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Remarks on automorphisms of Nambu-Poisson structures (**)

Introduction

The theory of Poisson manifolds (cf. [7], [21], [23]) constitutes an extension of the symplectic geometry to the nontransitive case. Numerous problems which arise in physics are well described on the ground of this theory. When a Poisson manifold is interpreted as the phase-space the group of all Poisson automorphisms plays the role of the symmetry group.

Y. Nambu in [11] proposed a more general setting of Hamiltonian mechanics. After years of oblivion some recent papers (e.g. [19], [4], [5]) have given a new interest to Nambu's ideas. From the geometric point of view a so-called Nambu-Poisson structure defined by R. Ibáñez, M.de León, J. C. Marrero and D. Martin de Diego in [5] constitutes an interesting generalization of the Poisson geometry. As mentioned in [5] some further extension is still possible. In the sequel we shall appeal to some definitions and facts established in [5].

The aim of this note is to study some aspects of automorphisms of Nambu-Poisson manifolds. In Sections 1 and 2 we recall some preliminary facts concerning Nambu-Poisson structures. Sections 3 and 4 are devoted to the introduction of the flux homomorphism (Theorem 2) and the study of its relation with isotopies (Theorem 3). This extends the symplectic case ([3], [1], [9]) as well as the case of regular Poisson manifolds [17]. In the remaining part we prove that the geometric structure of a Nambu-Poisson manifold restricted to the set of all regular points is uniquely determined by the group of its automorphisms (Theorem 5). This result can be viewed as a modern contribution to the Erlangen Program of F. Klein [6] (see also [2], [13], [14], [16], [24] and references therein). The proof of it

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follows a pattern from my previous papers. In contrast to a traditional argument as in [24], [2] our refinement of it works also in the case of nontransitive groups of diffeomorphisms. Another advantage of it is that an inspection concerning the perfectness of the group in question is not necessary. Nota bene such a theorem is unknown (and probably difficult) in the case of generalized Poisson geometry.

It is worth noting that all the presented facts generalize as well those for the volume preserving transformation group [20], [2].

All manifolds, tensors, diffeomorphisms and so on are assumed to be of the class C^∞. Certain facts presented in this note are no longer true in the real analytic category.

1 - Nambu-Poisson structures

Let us recall first the concept of Poisson manifolds. Let M be a smooth manifold of dimension m, and let $C^\infty(M)$ (resp. $\mathfrak{X}(M)$; resp. $\Omega^r(M)$) denote the ring of all $\boldsymbol{R}$-valued smooth functions on M (resp. the Lie algebra of all vector fields on M; resp. the space of all r-forms on M).

A Poisson structure can be introduced by a skew-symmetric (2, 0)-tensor Λ on M such that $[\Lambda, \Lambda] = 0$, where $[.,.]$ is the Schouten-Nijenhuis bracket. Then the rank of Λ_p may vary but it is even everywhere. The ring $C^\infty(M)$ can be then given a Lie algebra structure by means of the bracket

$$\{u, v\} = \Lambda(\mathrm{d}u, \mathrm{d}v) \qquad \text{for any } u, v \in C^\infty(M) \tag{1.1}$$

and every adjoint homomorphism of this bracket is a derivation of $C^\infty(M)$ (the Leibniz rule)

$$\{uv, w\} = u\{v, w\} + v\{u, w\} \qquad \text{for any } u, v, w \in C^\infty(M). \tag{1.2}$$

Observe that a Poisson structure may be defined equivalently by means of the bracket $\{.,.\}: C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ which is 2-linear, skew-symmetric and satisfies (1.2). The bracket and the tensor Λ are then related by (1.1).

We have the *musical* bundle homomorphism associated with Λ

$$\sharp : \Omega^1(M) \to \mathfrak{X}(M) \qquad \beta(\alpha^\sharp) = \Lambda(\alpha, \beta)$$

where $\alpha^\sharp = \sharp(\alpha)$, for any $\alpha, \beta \in \Omega^1(M)$. In case Λ is nondegenerate (i.e. rank(Λ) equals $\dim(M)$), $\sharp$ is an isomorphism and we get a symplectic structure.

In general, the distribution $\sharp(T_x^* M)$, $x \in M$, integrates to a generalized foliation such that Λ restricted to any leaf induces a symplectic structure [21]. This foliation is called symplectic.

The first step towards generalization of Poisson structures is the following

Definition 1. A skew-symmetric n-linear mapping

$$\{\cdot, \dots, \cdot\} : C^\infty(M) \times \dots \times C^\infty(M) \to C^\infty(M)$$

is called a *generalized almost Poisson bracket* of order n if it verifies the Leibniz rule

$$\{u_1 v_1, \dots, u_n\} = u_1\{v_1, \dots, u_n\} + v_1\{u_1, \dots, u_n\} \tag{1.3}$$

for all $u_1, \dots, u_n, v_1 \in C^\infty(M)$. The pair $(M, \{\cdot, \dots, \cdot\})$ is then called a *generalized almost Poisson manifold (of order n).*

Recall that the skew-symmetry means that

$$\{u_1, \dots, u_n\} = (-1)^{\mathrm{sgn}(\sigma)}\{u_{\sigma(1)}, \dots, u_{\sigma(n)}\},$$

where $u_i \in C^\infty(M)$, σ is any permutation of n elements and $\mathrm{sgn}(\sigma)$ its parity.

Equivalently, a generalized almost Poisson manifold of order n is given as the pair (M, Λ), where Λ is a skew-symmetric $(n, 0)$-tensor on M. The relation between Λ and the n-bracket $\{\cdot, \dots, \cdot\}$ is expressed by the equality

$$\Lambda(\mathrm{d}u_1, \dots, \mathrm{d}u_n) = \{u_1, \dots, u_n\} \tag{1.4}$$

for all $u_1, \dots, u_n \in C^\infty(M)$.

The concept of the *musical* homomorphism $\sharp$ extends naturally to any generalized almost Poisson structure (M, Λ). Namely, we define a linear mapping

$$\sharp : \Omega^{n-1}(M) \to \mathfrak{X}(M) \qquad \text{by setting}$$

$$\langle \sharp(\alpha_1 \wedge \dots \wedge \alpha_{n-1}), \beta\rangle = \Lambda(\alpha_1, \dots, \alpha_{n-1}, \beta)$$

for any $\alpha_1, \dots, \alpha_{n-1}$ and for any $\beta \in \Omega^1(M)$.

Here $\langle , \rangle$ is the natural pairing on $\mathfrak{X}(M) \times \Omega^1(M)$. We shall write $(\alpha_1 \wedge \dots \wedge \alpha_{n-1})^\sharp$ instead of $\sharp(\alpha_1 \wedge \dots \wedge \alpha_{n-1})$.

