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A theorem of existence for the equations of the Winslow's effect (**)

1 - Introduction

Electrorheological fluids are slightly conducting suspensions whose viscosity varies by a factor as high as 10^5 if a strong electric field is applied. This effect was discovered by W. M. Winslow in 1947 [10] and bears his name. Electrorheological fluids captured the interest of many authors since a fluid that changes its bulk viscosity with the electric field offers a way to address many electromechanical problems and can be used to control vibration damping, see [5].

The current levels associated with the high voltages are typically in the order of a few micro-ampere; consequently, magnetic effects are negligible and we assume the constitutive equations of Electrohydrodynamics, for which we refer to [3].

In this paper we propose (Section 2) a quite general constitutive equation relating the stress tensor to the electric field via a viscosity tensor, whose form is determined by using the objectivity principle. We hope, in this way, to capture the anisotropic character of the Winslow's effect.

The main result is given in Section 3, where we prove that the resulting boundary value problem has at least one weak solution.

2 - Derivation of the basic equations

Our treatment is based on a representation theorem for tensor valued isotropic functions. We recall that a tensor valued function is said to be *isotropic* if the

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form of its components functions is the same for all orthonormal bases (see [8] for details). The following result is a corollary of a theorem of A. S. Wineman and A. C. Pipkin [9], but, to make this paper more readable and self contained, we present an elementary proof.

Theorem 1. *Every isotropic function, defined in the set of non-vanishing vectors* $\mathbf{E}$ *with values in the space of fourth order tensors* $\mu_{iklm} = \overline{\mu}_{iklm}(E_j)$ $i, k, l, m, j = 1, 2, 3$, *which satisfies the symmetry relations:*

$$\mu_{iklm} = \mu_{kilm} \qquad \mu_{iklm} = \mu_{ikml} \tag{1}$$

can be written as

$$\begin{aligned}\mu_{iklm} &= a_1 \delta_{ik} \delta_{lm} + a_2(\delta_{il} \delta_{km} + \delta_{im} \delta_{kl}) + a_3 E_i E_k \delta_{lm} + a_4 E_l E_m \delta_{ik} \\ &+ a_5(E_i E_l \delta_{km} + E_k E_m \delta_{il} + E_i E_m \delta_{kl} + E_l E_k \delta_{im}) + a_6 E_i E_k E_l E_m\end{aligned} \tag{2}$$

where the a_i *are arbitrary functions of* $|\mathbf{E}|$.

Proof. We estimate the dimension of the linear space $\mathcal{H}$ of isotropic functions and exhibit an explicit basis for it. Then we include the symmetry conditions. We have dim $\mathcal{H} \leq 81$. By using suitable orthonormal transformation of the Euclidean space $\mathbf{R}^3$ we will prove that the dimension of $\mathcal{H}$ is, in fact, smaller. We set $\mathbf{E} = |\mathbf{E}|\mathbf{e}_1$ and note that this assumption is not restrictive.

By a reflection with respect to the plane span $\{\mathbf{e}_1, \mathbf{e}_2\}$ and taking into account that the functions in $\mathcal{H}$ are isotropic, we find that certain components must be equal to their opposites and therefore vanish. This implies dim $\mathcal{H} \leq 21$. With a further reflection with respect to span $\{\mathbf{e}_1, \mathbf{e}_3\}$ we obtain dim $\mathcal{H} \leq 11$. Finally a rotation of $\pi/4$ about $\mathbf{e}_1$ proves that dim $\mathcal{H} \leq 10$.

On the other hand the following 10 tensors are isotropic as can be checked from the definition:

$$F^{(1)}_{iklm} = \delta_{ik} \delta_{lm} \qquad F^{(2)}_{iklm} = \delta_{il} \delta_{km} \qquad F^{(3)}_{iklm} = \delta_{im} \delta_{lk}$$

$$F^{(4)}_{iklm} = \delta_{lm} E_i E_k \qquad F^{(5)}_{iklm} = \delta_{ik} E_l E_m \qquad F^{(6)}_{iklm} = \delta_{km} E_i E_l$$

$$F^{(7)}_{iklm} = \delta_{il} E_k E_m \qquad F^{(8)}_{iklm} = \delta_{lk} E_i E_m \qquad F^{(9)}_{iklm} = \delta_{im} E_l E_k$$

$$F^{(10)}_{iklm} = E_i E_k E_l E_m.$$

We will prove the linear independence of $\{F^{(\gamma)}_{iklm}\}$ by a straightforward calculation of the Gram determinant

