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$C^{2,\beta}_{\text{loc}}$ regularity for degenerate elliptic equations (**)

1 - Introduction

Let Ω be an open, bounded, connected domain of $\boldsymbol{R}^n$. We consider $C^\infty(\boldsymbol{R}^n)$ vector fields X_i, $i = 1, 2, \ldots, m$, satisfying a Hörmander condition [9], [10], and the forms:

$$a(u, v) = \int_\Omega \sum_{i=1}^m X_i u(x)\, X_i v(x)\, dx \tag{1.1}$$

$$\int_\Omega \sum_{i=1}^m X_i u(x)\, X_i v(x)\, dx \tag{1.2}$$

$$+ \frac{\Gamma_0}{2} \int_{\Omega \times \Omega \setminus \text{diag}(\Omega)} [u(x) - u(y)][v(x) - v(y)]\, N(x, y)\, dx\, dy + \int_\Omega l(x)\, u(x)\, v(x)\, dx$$

where Γ_0 is a positive constant, $N = N(x, y)$ is a smooth, non negative, symmetric function, possibly singular on $\text{diag}(\Omega) = \{(x, x) \mid x \in \Omega\}$, and $l = l(x)$ is a measurable, non negative, bounded function.

The aim of this paper is to confront the forms (1.1) and (1.2) from the point of view of the regularity of the solutions of the related equations.

Recall that, according to the fundamental theory of A. Beuerling and J. Deny [1], [2], and its extensions due to M. Silverstein [16], [17], M. Fukushima [8], Y. Le Jan [12], any *regular Dirichlet form* ξ on the space $H = L^2(\boldsymbol{R}^n)$ can be expressed on its domain $D[\xi]$, a linear subspace of H, as follows

$$\xi(u, v) = b(u, v) + \tfrac{1}{2} \int_{R^n \times R^n \setminus \text{diag}(R^n)} [\tilde{u}(x) - \tilde{u}(y)][\tilde{v}(x) - \tilde{v}(y)]\, j(dx, dy)$$
$$+ \int_X \tilde{u}(x)\, \tilde{v}(x)\, k(dx)$$

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The functions $\tilde{u}$ and $\tilde{v}$ are the quasi-continuous modifications of u and v with respect to the capacity associated to ξ [8] and $u = \tilde{u}$, $v = \tilde{v}$ a.e. in $\boldsymbol{R}^n$. The first term in the right hand side of the above formula denotes the *diffusion part*, while the second and the third terms are respectively the *jump part* or the *non local part* and the *local part*. The measures $j(dx, dy)$ and $k(dx)$ are positive Radon measures uniquely associated to ξ and are called respectively the *jumping measure* and the *killing measure* of ξ.

2 - Preliminaries

Let $k \geq 1$ be integer. For any multi-index $s = (s_1, \ldots, s_k)$ such that $1 \leq s_i \leq m$, $i = 1, \ldots, k$, we define $X^s = X_{s_1} \ldots X_{s_k}$ and denote $|s| = k$. For any integer $k \geq 1$, $W^{k,p}(\Omega, X)$ denotes the space of all functions $u \in L^p(\Omega)$ such that $X^s u \in L^p(\Omega)$ for any $|s| \leq k$, with the norm $\|u\|_{k,p}^p = \|u\|_p + \sum_{|s| \leq k} \|X^s u\|_p$. $W_0^{k,p}(\Omega, X)$ will be the closure of $C_0^\infty(\Omega)$ with respect to the same norm.

Let Σ be an open, bounded, connected subset of $\boldsymbol{R}^n$ such that $\Omega \subset\subset \Sigma$. If d_a denotes the intrinsic metric [14] associated to the form a in Σ, then C. Fefferman and D. H. Phong [7] proved that d_a is linked to the Euclidean metric by the estimates

$$\frac{1}{C}|x - y| \leq d_a(x, y) \leq C|x - y|^\varepsilon \tag{2.1}$$

where $\varepsilon = \frac{1}{m}$ and C is a structural positive constant dependent also on $\overline{\Sigma}$.

By d_a we can define the intrinsic balls $B(x, r) = \{y \in R^n \,|\, d_a(x, y) < r\}$ satisfying the doubling condition [14]. Thus (Σ, d_a) is a homogeneous metric space in the sense of R. Coifman and G. Weiss [6]. Let v be the homogeneous dimension of the graded nilpotent group generated by the left invariant vector fields corresponding to the lifted vector fields $\{\widetilde{X}_i\}$ of $\{X_i\}$ [13].