A vector field X is called an *infinitesimal automorphism* (i.a.) of (M, Λ) if $\mathcal{L}_X \Lambda = 0$, where $\mathcal{L}$ denotes the Lie derivative. It can be checked ([5], Proposition 3.5) that this condition amounts to claiming that X is a derivation of the bracket $\{\cdot, \dots, \cdot\}$.

Next, a vector field X on (M, Λ) is said to be *Hamiltonian* if there are $n-1$ functions $u_1, \ldots, u_{n-1} \in C^\infty(M)$ such that $X = X_{u_1, \ldots, u_{n-1}}$ where

$$X_{u_1, \ldots, u_{n-1}} = (\mathrm{d}u_1 \wedge \ldots \wedge \mathrm{d}u_{n-1})^\sharp.$$

By $\mathcal{X}_\Lambda(M)$ (resp. $\mathcal{X}_H(M, \Lambda)$) we denote the space of all i.a. (resp. Hamiltonian vector fields) of (M, Λ). Likewise, $\mathcal{X}_\Lambda(M)_c$ (resp. $\mathcal{X}_H(M, \Lambda)_c$) stands for the Lie algebra of all compactly supported elements of $\mathcal{X}_\Lambda(M)$ (resp. $\mathcal{X}_H(M, \Lambda)$). Note that in view of the equality $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X, Y]}$ the space $\mathcal{X}_\Lambda(M)$ (or $\mathcal{X}_\Lambda(M)_c$) is actually a Lie algebra.

Now we formulate the clue notion in the paper.

Definition 2 [5]. A generalized almost Poisson manifold of order n is called a *Nambu-Poisson manifold* (*NP-manifold* in short) if the following integrability condition is fulfilled: for any functions $u_1, \ldots, u_{n-1} \in C^\infty(M)$ the Hamiltonian vector field $X_{u_1, \ldots, u_{n-1}}$ is a derivation of the bracket $\{\cdot, \ldots, \cdot\}$, i.e.

$$X_{u_1, \ldots, u_{n-1}}\{v_1, \ldots, v_n\} = \sum_{i=1}^{n} \{v_1, \ldots, X_{u_1, \ldots, u_{n-1}} v_i, \ldots, v_n\}$$

for all $u_1, \ldots, u_{n-1}, v_1, \ldots, v_n \in C^\infty(M)$.

Notice that for $n = 2$ the above condition is equivalent to $[\Lambda, \Lambda] = 0$ (cf. [5]). Consequently, the NP-manifolds of order 2 coïncide with Poisson manifolds.

As an immediate consequence of Definition 2 we get

Corollary 1. *Every Hamiltonian vector field on an NP-manifold is an infinitesimal automorphism.*

A basic example of an NP-manifold of order n is an oriented n-dimensional manifold. Indeed, by fixing a volume form ω on M we define the associated Nambu-Poisson bracket by

$$\{u_1, \ldots, u_n\}\, \omega = \mathrm{d}u_1 \wedge \ldots \wedge \mathrm{d}u_n. \tag{1.5}$$

The correspoding $(n, 0)$-tensor is denoted by Λ_ω. Conversely, it is easily seen that given an n-dimensional NP-manifold (M, Λ) of order n one can find a volume form ω on M such that $\Lambda = \Lambda_\omega$.

Proposition 1. *Let (M, Λ) be an NP-manifold. Then $\mathcal{X}_H(M, \Lambda)$ is a Lie subalgebra of $\mathcal{X}_\Lambda(M)$.*

In fact, we have

$$[X_{u_1, \dots, u_{n-1}}, X_{v_1, \dots, v_{n-1}}] = \sum_{i=1}^{n-1} X_{v_1, \dots, X_{u_1, \dots, u_{n-1}} v_i, \dots, v_{n-1}}.$$

Let us recall some concepts from [18] (see also [15]). A *(generalized) foliation* is a partition $\mathcal{F}$ of M into weakly imbedded submanifolds, called leaves, such that the following condition holds. If x belongs to a k-dimensional leaf, then there is a local chart (U, ϕ) with $\phi(x) = 0$, and $\phi(U) = V \times W$, where V is open in $\boldsymbol{R}^k$, and W is open in $\boldsymbol{R}^{n-k}$, such that if $L \in \mathcal{F}$ then $\phi(L|U) \cap (V \times W) = V \times l$, where $l = \{w \in W: \phi^{-1}(0, w) \in L\}$.

Next we say that a smooth distribution $\mathcal{D} = \{D_x\}_{x \in M}$ is completely integrable if there exists a foliation $\mathcal{F}$ such that $D_x = T_x \mathcal{F}$, $\forall x \in M$. Observe that the dimension of D_x (and of the leaves of $\mathcal{F}$) may vary and that a usual Frobenius theorem is no longer true unless foliations are regular (i.e. all the leaves have the same dimension).

Given a generalized almost Poisson tensor let us define a smooth distribution $\mathcal{D} = \{D_x\}_{x \in M}$ with D_x being the subspace of $T_x M$ spanned by all Hamiltonian vector fields $X_{u_1, \dots, u_{n-1}}$ evaluated at $x \in M$. Equivalently, $D_x = \sharp(\wedge^{n-1} T_x^* M)$. This distribution is called *characteristic*. For $n = 2$ $\mathcal{D}$ is always completely integrable and we get the symplectic foliation $\mathcal{F}(\Lambda)$.

In our case we have the following structural theorem.

Theorem 1 [5]. *Let (M, Λ) be an m-dimensional Nambu-Poisson manifold of order $n \geq 3$. Then:*

1. *The corresponding distribution $\mathcal{D}_x$ is completely integrable and, consequently, it defines a foliation, denoted by $\mathcal{F}(\Lambda)$. There are two kind of leaves of $\mathcal{F}(\Lambda)$:*

a. *if $\Lambda_x = 0$ then then leaf passing through x reduces to x itself*

b. *if $\Lambda_x \neq 0$ then the leaf meeting x has dimension n and Λ restricted to it induces a Nambu-Poisson structure which comes from a volume form.*

2. *In case* **b.** *there exists a distinguished chart $(x_1, \dots, x_n, y_1, \dots, y_q)$ at x $(q = m - n)$ such that $\Lambda = \partial_1 \wedge \dots \wedge \partial_n$ where $\partial_i = \partial / \partial x_i$.*

The points x with $\Lambda_x = 0$ are called *singular* while the remaining ones are called *regular*. By M^r we will denote the set of all regular points of (M, Λ).

Let L be a regular leaf. By Ω_L we shall denote the volume form on L which generates the Nambu-Poisson structure on L.

Observe that the existence of canonical coordinates for Poisson manifolds has been proved in [7], [23].