$$\det \operatorname{Gram}(F^{(1)}, \dots, F^{(10)})$$

$$= \det \begin{pmatrix} 9 & 3 & 3 & 3|\mathbf{E}|^2 & |\mathbf{E}|^2 & |\mathbf{E}|^2 & |\mathbf{E}|^2 & |\mathbf{E}|^2 & 3|\mathbf{E}|^2 & |\mathbf{E}|^4 \\ 3 & 9 & 3 & |\mathbf{E}|^2 & 3|\mathbf{E}|^2 & |\mathbf{E}|^2 & |\mathbf{E}|^2 & 3|\mathbf{E}|^2 & |\mathbf{E}|^2 & |\mathbf{E}|^4 \\ 3 & 3 & 9 & |\mathbf{E}|^2 & |\mathbf{E}|^2 & 3|\mathbf{E}|^2 & 3|\mathbf{E}|^2 & |\mathbf{E}|^2 & |\mathbf{E}|^2 & |\mathbf{E}|^4 \\ 3|\mathbf{E}|^2 & |\mathbf{E}|^2 & |\mathbf{E}|^2 & 3|\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^6 \\ |\mathbf{E}|^2 & 3|\mathbf{E}|^2 & |\mathbf{E}|^2 & |\mathbf{E}|^4 & 3|\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^6 \\ |\mathbf{E}|^2 & |\mathbf{E}|^2 & 3|\mathbf{E}|^2 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & 3|\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^6 \\ |\mathbf{E}|^2 & |\mathbf{E}|^2 & 3|\mathbf{E}|^2 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & 3|\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^6 \\ |\mathbf{E}|^2 & 3|\mathbf{E}|^2 & |\mathbf{E}|^2 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & 3|\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^6 \\ 3|\mathbf{E}|^2 & |\mathbf{E}|^2 & |\mathbf{E}|^2 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & 3|\mathbf{E}|^4 & |\mathbf{E}|^6 \\ |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^4 & |\mathbf{E}|^6 & |\mathbf{E}|^6 & |\mathbf{E}|^6 & |\mathbf{E}|^6 & |\mathbf{E}|^6 & |\mathbf{E}|^6 & |\mathbf{E}|^8 \end{pmatrix}$$

$$= 2048\,|\mathbf{E}|^{32} \neq 0\,.$$

Hence $\{F^{(\gamma)}_{iklm}\}$ give a basis and all the elements $\mu_{iklm} \in \mathcal{H}$ are given by

$$\mu_{iklm} = \sum_{\gamma=1}^{10} a_\gamma(|\mathbf{E}|)\, F^{\gamma}_{iklm}$$

where the a_γ must be functions of $|\mathbf{E}|$, which is the only invariant (under orthogonal transformations) of $\mathbf{E}$.

Taking into account the symmetry conditions we see that $c_2 = c_3$ and that $c_6 = c_7 = c_8 = c_9$, and the result follows.

Let $\mathbf{E}$ be the electric field and ε the dielectric constant. We decompose the stress tensor σ_{ik} in the following way

$$\sigma_{ik} = -p\delta_{ik} + \sigma^{\mathrm{M}}_{ik} + \sigma^{\mathrm{W}}_{ik} \tag{3}$$

where p is the pressure and σ^{M}_{ik} the Maxwell's stress tensor, i.e.

$$\sigma^{\mathrm{M}}_{ik} = \varepsilon\left(E_i E_k - \frac{1}{2}|\mathbf{E}|^2 \delta_{ik}\right). \tag{4}$$

With the term σ^{W}_{ik} we try to describe the modification induced by the Winslow's effect on the viscosity. To this end we assume that

$$\sigma^{W}_{ik} = \overline{\mu}_{iklm}(\mathbf{E})\, V_{lm} \tag{5}$$

where $V_{lm} = \dfrac{v_{l,m} + v_{m,l}}{2}$, are the components of the rate of deformation tensor corresponding to the velocity field v_i and $\overline{\mu}_{iklm}(\mathbf{E})$ is a tensor viscosity which satisfies the symmetry conditions (1) which follows from the symmetry of σ_{ik} and V_{lm}.

By the principle of objectivity we require the function $\overline{\mu}_{iklm}(\mathbf{E})$ to be isotropic. Then Theorem 1 applies and the tensor is given by (2).

Under steady condition there exists an electric potential Φ such that

$$\mathbf{E} = -\nabla\Phi\,. \tag{6}$$

From (4) and (6) it follows the expression of the body forces of electric origin

$$f_i = \sigma^{M}_{ik,\,k} = qE_i \tag{7}$$

where q is the charge density related to the potential Φ by the equation

$$-\varepsilon\Delta\Phi = q\,. \tag{8}$$

A second constitutive equation is given by

$$\mathbf{J} = -D'\nabla q + Kq\mathbf{E} + q\mathbf{v} \tag{9}$$

where $\mathbf{J}$ is the current density, D' is a coefficient of diffusion and K the ionic mobility, see A. C. Eringen and G. A. Maugin [3].