D. Jerison and A. Sanchez-Calle [11] proved the *Poincaré inequality*:

There exists a positive constant Γ_1 such that, for any $u \in C_0^\infty(\Omega)$

$$\|u\|_{L^2(\Omega)} \leq \Gamma_1 \|Xu\|_{L^2(\Omega)} \tag{2.2}$$

L. Hörmander [9], L. P. Rotschild and M. Stein [15] proved that the form (1.1) is subelliptic of order ε, that is there exists a positive constant Γ_2 such that, for any $u \in C_0^\infty(\Omega)$,

$$\|u\|_{H^\varepsilon(\Omega)} \leq \Gamma_2 \|u\|_{W_0^{1,2}(\Omega, X)} \tag{2.3}$$

where $H^\varepsilon(\Omega) = W^{\varepsilon, 2}(\Omega)$ is the usual fractional Sobolev space of order ε.

We will take

$$N(x, y) = \frac{c}{d_a(x, y)^m} \qquad m = \frac{1}{2}(n + 2\alpha) \tag{2.4}$$

for $2\alpha < 2\varepsilon + n$ ($m < n + \varepsilon$). Moreover, c is a positive constant such that $N(x, y) \leq \dfrac{1}{|x - y|^m}$. By (2.1), (2.3), (2.4), the form (1.2) is well defined on $W^{1,2}(\Omega, X)$. In fact, if $u, v \in W^{1,2}(\Omega, X)$, then

$$\frac{1}{2} \int_{\Omega \times \Omega \setminus \mathrm{diag}(\Omega)} [u(x) - u(y)][v(x) - v(y)]\, N(x, y)\, dx\, dy = 2 \int_\Omega Iu(x)\, v(x)\, dx \tag{2.5}$$

where:

$$Iu(x) = \int_\Omega [u(x) - u(y)]\, N(x, y)\, dy \tag{2.6}$$

$$\|Iu(x)\|_{L^2(\Omega)} \leq C(\alpha, \varepsilon) \|u\|_{H^\varepsilon(\Omega)} \tag{2.7}$$

where $C(\alpha, \varepsilon) = \left(\dfrac{R^{n + 2(\varepsilon - \alpha)}}{n + 2(\varepsilon - \alpha)} \omega_n \right)^{\frac{1}{2}}$ and R is a positive constant such that $R < \mathrm{diam}(\Omega)$.

Let $Lu = - \sum_{i=1}^m X_i^2 u$. As consequence of (2.2), each of the problems

$$Lu_1 = F \qquad u_1 \in W_0^{1,2}(\Omega, X) \tag{2.8}$$

$$Lu_2 + \Gamma_0 I u_2 + l u_2 = F \qquad u_2 \in W_0^{1,2}(\Omega, X) \tag{2.9}$$

has one and only one solution for any $F \in L^2(\Omega)$.

Observe that, from (2.2), we have

$$\|u_1\|_{W_0^{1,2}(\Omega, X)} \leq \Gamma_1 \|F\|_{L^2(\Omega)}. \tag{2.10}$$

3 - $C^{2,\beta}_{\rm loc}$ regularity

For any integer $k \geq 1$ and any real number $\beta \in (0, 1)$, $C^{k,\beta}_{(\rm loc)}(\Omega)$ will denote the space of all functions which are (locally) Hölder continuous in Ω with respect to the Euclidean metric. Moreover $S^{k,\beta}_{(\mathrm{loc})}(\Omega)$ will denote the space of all functions which are (locally) Hölder continuous in Ω with respect to the intrinsic metric d_α. C. J. Xu [18] proved that $C^{0,\beta}(\Omega)\subset S^{0,\beta}(\Omega)\subset C^{0,\beta\varepsilon}(\Omega)$. Moreover, if we have $F\in L^\infty(\Omega)\cap S^{0,\beta}_{\mathrm{loc}}(\Omega)$, then for any weak solution u of the equation $Lu=F$, if $u\in C^0(\Omega)$, then $u\in S^{2,\beta}_{\mathrm{loc}}(\Omega)$.

Proposition 1 ([4], [5]). *Let $F\in L^q(\Omega)$, where $q>(\frac{v}{2}\vee 2)$. Then the solution u_1 of* (2.8) *belongs to $L^\infty(\Omega)$.*

Proposition 2. *Let $F\in L^q(\Omega)$, where $q>(\frac{v}{2}\vee 2)$. Then the solution u_2 of* (2.9) *belongs to $L^\infty(\Omega)$.*

Proof. Thanks to (2.2), (2.3), (2.7), if $b(u,v)$ denotes the form (1.2), then there exists a positive constant c' such that, for any $u\in W_0^{1,2}(\Omega)$, we have $b(u,u)\leqslant c'a(u,u)$. It suffices now to repeat the proof of Proposition 1.