2 – Preliminaries on Poisson automorphism and isotopies

We begin with the following

Definition 3. Let $(M_i, \{\cdot, \dots, \cdot\}_i)$, $i = 1, 2$, be a generalized Poisson manifold of order n. A smooth mapping $f: M_1 \to M_2$ is called a *Poisson morphism* if for any $u_1, \dots, u_n \in C^\infty(M)$ we have

$$\{u_1 \circ f, \dots, u_n \circ f\}_1 = \{u_1, \dots, u_n\}_2 \circ f.$$

By a straightforward computation we have

Proposition 2. *The following conditions are equivalent*:

1. *$f: (M_1, \Lambda_1) \to (M_2, \Lambda_2)$ is a Poisson morphism.*

2. *Λ_1, Λ_2 are f-related, that is*

$$\Lambda_1(f^*\alpha_1, \dots, f^*\alpha_n) = \Lambda_2(\alpha_1, \dots, \alpha_n)$$

for all $\alpha_1, \dots, \alpha_n \in \Omega^1(M_2)$.

3. *For any $u_1, \dots, u_{n-1} \in C^\infty(M_2)$ one has*

$$f_* X_{u_1, \dots, u_{n-1}} = X_{u_1 \circ f, \dots, u_{n-1} \circ f}$$

that is the Hamiltonian vector fields $X_{u_1, \dots, u_{n-1}}$ and $X_{u_1 \circ f, \dots, u_{n-1} \circ f}$ are f-related.

Let us recall that there is a bijective correspondence between smooth isotopies f_t in $\mathrm{Diff}_c^\infty(M)$ satisfying $f_0 = \mathrm{id}$ and smooth families of compactly supported vector fields X_t (see e.g. [10]). This correspondence is given by the equality

$$\frac{df_t}{dt} = X_t \circ f_t. \tag{2.1}$$

In particular, a time-independent compactly supported vector field corresponds to its flow. Furthermore, if $G(M)$ is a locally arcwise connected group of diffeomorphisms (this is the case of all *classical* groups of diffeomorphisms [14]) then the corresponding smooth families of vector fields belong to the Lie algebra of $G(M)$.

From now on we assume that diffeomorphism groups are locally contractible. By $G(M)_0$ we denote the subgroup of all f which can be joined with the identity by a compactly supported isotopy f_t in $G(M)$. Then $G(M)_0$ is the connected component of the identity iff M is compact.

For an NP-manifold (M, Λ) the symbol $G(M, \Lambda)$ stands for the group of all leaf preserving diffeomorphisms satisfying $f^*\Lambda = \Lambda$ (i.e. being Poisson morphisms of (M, Λ) onto itself). Hence $G(M, \Lambda)_0$ denotes the subgroup of $G(M, \Lambda)$ of all f such that there is a compactly supported isotopy f_t in $G(M, \Lambda)$ with $f_0 = \mathrm{id}$ and $f_1 = f$.

Next, by $\mathcal{X}(M, \Lambda)$ (resp. $\mathcal{X}(M, \Lambda)_c$) we denote the Lie subalgebra of $\mathcal{X}_\Lambda(M)$ (resp. $\mathcal{X}_\Lambda(M)_c$) of all elements tangent to $\mathcal{F}(\Lambda)$. Clearly, $\mathcal{X}_H(M, \Lambda) \subset \mathcal{X}(M, \Lambda)$, $\mathcal{X}_H(M, \Lambda)_c \subset \mathcal{X}(M, \Lambda)_c$, and $\mathcal{X}(M, \Lambda)_c \cap \mathcal{X}_H(M, \Lambda) = \mathcal{X}_H(M, \Lambda)_c$.

We have

Proposition 3. *Suppose that f_t, X_t are related by the equation* (2.1). *Then $f_t \in G(M, \Lambda)_0$ for each t if and only if $X_t \in \mathcal{X}(M, \Lambda)_c$ for each t.*

Proof. When restricting f_t to a regular leaf L we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t^* \Omega_L = f_t^* (\mathcal{L}_{X_t} \Omega_L) = f_t^* (\iota(X_t)\, \mathrm{d}\Omega_L + \mathrm{d}(\iota(X_t) \Omega_L)) = f_t^*\, \mathrm{d}(\iota(X_t)\, \Omega_L).$$

It follows that the claim holds true on any leaf, and consequently so does on the whole M.

Definition 4.

1. A smooth path satisfying Proposition 3 is called a *Poisson isotopy.*

2. A Poisson isotopy f_t is said to be *Hamiltonian* if the corresponding $X_t \in \mathcal{X}_H(M, \Lambda)_c$ for each t.

3. A diffeomorphism f of (M, Λ) is called *Hamiltonian* if it can be written as a finite product of $\exp(X)$ where $X \in \mathcal{X}_H(M, \Lambda)_c$. The group of all Hamiltonian diffeomorphisms is denoted by $G^*(M, \Lambda)$.

Clearly $G^*(M, \Lambda)_0 = G^*(M, \Lambda)$. However the following questions arise. Let $\widehat{G}(M, \Lambda)$ be the group generated by all $\exp(X)$ with $\mathcal{X}(M, \Lambda)_c$. Next, let $\overline{G}^*(M, \Lambda)$ be the totality of f which can be joined with id by a Hamiltonian isotopy. It is not known whether $G(M, \Lambda)_0 = \widehat{G}(M, \Lambda)$ and $G^*(M, \Lambda) = \overline{G}^*(M, \Lambda)$. Note that equalities are satisfied for symplectic manifolds (cf. [8]) as consequences of a difficult simplicity theorem in [1].

Proposition 4. *$G^*(M, \Lambda)$ is a normal subgroup of $G(M, \Lambda)$.*

Indeed, if $f = \exp(X)$ where $X = (du_1 \wedge \dots \wedge du_{n-1})^\sharp$, then for any $g \in G(M, \Lambda)$ we have $g^{-1} \circ f \circ g = \exp(Y)$ where $Y = (d(u_1 \circ g) \wedge du_2 \wedge \dots \wedge du_{n-1})^\sharp$.