By the balance of the momentum and by the conservation of charge, we have

$$\varrho v_{i,k} v_k = \sigma_{ij,j} \qquad J_{i,i} = 0 \tag{10}$$

where ϱ is the density of mass. We arrive to

Problem 1.

$$\varrho v_{i,k} v_k = \sigma_{ij,j} \qquad v_{i,i} = 0 \qquad \text{in } \Omega \tag{11}$$

$$v = 0 \quad \text{on } \partial\Omega \tag{12}$$

$$-D'\Delta q + (\mathbf{v} - K\nabla\phi)\cdot\nabla q + \frac{K}{\varepsilon} q^2 = 0 \qquad -\varepsilon\Delta\phi = q \qquad \text{in } \Omega \tag{13}$$

$$q = q_b \qquad \phi = \phi_b \quad \text{on } \partial\Omega\,. \tag{14}$$

Equation $(13)_1$ follows from $(10)_2$ taking into account (8) and $(11)_2$.

Let us examine the consequences of (2), (4) and (5). By the incompressibility condition $(11)_2$ we have:

$$a_1(|\mathbf{E}|)\,\delta_{ik}\delta_{lm}\,\frac{v_{l,m}+v_{m,l}}{2}=0 \tag{15}$$

$$a_3(|\mathbf{E}|)E_iE_k\delta_{lm}\,\frac{v_{l,m}+v_{m,l}}{2}=0\,. \tag{16}$$

Redefining the pressure as $-p+\frac{1}{2}a_4(|\mathbf{E}|)\,E_lE_m(v_{l,m}+v_{m,l})$ the term corresponding to a_4 disappears. Hence by (15) and (16) we can rewrite (3) as follows

$$\begin{aligned}\sigma_{ik} &= -\pi\delta_{ik}+\sigma^{\mathrm{M}}_{ik}+\alpha(|\mathbf{E}|)(v_{i,k}+v_{k,i})\\ &+\beta(|\mathbf{E}|)[E_iE_l(v_{l,k}+v_{k,l})+E_kE_l(v_{l,i}+v_{i,l})]+\gamma(|\mathbf{E}|)\,E_iE_kE_lE_m v_{l,m}.\end{aligned} \tag{17}$$

We recover the usual Navier-Stokes equations with constant viscosity and body forces of electric origin if

$$\alpha(|\mathbf{E}|)=\alpha_0>0 \qquad \beta(|\mathbf{E}|)=0 \qquad \gamma(|\mathbf{E}|)=0\,.$$

Let d and Φ_0 be a characteristic length and potential. To write the problem in non dimensional form, we define the following non dimensional starred quantities:

$$\mathbf{x}=\mathrm{d}\mathbf{x}^* \qquad \phi=\Phi_0\phi^* \qquad \mathbf{E}=\frac{\Phi_0}{\mathrm{d}}\mathbf{E}^*$$

$$\mathbf{v}=\frac{K\Phi_0}{\mathrm{d}}\mathbf{v}^* \qquad q=\frac{\varepsilon\Phi_0}{\mathrm{d}^2}q^* \qquad \pi=\frac{\Phi_0^2K^2\varrho}{\mathrm{d}^2}\pi^*.$$

Redefining the functions $\alpha(|\mathbf{E}|)$, $\beta(|\mathbf{E}|)$, $\gamma(|\mathbf{E}|)$ in a suitable way, and suppressing the stars, we arrive to

Problem 2.

$$v_{i,k}v_k=\sigma_{ij,j} \qquad v_{i,i}=0 \qquad \text{in } \Omega \tag{18}$$

$$v=0 \quad \text{on } \partial\Omega \tag{19}$$

$$-D\Delta q+(\mathbf{v}-\nabla\phi)\cdot\nabla q+q^2=0 \qquad -\Delta\phi=q \qquad \text{in } \Omega \tag{20}$$

$$q=q_b \qquad \phi=\phi_b \qquad \text{on } \partial\Omega \tag{21}$$

where $D = \dfrac{D'}{K\Phi_0}$ and σ_{ik}, now non dimensional, is still formally given by (17).

