of an hyperplane and $K_u P$ corresponds to $\mathfrak{k} = \mathfrak{u}_0 \oplus \mathfrak{s}$. For any $\mathfrak{u}$-normal CR-structure Φ, the sum $J_0 \oplus \Phi$ defines an almost-CR-structure. This fact is a consequence of Lemmas 2.1, 3.4 and 4.1.

Finally, Theorem 4.3 is true even in this case.

Notice that in the even cases, the maximal even-dimensional linear subspace of $\mathbb{R}^{2n}$ is unique and coincides with $\mathbb{R}^{2n}$ itself. Thus, we may give a statement which does not depend on the dimension of M.

Theorem 4.4. *Let (M, g) be an m-dimensional $(m > 4)$ Riemannian manifold. Take the almost CR-structures on $P = SO_g(M)$ of the form*

$$E^h P \oplus KP\,, \qquad \mathbb{J}_h = J_0 \oplus \Phi^*.$$

(in the even-dimensional case $\mathbb{J}_h$ is unique, in the odd there are m). Then, the sectional curvature of (M, g) is constant if and only if all the $\mathbb{J}_h$ are integrable.

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Abstract

Let (M, g) *be an m-dimensional orientable riemannian manifold. A family of almost-*CR*-structures is constructed on the principal bundle* $SO_g(M)$*. Their integrability is studied, obtaining that it is equivalent to* (M, g) *being of constant sectional curvature* (*Theorem* 4.4).

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