3 - Foliated forms and foliated cohomology

Let $(\Omega^*(M), \mathrm{d})$ be the De Rham complex of a smooth manifold M. Given a (not necessarily regular) foliation $\mathcal{F}$ on M we define the subcomplex $\Omega^*(M, \mathcal{F})$ as follows: $\omega \in \Omega^r(M, \mathcal{F})$ if and only if $\omega \in \Omega^r(M)$ and $\omega(X_1, \dots, X_r) = 0$ for any vector fields $X_1, \dots, X_r$ tangent to $\mathcal{F}$. Then we set

$$\Omega^*(\mathcal{F}) = \Omega^*(M)/\Omega^*(M, \mathcal{F})$$

and by $\overline{\omega} \in \Omega^r(\mathcal{F})$ we denote the class of $\omega \in \Omega^r(M)$. It is easily seen that if $\overline{\omega}_1 = \overline{\omega}_2$ then $\overline{d\omega_1} = \overline{d\omega_2}$. Consequently, the differential d descends to the differential $\overline{\mathrm{d}}$ defined on $\Omega^*(\mathcal{F})$. Thus we get a new differential complex $(\Omega^*(\mathcal{F}), \overline{\mathrm{d}})$, the complex of foliated forms. By $H^*(\mathcal{F})$ we denote the cohomology of $\Omega^*(\mathcal{F})$, and by $[\overline{\omega}]$ the cohomology class of $\overline{\omega}$. Clearly $H^r(\mathcal{F}) = 0$ if $r > \dim \mathcal{F}$, where $\dim \mathcal{F}$ is the maximal dimension of the leaves of $\mathcal{F}$.

It is visible that the exterior product $\wedge$ in $\Omega^*(M)$ descends to $\Omega^*(\mathcal{F})$. Next, it is easily seen that $\iota(X)\,\overline{\omega} = \overline{\iota(X)\omega}$ and $f^*\overline{\omega} = \overline{f^*\omega}$ are correct definitions whenever X is tangent to $\mathcal{F}$ and f is a leaf preserving smooth map. The former enables us to define the Lie derivative by

$$\mathcal{L}_X\overline{\omega} = \iota(X)\,\overline{\mathrm{d}}\overline{\omega} + \overline{\mathrm{d}}\,\iota(X)\,\overline{\omega}$$

for X tangent to $\mathcal{F}$. Moreover, for a smooth family $\omega_t \in \Omega^r(M)$, $t \in I$, we define

$$\int_0^1 \overline{\omega}_t\, dt = \overline{\int_0^1 \omega_t\, dt} \qquad \text{and} \qquad \int_0^1 [\overline{\omega}_t]\, dt = [\overline{\int_0^1 \omega_t\, dt}].$$

Definition 5. Let (M, Λ) be an NP-manifold of order n, let M^r be the set of all its regular points and let $\mathcal{F}(\Lambda)^r = \mathcal{F}(\Lambda) \,|\, M^r$. Then by $\overline{\Omega} = \overline{\Omega}(\Lambda)$ we denote the $\mathcal{F}(\Lambda)^r$-foliated n-form on M^r such that

$$\{u_1, \dots, u_n\}\, \overline{\Omega} = \overline{du_1 \wedge \dots \wedge du_n} = \overline{du_1} \wedge \dots \wedge \overline{du_n}$$

for all $u_1, \dots, u_n \in C^\infty(M)$. Here $\{\cdot, \dots, \cdot\}$ is related to Λ by (1.4). We call $\overline{\Omega}$ *the foliated n-form associated with* Λ.

Observe that in a canonical chart $(x_1, \dots, x_n, y_1, \dots, y_q)$ at a regular point we have $\overline{\Omega} = \overline{dx_1 \wedge \dots \wedge dx_n}$.

The above definition enables us to introduce the notion of exactness. Namely, one says that (M, Λ) is *exact* if there is $\overline{\varpi} \in \Omega^{n-1}(\mathcal{F}(\Lambda))$ such that $\bar{d}\overline{\varpi} = \overline{\Omega}$ on M^r.

Unfortunately, there is no a one-to-one correspondence between $(n, 0)$-tensors and (foliated) n-forms as in the symplectic or regular Poisson case [17]. This is revealed e.g. by the following

Example 1. Let $M = \boldsymbol{R}^2$ and $\Lambda = \alpha \partial_1 \wedge \partial_2$, where $\partial_i = \partial/\partial x_i$ and $\alpha \in C^\infty(M)$. Then $\mathcal{F}(\Lambda)$ is generated by $X_{x_1} = \alpha \partial_2$ and $X_{x_2} = -\alpha \partial_1$. The zeros of α are the singular leaves of $\mathcal{F}(\Lambda)$ while the connected components of $\boldsymbol{R}^2 - \alpha^{-1}(0)$ are the regular leaves. The corresponding foliated 2-form $\Omega = \dfrac{1}{\alpha} dx_1 \wedge dx_2$ cannot be extended to singular leaves.

4 - The existence of the flux homomorphism

By an argument similar to that for symplecthomorphisms ([22], [17]) one can show that the group $G(M, \Lambda)_0$ is locally contractible. Therefore $\widetilde{G}(M, \Lambda)_0$, the universal covering of $G(M, \Lambda)_0$, is the totality of pairs $(f, \{f_t\})$ where $f = f_1$ of $G(M, \Lambda)$ and $\{f_t\}$ is the homotopy rel. endpoints class of the isotopy f_t, $t \in I$. Then $\widetilde{G}(M, \Lambda)_0$ is given a group structure and the multiplication of it can be defined either by the pointwise multiplication over I of representants or, equivalently, by the juxtaposition of representants. The latter means that $\{g_t\} . \{f_t\} = \{h_t\}$ where

$$h_t = \begin{cases} f_{2t} & \text{for } 0 \le t \le \dfrac{1}{2} \\ g_{2t-1} \circ f_1 & \text{for } \dfrac{1}{2} \le t \le 1 . \end{cases}$$

Now we wish to generalize the concept of the flux homomorphism (first introduced by E. Calabi in [3] for symplecthomorphisms) to Nambu-Poisson manifolds. Let $\overline{\Omega} = \overline{\Omega}(\Lambda)$ be the foliated n-form living on M^r corresponding to Λ. For any

Poisson isotopy f_t we set

$$\text{Flux}(\{f_t\}) = \int_0^1 [\iota(X_t)\,\overline{\Omega}]\,dt \in H^{n-1}(\mathcal{F}(\Lambda)^r) \tag{4.1}$$

where the family X_t is defined by (2.1).

Theorem 2. *Let* (M, Λ) *be any NP-manifold. The equality* (4.1) *defines a continuous homomorphism* $\text{Flux}: \tilde{G}(M, \Lambda)_0 \to H^{n-1}(\mathcal{F}(\Lambda)^r)$. *Moreover, if* ϕ_t *is a flow of* $X \in \mathcal{X}(M, \Lambda)_c$ *then* $\text{Flux}(\{\phi_t\}) = [\iota(X)\,\overline{\Omega}]$.

Proof. The proof follows the lines of [1], p. 182-3 (the symplectic case; see also [17] for regular Poisson manifolds), and we reproduce it here for the sake of completeness only. First observe that any Poisson isotopy of (M, Λ) is uniquely determined on M^r as it stabilizes at the singular points.