Let Ω be an open, bounded and connected subset of $\boldsymbol{R}^3$ with a regular boundary $\partial\Omega$. We use the customary spaces $\mathcal{C}^{m,\alpha}(\overline{\Omega})$, $L^p(\Omega)$ and $H^{m,p}(\Omega)$, referring to the book [1] for definitions and properties. Let

$$\boldsymbol{L}^p(\Omega) = (L^p(\Omega))^3 \qquad \boldsymbol{H}^{m,p}(\Omega) = (H^{m,p}(\Omega))^3 .$$

Norm and scalar product are denoted $|\,.\,|$, $(\ ,\ )$ in both $L^2(\Omega)$ and $\boldsymbol{L}^2(\Omega)$. Let $\mathcal{V} = \{\mathbf{v} \in (\mathcal{C}_0^\infty(\Omega))^3 \colon v_{i,i} = 0\}$. As usual, H and V are the closure of $\mathcal{V}$ in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$ respectively.

3 - Existence of weak solutions

We set, whenever the integrals exist,

$$a(\mathbf{E};\mathbf{v},\mathbf{w}) = \int_\Omega \alpha(|\mathbf{E}|)(v_{i,k} + v_{k,i})\, w_{i,k}\,\mathrm{d}x$$

$$b(\boldsymbol{u};\mathbf{v},\mathbf{w}) = \int_\Omega u_k v_{i,k} w_i\,\mathrm{d}x$$

$$d(\mathbf{E};\mathbf{v},\mathbf{w}) = \int_\Omega \beta(|\mathbf{E}|)[E_i E_l (v_{k,l} + v_{l,k}) + E_k E_l (v_{i,l} + v_{l,i})]\, w_{i,k}\,\mathrm{d}x$$

$$g(\mathbf{E};\mathbf{v},\mathbf{w}) = \int_\Omega \gamma(|\mathbf{E}|)\, E_i E_k E_l E_m v_{l,m} w_{i,k}\,\mathrm{d}x .$$

We assume that

$$\alpha(\xi),\ \beta(\xi),\ \gamma(\xi) \in L^\infty(\boldsymbol{R}) \cap \mathcal{C}(\boldsymbol{R}) \tag{22}$$

$$\alpha(\xi) \geq \alpha_0 > 0 \qquad \beta(\xi) \geq 0 \qquad \gamma(\xi) \geq 0 \qquad \forall \xi \in \boldsymbol{R} . \tag{23}$$

Lemma 1. *Let* (22) *and* (23) *hold, then for all* $\mathbf{v} \in V$ *and* $\mathbf{E} \in \boldsymbol{L}^\infty(\Omega)$ *we have*

$$a(\mathbf{E};\mathbf{v},\mathbf{v}) \geq \alpha_0 \|\mathbf{v}\|_V^2 \qquad d(\mathbf{E};\mathbf{v},\mathbf{v}) \geq 0 \qquad g(\mathbf{E};\mathbf{v},\mathbf{v}) \geq 0 . \tag{24}$$

Proof. Assume $\mathbf{v} \in \mathcal{V}$ and $\mathbf{E} \in \boldsymbol{L}^\infty(\Omega)$. By (23) we have

$$a(\mathbf{E}; \mathbf{v}, \mathbf{v}) = \frac{1}{2}\int_\Omega \alpha(|\mathbf{E}|)(v_{i,k} + v_{k,i})^2 \,\mathrm{d}x \geq \frac{\alpha_0}{2}\int_\Omega (v_{i,k} + v_{k,i})^2 \,\mathrm{d}x$$

$$= \alpha_0 \int_\Omega v_{i,k}^2 \,\mathrm{d}x + \alpha_0 \int_\Omega v_{i,k} v_{k,i} \,\mathrm{d}x = \alpha_0 \|\mathbf{v}\|_V^2.$$

Again from (23) we obtain

$$d(\mathbf{E}; \mathbf{v}, \mathbf{v}) = \int_\Omega \beta(|\mathbf{E}|)[E_i E_l (v_{k,l} + v_{l,k}) + E_k E_l (v_{i,l} + v_{l,i})]\, v_{i,k} \,\mathrm{d}x$$

$$= \int_\Omega \beta(|\mathbf{E}|)[(E_i v_{i,k})^2 + 2(E_i v_{i,k})(E_l v_{k,l}) + (E_i v_{k,i})^2] \,\mathrm{d}x$$

$$= \int_\Omega \beta(|\mathbf{E}|)(E_i v_{i,k} + E_l v_{k,l})^2 \,\mathrm{d}x \geq 0$$

$$g(\mathbf{E}; \mathbf{v}, \mathbf{v}) = \int_\Omega \gamma(|\mathbf{E}|)\, E_i E_k E_l E_m v_{l,m} v_{i,k} \,\mathrm{d}x = \int_\Omega \gamma(|\mathbf{E}|)(E_i E_k v_{i,k})^2 \,\mathrm{d}x \geq 0\,.$$

Since $\mathcal{V}$ is dense in V, the conclusion follows.