Proposition 3 ([4], [5])]*Let $F\in L^q(\Omega)$, where $q>(\frac{v}{2}\vee 2)$. Then the solution u_1 of* (2.8) *is locally Hölder-continuous in Ω. The exponent of the Hölder continuity of u_1 is a structural constant $\gamma\in(0,1)$.*

Proposition 4. *Let $F\in L^q(\Omega)$, where $q>(\frac{v}{2}\vee 2)$ and let $2\alpha<n$. Then the solution u_2 of* (2.9) *is locally Hölder-continuous in Ω with the same exponent γ of the solution u_1 of* (2.8).

Proof. If $F\in L^q(\Omega)$, where $q>(\frac{v}{2}\vee 2)$, then, by Proposition 2, $u_2\in L^\infty(\Omega)$ and then $Iu_2\in L^\infty(\Omega)$. For any $\varphi\in L^\infty(\Omega)$ we have $F_\varphi=F-I\varphi-l\varphi\in L^q(\Omega)$. By Proposition 3 the solution $u\in W_0^{1,2}(\Omega,X)$ of the problem $Lu=F_\varphi$, belongs to $C^{0,\gamma}_{\mathrm{loc}}(\Omega)$. If $\varphi=u_2$, then $u=u_2$. So $u_2\in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$.

Proposition 5. *Let $2(\alpha+\beta)<n$ for $\beta\in(0,1)$. If $u\in L^\infty(\Omega)\cap C^{0,\beta}_{\mathrm{loc}}(\Omega)$, then $Iu\in L^\infty(\Omega)\cap C^{0,\beta}_{\mathrm{loc}}(\Omega)$.*

Proof. Let $x,x'\in\Omega$. If $|x-x'|=\frac{\delta}{2}>0$, we consider the sets $A_\delta=B(x',\delta)\cap\Omega$ and $C_\delta=\Omega\backslash B(x',\delta)$. We have

$$\Big|\int_{A_\delta}\frac{u(x)-u(y)}{|x-y|^m}\,dy\Big|\leqslant\|u\|_{C^{0,\beta}(A_\delta)}|x-x'|^\beta\int_{A_\delta}\frac{dy}{|x-y|^m}\leqslant C(R,\alpha)\|u\|_{C^{0,\beta}(A_\delta)}|x-x'|^\beta$$

where $C(R,\alpha)=\dfrac{2\omega_n R^m}{n-2\alpha}$ and $m=\frac{1}{2}(n+2\alpha)$.

We can estimate $\left|\int_{A_\delta} \frac{u(x')-u(y)}{|x'-y|^m}\,dy\right|$ in a similar way. So we obtain

$$(3.1)\qquad \left|\int_{A_\delta} \frac{u(x)-u(y)}{|x-y|^m}\,dy - \int_{A_\delta} \frac{u(x')-u(y)}{|x'-y|^m}\,dy\right| \leq 2C(R,\alpha)\|u\|_{C^{0,\beta}(A_\delta)}\,|x-x'|^\beta.$$

On the other hand

$$\left|\int_{C_\delta} \frac{u(x)-u(y)}{|x-y|^m}\,dy - \int_{C_\delta} \frac{u(x')-u(y)}{|x'-y|^m}\,dy\right|$$

$$\leq \int_{C_\delta} \frac{|u(x)-u(x')|}{|x-y|^m}\,dy + \int_{C_\delta} |u(y)-u(x')|\left|\frac{1}{|x-y|^m} - \frac{1}{|x'-y|^m}\right|dy$$

$$(3.2)\qquad \leq C(\alpha)\|u\|_{C^{0,\beta}(A_\delta)}|x-x'|^\beta + 2\|u\|_{L^\infty(\Omega)}\int_{C_\delta} |x-x'|\,\frac{\max\{|x-y|,\,|x'-y|\}^{m-1}}{|x-y|^m\,|x'-y|^m}\,dy$$

$$\leq C(\alpha)\|u\|_{C^{0,\beta}(A_\delta)}|x-x'|^\beta + 2\|u\|_{L^\infty(\Omega)}|x-x'|^\beta\int_{C_\delta}\left[\frac{1}{|x-y|^{m+\beta}} + \frac{1}{|x'-y|^{m+\beta}}\right]dy$$

$$\leq [C(\alpha)\|u\|_{C^{0,\beta}(A_\delta)} + 4C(\alpha+\beta)\|u\|_{L^\infty(\Omega)}]\,|x-x'|^\beta.$$

By (3.1) and (3.2) we conclude the proof.