Let f_t, g_t be two Poisson isotopies such that $\{f_t\} = \{g_t\}$ and $f_1 = g_1$. Therefore there is $F_{s,t}$, a smooth 2-parameter family in $G(M, \Lambda)_0$ satisfying $F_{0,t} = f_t$, $F_{1,t} = g_t$, $F_{s,0} = \text{id}$, $F_{s,1} = f_1 = g_1$ for any $s, t \in I$. We then set

$$X_{s,t} = \frac{\partial F_{s,t}}{\partial t} \circ F_{s,t}^{-1} \qquad Y_{s,t} = \frac{\partial F_{s,t}}{\partial s} \circ F_{s,t}^{-1}$$

and it is visible that $t \mapsto X_{s,t}$ corresponds by (2.1) to $t \mapsto F_{s,t}$.

We have

$$\begin{aligned}
\frac{\partial}{\partial s}\int_0^1 \iota(X_{s,t})\,\overline{\Omega}\,dt &= \int_0^1 \iota\left(\frac{\partial X_{s,t}}{\partial s}\right)\overline{\Omega}\,dt \\
&= \int_0^1 \iota\left(\frac{\partial Y_{s,t}}{\partial t}\right)\overline{\Omega}\,dt + \int_0^1 \iota([X_{s,t}, Y_{s,t}])\,\overline{\Omega}\,dt \\
&= \int_0^1 \frac{\partial}{\partial t}(\iota(Y_{s,t})\,\overline{\Omega})\,dt + \int_0^1 \mathcal{L}_{X_{s,t}}(\iota(Y_{s,t}\overline{\Omega})\,dt \\
&= \int_0^1 \iota([X_{s,t}, Y_{s,t}])\,\overline{\Omega}\,dt = \overline{d}\left(\int_0^1 \overline{\Omega}(Y_{s,t}, X_{s,t})\,dt\right).
\end{aligned}$$

Let us explain the above equalities. The second one follows by the equality $\dfrac{\partial X_{s,t}}{\partial s} = \dfrac{\partial Y_{s,t}}{\partial t} + [X_{s,t}, Y_{s,t}]$ (cf. [1]). Next, the third is a consequence of the formula $\iota([X, Y])\,\overline{\Omega} = [\mathcal{L}_X, \iota(Y)]$ and Proposition 3. The fourth equality follows by $Y_{s,0} = Y_{s,1} = 0$ and by $\mathcal{L}_X\overline{\Omega} = \overline{\mathrm{d}}\iota(X)\,\overline{\Omega}$.

Hence we get

$$\mathrm{Flux}(\{g_t\}) - \mathrm{Flux}(\{f_t\}) = \int_0^1 \overline{\mathrm{d}}\left(\int_0^1 \overline{\Omega}(Y_{s,t}, X_{s,t})\,\mathrm{d}t\right)\mathrm{d}s$$

$$= \overline{\mathrm{d}}\left(\int_{I\times I} \overline{\Omega}(Y_{s,t}, X_{s,t})\,\mathrm{d}t\wedge\mathrm{d}s\right).$$

which implies that Flux is well defined. Next, Flux is a homomorphism since the multiplication in $\widetilde{G}(M, \Lambda)$ can be represented by the juxtaposition. Finally, Flux is continuous by a standard argument.

The last assertion holds since X corresponding to ϕ_t is time-independent.

Remark. Contrary to [3], [1], [17] Flux is not surjective.

Proposition 5. *If* $\overline{\Omega} = -\overline{\mathrm{d}}\overline{\omega}$ *(i.e.* $\overline{\Omega}$ *is exact) and* f_t *is a Poisson isotopy then* $\mathrm{Flux}(\{f_t\}) = [\overline{\omega} - f_1^*\overline{\omega}]$.

Proof. Let X_t be related to f_t by (2.1). Then we have

$$[\iota(X_t)\,\overline{\Omega}] = [f_t^*\,\iota(X_t)\,\overline{\Omega}] = -[f_t^*\,\iota(X_t)\,\overline{d}\overline{\omega}] = -[f_t^*\,\mathcal{L}_{X_t}\overline{\omega}] = -\frac{\mathrm{d}}{\mathrm{d}t}[f_t^*\overline{\omega}].$$

The assertion follows by integrating over the interval $[0,1]$.

The significance of the flux homomorphism consists in the fact that it characterizes isotopies. To obtain an analogue with the symplectic case (cf. [9]) we have to introduce additional Lie algebras of i.a. (A full analogue does not hold because of situations as in Example 1.)

Let $\mathcal{X}_r(M, \Lambda)$ be the subalgebra of all $X \in \mathcal{X}(M, \Lambda)_c$ such that $\mathrm{supp}\,(X|M^r)$ is compact in M^r. Then we have the isomorphism

$$*: \mathcal{X}_r(M, \Lambda) \ni X \mapsto {*X} = \iota(X)\,\overline{\Omega} \in \Omega^{n-1}_{cl}(\mathcal{F}(\Lambda)^r),$$

where $\Omega^{n-1}_{cl}(\mathcal{F}(\Lambda)^r)$ is the subspace of $\Omega^{n-l}(\mathcal{F}(\Lambda)^r)$ of all $\overline{\mathrm{d}}$-closed forms with compact support in M^r. Now by $\mathcal{X}^*_r(M, \Lambda)$ we denote the subalgebra of all X of $\mathcal{X}_r(M, \Lambda)$ such that $*X$ is $\overline{\mathrm{d}}$-exact.

Theorem 3. *Let (M, Λ) be an NP-manifold and let f_t be a Poisson isotopy such that the corresponding $X_t \in \mathcal{X}_r(M, \Lambda)$. Then f_t is homotopic rel. endpoints to an isotopy g_t with the corresponding $Y_t \in \mathcal{X}_r^*(M, \Lambda)$, if and only if* $\mathrm{Flux}(\{f_t\}) = 0$.

Proof. We may assume that $X_t \in \mathcal{X}_r^*(M, \Lambda)$. Then $\iota(X_t)\,\overline{\Omega} = \overline{\mathrm{d}}\overline{\omega}_t$ for $\overline{\omega}_t \in \Omega^{n-2}(\mathcal{F}(\Lambda)^r)$. Hence $\mathrm{Flux}(\{f_t\}) = 0$ as $[\overline{\mathrm{d}}\overline{\omega}_t] = 0$.

Conversely, by assumption we have $\int_0^1 \iota(X_t)\,\overline{\Omega}\,\mathrm{d}t = \overline{\mathrm{d}}\overline{\omega}$, for some $\overline{\omega}$ of $\Omega^{n-2}(\mathcal{F}(\Lambda)^r)$. Observe that $\overline{\omega}$ can be chosen compactly supported since all $\mathrm{supp}\, X_t$ are in a fixed compact subset of M^r. Let $X_{\overline{\omega}}$ be the vector field satisfying ${}^*X_{\overline{\omega}} = \overline{\mathrm{d}}\overline{\omega}$ and let ϕ_t be the flow of $X_{\overline{\omega}}$. It is visible that it suffices to consider $\phi_1^{-1} \circ f$ instead of f. Therefore after a possible reparametrization we may and do assume that $\int_0^1 X_t\,\mathrm{d}t = 0$. Next we set

$$Z_t = -\int_0^t X_\tau\,\mathrm{d}\tau\,. \tag{4.2}$$

Let $s \mapsto \psi_t^s$ be the flow of Z_t. We put $g_t = \psi_t^1 \circ f_t$. Then $f_1 = g_1$ and

$$\mathrm{Flux}(\{g_\tau\}_{0 \le \tau \le t}) = 0 \tag{4.3}$$

for each t. In fact, $\mathrm{Flux}(\{\psi_\tau^1\}_{0 \le \tau \le t}) = \mathrm{Flux}(\{\psi_t^s\}_{0 \le s \le 1}) = [\iota(Z_t)\,\overline{\Omega}]$ due to the homotopy invariance. Thus by (4.2)

$$\begin{aligned}\mathrm{Flux}(\{g_\tau\}_{0 \le \tau \le t}) &= \mathrm{Flux}(\{\psi_\tau^1\}_{0 \le \tau \le t}) + \mathrm{Flux}(\{f_\tau\}_{0 \le \tau \le t})\\ &= [\iota(Z_t)\,\overline{\Omega}] + \int_0^t [\iota(X_\tau)\,\overline{\Omega}]\,\mathrm{d}\tau = 0\,.\end{aligned}$$

Finally, it is straightforward that (4.3) implies that Y_t related to g_t belongs to $\mathcal{X}_r^*(M, \Lambda)$.

5 - Locality and pseudo-k-transitivity

In this section we make some preparations to prove that the regular part of any *NP*-manifold is uniquely determined by the group of its automorphisms (Theorem 5). First we consider two important concepts concerning the automorphism group of a geometric structure: the locality and the pseudo-k-transitivity.

Definition 6. A diffeomorphism group $G(M)$ satisfies $\boldsymbol{L}$-condition (*locality*) if for any open relatively compact U, $V \subset M$ with $\overline{U} \subset V$, and a smooth diffeotopy $\{f_t\}$ in $G(M)$ with $f_0 = \mathrm{id}$, there exist $\varepsilon > 0$ and a smooth diffeotopy $\{g_t\}$ such that $g_t = f_t$ on U for $|t| < \varepsilon$, and $\mathrm{supp}(g_t) \subset V$.

Then any orbit of $G(M)_0$ is a weakly imbedded submanifold of M.

Definition 7. A diffeomorphism group $G(M)$ is *pseudo-k-transitive* if for any two k-tuples of pairwise distinct points $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ of M such that x_i, y_i belong to the same orbit of $G(M)_0$ and each orbit of dimension $\leqslant 1$ contains at most one x_i there exists $f \in G(M)$ satisfying $f(x_i) = y_i$, $i = 1, \dots, k$.

Notice that this definition coincides with the k-transitivity (i.e. $T(k)$-property in [2], [13]) if $G(M)_0$ acts transitively. (We assume here $\dim(M) > 1$; in dim 1 the k-transitivity is somewhat differently formulated.)

The following theorem, which generalizes a well-known theorem of W.Boothby, relates the two above concepts.

Theorem 4. *Let* $G(M) \subset \mathrm{Diff}^{\infty}(M)$ *satisfy the* $\boldsymbol{L}$*-condition. Then* $G(M)$ *is pseudo-k-transitive for each* $k \geqslant 1$.

For the proof, see [15].

Corollary 2. *Let* (M, Λ) *be an NP-manifold. Then* $G^*(M, \Lambda)_c$ *(and a fortiori* $G^*(M, \Lambda)$, $G(M, \Lambda)_c$, $G(M, \Lambda)$*) is pseudo-k-transitive for each* $k \geqslant 1$.

This is so since the group $G^*(M, \Lambda)$ (and $G^*(M, \Lambda)_c$) satisfies the $\boldsymbol{L}$-condition. In fact, let $f_t \in G^*(M, \Lambda)$. Then one can assume that $f_t = \exp(X_t)$ with $X_t = (du_1 \wedge \dots \wedge du_{n-1})^{\sharp} \in \mathcal{X}_H(M, \Lambda)$. Choose a smooth function λ with $\mathrm{supp}\,\lambda \subset V$ and $\lambda = 1$ on U. Then g_t corresponding to

$$Y_t = (\mathrm{d}(\lambda u_1) \wedge \dots \wedge \mathrm{d}(\lambda u_{n-1}))^{\sharp}$$

satisfies Definition 6.

Definition 8 (*Fragmentation property*). Let $G(M)$ be a diffeomorphism group. For any finite family of open balls $\{U_i\}$ and any $f \in G(M)_0$ such that $\mathrm{supp}(f) \subset \bigcup U_i$ there exists a decomposition $f = f_s \circ \dots \circ f_1$ such that $\mathrm{supp}(f_j) \subset U_{i(j)}$ for $j = 1, \dots, s$.

Proposition 6. *Any diffeomorphism group satisfying the* **L**-*condition possesses the fragmentation property. In particular, so does* $G^*(M, \Lambda)$.

Proof. Let $f \in G(M)_0$, let f_t be an isotopy joining f with id, and let X_t be the corresponding family in $\mathcal{X}_G(M)$, the Lie algebra of $G(M)$. If we take $f_{(p/m)t} f^{-1}_{(p-1/m)t}$, $p = 1, \dots, m$, instead of f_t for m sufficiently large, we may assume that f_t is close to the identity.

First we choose a new family of open balls, $\{V_j\}_{j=1}^s$, satisfying $\operatorname{supp}(f_t) \subset V_1 \cup \dots \cup V_s$ for each t and which is starwise finer that $\{U_i\}$, that is

$$(\forall j)\ (\exists i)\ \operatorname{star}(V_j) \subset U_{i(j)} \qquad \text{where} \quad \operatorname{star}(V_j) = \bigcup_{\overline{V}_j \cap \overline{V}_k \neq \emptyset} V_k.$$

Let $(\lambda_j)_{j=1}^s$ be a partition of unity subordinate to (V_j), and let $Y_t^j = \lambda_j X_t$. We set

$$X_t^j = Y_t^1 + \dots + Y_t^j \qquad j = 1, \dots, s$$

and $X_t^0 = 0$. Each of the smooth families X_t^j integrates to an isotopy g_t^j with support in $V_1 \cup \dots \cup V_j$. We get the factorization $f_t = g_t^s = f_t^s \circ \dots \circ f_t^1$, where $f_t^j = g_t^j \circ (g_t^{j-1})^{-1}$. Then the required inclusions

$$\operatorname{supp}(f_t^j) = \operatorname{supp}(g_t^j \circ (g_t^{j-1})^{-1}) \subset \operatorname{star}(V_j) \subset U_{i(j)}$$

hold whenever f_t is sufficiently near the identity.