Theorem. *Let* $2(\alpha+\beta) < n$ *for* $\beta \in (0,1)$. *If* $F \in L^\infty(\Omega)\cap C^{0,\beta}_{\mathrm{loc}}(\Omega)$, *then the solution* u_1 *of the problem* (2.8) *belongs to* $C^{2,\beta\varepsilon}_{\mathrm{loc}}(\Omega)$ *and the solution* u_2 *of the problem* (2.9) *belongs to* $C^{2,(\beta\wedge\gamma)\varepsilon}_{\mathrm{loc}}(\Omega)$, *where* γ *is the constant of Proposition* 3.

Proof. If $F \in L^\infty(\Omega)\cap C^{0,\beta}_{\mathrm{loc}}(\Omega)$, then, by Propositions 1, 3 and by [18] we have $u_1 \in C^{2,\beta\varepsilon}_{\mathrm{loc}}(\Omega)$. Moreover Propositions 2 and 4 give $u_2 \in L^\infty(\Omega)\cap C^{0,\gamma}_{\mathrm{loc}}(\Omega)$. Hence, by Proposition 5, $Iu_2 \in L^\infty(\Omega)\cap C^{0,\gamma}_{\mathrm{loc}}(\Omega)$. The technique used to prove Proposition 4 gives $u_2 \in C^{2,(\beta\wedge\gamma)\varepsilon}_{\mathrm{loc}}(\Omega)$.

References

[1] A. Beuerling and J. Deny, *Espaces de Dirichlet*, Acta Math. **99** (1958), 203-224.

[2] A. Beuerling and J. Deny, *Dirichlet spaces*, Proc. Nat. Acad. Sci. USA 45 (1958), 208-215.

[3] M. BIROLI and U. MOSCO, *Formes de Dirichlet et estimations structurelles dans les milieux discontinus*, C. R. Acad. Sci. Paris **313** (1991), 593-598.

[4] M. BIROLI and U. MOSCO, *A Saint-Venant principle for Dirichlet forms on discontinuous media*, Ann. Mat. Pura Appl. **169** (1995), 125-181.

[5] M. BIROLI and N. A. TCHOU, *Asymptotic bahaviour of relaxed Dirichlet problems involving a Dirichlet-Poincaré form*, preprint.

[6] R. R. COIFMAN and G. WEISS, *Analyse armonique sur certaines espaces homogènes*, Lectures Notes in Math. **242**, Springer, Berlin 1971.

[7] C. FEFFERMAN and D. H. PHONG, *Subelliptic eigenvalues problems*, Conference on Harmonic Analysis in honor of A. Zygmund 2, Wadsworth 1983, 590-606.

[8] M. FUKUSHIMA, *Dirichlet forms and Markov processes*, North Holland **23**, Amsterdam 1980.

[9] L. HÖRMANDER, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147-171.

[10] L. HÖRMANDER, *Subelliptic operators*, Ann. of Math., Stud. **91**, Princeton Univ. Press, Princeton, USA 1967.

[11] D. JERISON and A. SANCHEZ-CALLE, *Subelliptic second order differential operators*, Lectures Notes in Math. **1277**, Springer, Berlin 1987.

[12] Y. LE JAN, *Mesures associées à une forme de Dirichlet. Applications*, Bull. Soc. Math. France **106** (1978), 61-112.

[13] G. LU, *Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications*, Rev. Mat. Iberoamericana 8 (1992), 367-439.

[14] A. NAGEL, E. M. STEIN and S. WEINGER, *Balls and metrics defined by vector fields, I: Basic properties*, Acta Math. **155** (1985), 103-147.

[15] L. P. ROTHSCHILD and M. STEIN, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), 247-320.

[16] M. SILVERSTEIN, *Symmetric Markov processes*, Lecture Notes in Math. **426** Springer, Berlin 1974.

[17] M. SILVERSTEIN, *Boundary theory for symmetric Markov processes*, Lecture Notes in Math. **516**, Springer, Berlin 1976.

[18] C. J. XU, *Regularity for quasilinear second order subelliptic equations*, Comm. Pure Appl. Math. (1992), 77-96.

Sommario

In questo articolo si danno condizioni di regolarità $C^{2,\beta}_{\mathrm{loc}}$ *per le soluzioni di alcune equazioni integro-differenziali relative a campi vettoriali di Hörmander.*

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