Remark. The above proof shows that actually the fragmentation property holds as well for isotopies.

For any $f \in G(M)$ we denote by $\operatorname{Fix}(f)$ the set of all $x \in M$ fixed by f.

Proposition 7. *For any sufficiently small neighborhood V of a regular point $x \in M$ there exists $f \in G^*(M, \Lambda)$ such that* $\operatorname{Fix}(f) = (M - U) \cup \{x\}$ *for some open ball U with $\overline{U} \subset V$.*

Proof. Suppose $(V, x_1, \dots, x_n, y_1, \dots, y_q)$ is a canonical chart at x. In this chart let $r > 0$ such that $\overline{B(0, r)} \subset V$. Choose a smooth $\lambda: \boldsymbol{R} \to \boldsymbol{R}$ such that $\lambda(x) = 0$ for $x \leq 0$, and $\lambda(x) > 0$ for $x > 0$. We define

$$u(x_1, \dots, x_n, y_1, \dots, y_q) = \lambda((x_1^2 + \dots + y_q^2)(r^2 - (x_1^2 + \dots + y_q^2))).$$

Next we take $\alpha_1, \dots, \alpha_{n-1} \in \Omega^1(M)$ such that $(\alpha_1 \wedge \dots \wedge \alpha_{n-1})^\sharp \neq 0$ everywhere on V. This can be done for V sufficiently small. Then we put $f = \exp(X)$, where $X = (u\alpha_1 \wedge \ldots \wedge \alpha_{n-1})^\sharp$.

6 – Isomorphism between Poisson automorphism groups

Recall that a diffeomorphism between two foliated manifolds is leaf preserving, if it maps each leaf of the first manifold onto a leaf of the second manifold.

Theorem 5. *Let (M_i, Λ_i), $i = 1, 2$, be an NP-manifold. Then any group isomorphism $\Phi: G(M_1, \Lambda_1) \to G(M_2, \Lambda_2)$ is induced by a unique leaf preserving diffeomorphism $\phi: M_1^r \to M_2^r$ in the sense that $\Phi(f)\,|M_2^r = \phi f \phi^{-1}$ $\forall f \in G(M_1, \Lambda_1)$. Moreover, $\phi^* \Lambda_2 = \alpha \Lambda_1$ where α is a smooth function constant on the leaves.*

From now on, for simplicity, we denote $G_i = G(M_i, \Lambda_i)$, $i = 1, 2$. By $S_x G_i$ we denote the isotropy subgroup (or the stabilizer) of G_i at $x \in M_i$. Next we let $F_y^1 = \Phi^{-1}(S_y G_2)$ for $y \in M_2$.

The proof of the following lemma is due to J. Whittaker [24]. The nontransitive version is in [14], Sec. 3 and it applies to our case. This is so since $G(M, \Lambda)$ is pseudo-3-transitive (Corollary 2).

Lemma 1.

i. *Let $x \in M_i^r$. Then $S_x G_i$ is a maximal subgroup of G_i.*

ii. *Let $y \in M_2^r$ and let C be a closed nonempty subset of M_1^r satisfying $C \cap L \neq L$ for any regular leaf $L \in \mathcal{F}(\Lambda_1)$ and such that $f(C) = C$ for any $f \in F_y^1$.*

Then $C = \{x\}$ for some $x \in M_1^r$ and $F_y^1 = S_x G_1$.

The proof of Theorem 5 consists mainly in showing the following

Theorem 6. *For any $y \in M_2^r$ there exists a unique $x \in M_1^r$ such that $\Phi(S_x G_1) = S_y G_2$.*

Proof. We introduce the following notation. For any open ball U in M_i let $G_i(U)$ (resp. $G_i^*(U)$) be the totality of elements of $G(M_i, \Lambda_i)$ (resp. $G^*(M_i, \Lambda_i)$) compactly supported in U. Next, for $y \in M_2^r$ we denote by $\mathcal{C}_y$ the totality of open balls U of M_1 satisfying

$$G_1^*(U) \subset F_y^1 = \Phi^{-1}(S_y G_2).$$

Let $C_y = M_1 - \bigcup \mathcal{C}_y$. It is easily seen that the closed subset C_y is preserved by elements of F_y^1.

The following statement (cf. [**13**], [**14**]) is a clue part of the proof.

For any $y \in M_2^r$ and for any regular leaf $L \in \mathcal{F}(\Lambda_1)$ there is an open ball U on M_1 such that $U \cap L \neq \emptyset$ and $G_1^*(U) \subset F_y^1$.

To show the above property fix $y \in M_2^r$. For any open ball V on M_1 satisfying $V \cap L \neq \emptyset$ we may assume the existence of $f_1 \in G_1^*(V)$ such that $f_2(y) \neq y$ where $f_2 = \Phi(f_1)$; otherwise we are done. We can then take two open sets V, W and $\tilde{f}_1, \bar{f}_1 \in G_1^*(V)$, $\tilde{g}_1, \bar{g}_1 \in G_1^*(W)$ such that $\overline{V} \cap \overline{W} = \emptyset$, $L - \overline{V \cup W} \neq \emptyset$, and

$$y \neq \tilde{f}_2(y) \neq \bar{f}_2(y) \neq y \qquad y \neq \tilde{g}_2(y) \neq \bar{g}_2(y) \neq y$$

where $\tilde{f}_2 = \Phi(\tilde{f}_1)$, $\bar{f}_2 = \Phi(\bar{f}_1)$, $\tilde{g}_2 = \Phi(\tilde{g}_1)$, $\bar{g}_2 = \Phi(\bar{g}_2)$ (cf. [**14**], Lemma 4.1; in our case the proof is the same).

Due to Proposition 7 one can choose $h_2 \in G_2$ such that $\mathrm{Fix}(h_2) = (M - B) \cup \{y\}$ for some open ball B at y so small that the equalities

$$B \cap \tilde{f}_2(B) = \emptyset, \qquad B \cap \bar{f}_2(B) = \emptyset, \qquad B \cap \tilde{g}_2(B) = \emptyset, \qquad B \cap \bar{g}_2(B) = \emptyset$$

hold. Let $h_1 = \Phi^{-1}(h_2)$. Then we have either $\overline{V \cup h_1(V)} \not\supset L$, or $\overline{V \cup h_1(V)} \supset L$.

In first case we choose an open ball U such that $\overline{U} \cap \overline{(V \cup h_1(V))} = \emptyset$, and $U \cap L \neq \emptyset$. For any $k_1 \in G_1^*(U)$ one has $[k_1, [\tilde{f}_1, h_1]] = \mathrm{id}$ and $[k_1, [\bar{f}_1, h_1]] = \mathrm{id}$, as $\mathrm{supp}([\tilde{f}_1, h_1])$ and $\mathrm{supp}([\bar{f}_1, h_1])$ are contained in $V \cup h_1(V)$. Hence for $k_2 = \Phi(k_1)$ we have

$$[k_2, [\tilde{f}_2, h_2]] = \mathrm{id} \qquad [k_2, [\bar{f}_2, h_2]] = \mathrm{id}.$$

By definition we get $\mathrm{Fix}([\tilde{f}_2, h_2]) = M - (B \cup \tilde{f}_2(B)) \cup \{y, \tilde{f}_2(y)\}$ and similarly for $\mathrm{Fix}([\bar{f}_2, h_2])$ with $\bar{f}_2$ instead of $\tilde{f}_2$.

Observe that $f(\mathrm{Fix}(g)) = \mathrm{Fix}(g)$ whenever f, g commute. Hence either $k_2(y) = y$ or $k_2(y) = \tilde{f}_2(y)$. Analogously for $\bar{f}_2$ instead of $\tilde{f}_2$ we get either $k_2(y) = y$ or $k_2(y) = \bar{f}_2(y)$. Consequently we have $k_2(y) = y$ as $\tilde{f}_2(y) \neq \bar{f}_2(y)$. Thus $G_1^*(U) \subset F_y^1$, as required.

In the second case we take W instead of V and we apply an analogous argument. This proves the statement.

Now we apply Lemma 1. By the result we have just achieved C_y cannot contain any leaf of $\mathcal{F}(\Lambda_1)$. Arguing by contradiction suppose $C_y = \emptyset$, i.e. $M_1^r \subset \bigcup U_j$ such that $G_1^*(U_j) \subset F_y^1$. By Proposition 6 we get $G^*(M_1, \Lambda_1) \subset F_y^1$. Then, by

Proposition 4

$$G^*(M_1, \Lambda_1) \subset \bigcup_{g \in G(M_1, \Lambda_1)} g^{-1} \circ F_y^1 \circ g = \{\mathrm{id}\}.$$

This contradiction proves that $C_y \neq \emptyset$. By Lemma 1 for $C = C_y$ we get $F_y^1 = S_x G_1$ for some $x \in M_1^r$.

Proof of Theorem 5. Theorem 6 and its symmetric version determine uniquely a bijection $\phi: M_1^r \to M_2^r$ verifying $\Phi(S_x G_1) = S_{\phi(x)} G_2$. It follows from the proof that ϕ satisfies

$$\Phi(f)\,|M_2^r = \phi f \phi^{-1} \qquad \forall f \in G_1 \tag{6.1}$$

and that ϕ is leaf preserving. It is visible from (6.1) that ϕ is a homeomorphism. As in [13] we prove that ϕ is a diffeomorphism as well.

To show the uniqueness let ψ be another bijection such that $\Phi(f)\,|M_2^r = \psi f \psi^{-1}$ $\forall f \in G_1$. Assume that $\phi \neq \psi$ and set $\chi = \psi^{-1}\phi \neq \mathrm{id}$. We have $\chi f \chi^{-1} = f|M_1^r$ for any $f \in G_1$. There is $x \in M_1$ such that $y = \phi(x) \neq x$. Let $z \in L_x$, $z \neq x$, $z \neq y$ such that y, z lie in the same component of $L_x - \{x\}$ if $\dim(L_x) = 1$. By the pseudo-2-transitivity of G_1 there is $f \in S_x G_1$ such that $f(y) = z$. It follows that $\chi f \chi^{-1}(y) = \chi f(x) = \chi(x) = y$ while $f(y) = z \neq y$; a contradiction.

Finally, to state that ϕ interchanges the geometric structures Λ_1 and Λ_2 we need first that ϕ_* is a Lie algebra isomorphism of $\mathcal{X}(M_1, \Lambda_1)_c$ onto $\mathcal{X}(M_2, \Lambda_2)_c$. This can be done exactly as in [14], p. 540. Therefore, if $L_1 \in \mathcal{F}(\Lambda_1)$, $L_2 = \phi(L_1)$ and Ω_{L_i} is the volume form living on L_i, then the equality

$$\mathrm{d}\iota(X)\, \Omega_{L_1} = 0 \tag{6.2}$$

implies the equality

$$\mathrm{d}\iota(\phi_* X)\, \Omega_{L_2} = 0\,. \tag{6.3}$$

Now let $x \in M_1^r$. We choose a canonical chart (U, x_i, y_j), $i \leq n, j \leq q$ at x such that $\Omega_{L_1} = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$ in U. Now let $(V = \phi(U), x_i', y_j')$ be a chart at $y = \phi(x)$ such that $x_i' \circ \phi = x_i$, $y_j' \circ \phi = y_j$. In particular, $\phi_*(\partial_i) = \partial_i'$ where $\partial_i = \partial/\partial x_i$ and $\partial_i' = \partial/\partial x_i'$. Let us write $\Omega_{L_2} = \alpha\, \mathrm{d}x_1' \wedge \ldots \wedge \mathrm{d}x_n'$. It is easily seen that suitable extensions of ∂_i verify (6.2). Consequently, suitable extensions of ∂_i' satisfy (6.3). Hence the function α is constant. Therefore we get the equality $\phi^* \Lambda_2 = \alpha \Lambda_1$ with α depending on leaves only. This implies that α is a smooth function.

This completes the proof.

Final remarks.

1. Theorem 5 still holds for $G(M, \Lambda)_c$, $G^*(M, \Lambda)$, or $G^*(M, \Lambda)_c$ instead of $G(M, \Lambda)$. The proof is essentially the same.

2. Clearly, the topological and smooth structure of the set $M - M^r$ cannot be deduced from $G(M, \Lambda)$. The possible exception is when the set of singular points is sufficiently *thin*, e.g. finite.

3. An infinitesimal version of Theorem 5 still holds. It states that if there exists a Lie algebra isomorphism $\Psi : \mathcal{X}(M_1, \Lambda_1) \to \mathcal{X}(M_2, \Lambda_2)$ then there exists a unique Poisson diffeomorphism $\phi : M_1^r \to M_2^r$ such that $\phi_* = \Psi$. The result and its proof is a generalization of that for the unimodular structures in [**12**].

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Sommario

Le strutture di Nambu-Poisson costituiscono una nozione base nella geometria multisimplettica. Vengono qui ottenuti alcuni risultati relativi al gruppo degli automorfismi di queste strutture. Si propone anzi tutto una definizione di omomorfismo flusso (flux homomorphism). Si mostra poi che il gruppo degli automorfismi determina completamente la struttura sottogiacente sull'insieme dei punti regolari.

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