As weak formulation of Problem 2 we take

Problem 3. To find $\{\mathbf{v}, q, \phi\} \in V \times H^1(\Omega) \times H^1(\Omega)$ such that

(25) $$a(\mathbf{E}; \mathbf{v}, \mathbf{w}) + b(\mathbf{v}; \mathbf{v}, \mathbf{w}) + d(\mathbf{E}; \mathbf{v}, \mathbf{w}) + g(\mathbf{E}; \mathbf{v}, \mathbf{w}) = (q\mathbf{E}, \mathbf{w})$$

for any $\mathbf{w} \in V$

$$\phi - \phi_b \in H_0^1(\Omega) \qquad (\nabla\phi, \nabla\zeta) = (q, \zeta) \qquad \forall \zeta \in H_0^1(\Omega)$$

$$\mathbf{E} = -\nabla\phi$$

$$q - q_b \in H_0^1(\Omega) \qquad D(\nabla q, \nabla\eta) + ((\mathbf{v} - \nabla\phi)\cdot q, \eta) + (q^2, \eta) = 0$$

for any $\eta \in H_0^1(\Omega)$.

We have

Theorem 2. *Let* $q_b \in H^1(\Omega)$ *with* $0 \leq q_b \leq q_0 = \sup_{\partial\Omega} q_b < +\infty$. *Assume* $\phi_b \in H^1(\Omega)$ *and* $\mathbf{v} \in V \cap \mathcal{C}^{0,\alpha}(\Omega)$, *then the following problem:*

$$(q, \phi) \in H^1(\Omega) \times H^1(\Omega)$$

$$\phi - \phi_b \in H_0^1(\Omega) \qquad (\nabla\phi, \nabla\zeta) = (q, \zeta) \qquad \forall \zeta \in H_0^1(\Omega)$$

$$q - q_b \in H_0^1(\Omega) \qquad D(\nabla q, \nabla\eta) - (q\mathbf{v}, \nabla\eta) + (q\nabla\phi, \nabla\eta) = 0 \qquad \forall \eta \in H_0^1(\Omega)$$

has at least one solution that satisfies

$$0 \leq q \leq q_0. \tag{26}$$

Proof. We suppose $q_b \in \mathcal{C}^1(\overline{\Omega})$. Let $\Sigma = \{q \in \mathcal{C}^{0,\alpha}(\Omega);\ 0 \leq q \leq q_0\}$ and define the map

$$T\colon \Sigma \to \mathcal{C}^{0,\alpha}(\Omega) \qquad T\colon q \mapsto \overline{q},$$

via the linear problem

$$-\Delta\phi = q \qquad -D\Delta\overline{q} + (\mathbf{v} - \nabla\phi)\cdot\nabla\overline{q} + q\overline{q} = 0 \quad \text{in } \Omega$$

$$\phi = \phi_b \qquad q = q_b \qquad \text{on } \partial\Omega\,.$$

By the Hölder results for linear elliptic equations (Courant-Hilbert [**2**]) we see that this problem has a unique solution such that

$$\|\nabla\phi\|_{(\mathcal{C}^{0,\alpha}(\Omega))^3} \leq C(q, \phi_b, \Omega, \alpha) \qquad \text{and} \qquad \|\overline{q}\|_{\mathcal{C}^{0,\alpha}(\Omega)} \leq B(q_b, \|V\|_{\mathcal{C}^{0,\alpha}(\Omega)}, \phi, \alpha)\,.$$

By the classical maximum principle (see D. Gilbarg and N. S. Trudinger [**4**]) we prove that condition (26) is satisfied by $\overline{q}$.

We conclude that T maps Σ into a compact subset of itself. Hence, by Schauder's fixed point theorem, T has a fixed point. We complete the proof constructing an approximating sequence $\{q_{bn}\}$ such that $q_{bn} \in \mathcal{C}^1(\overline{\Omega})$, with

$$\|q_{bn}\|_{L^\infty(\Omega)} \leq q_0 \qquad q_{bn} \to q_b \qquad \text{in } H_0^1(\Omega)\,. \tag{27}$$

Let now (q_n, ϕ_n) be the solution obtained by the previous argument. Define $\beta_n = q_n - q_{bn} \in H_0^1(\Omega)$. We have $\|\beta_n\|_{L^\infty(\Omega)} \leq 2q_0$. Moreover, we have

$$D(\nabla(\beta_n + q_{bn}), \nabla\eta) - ((\beta_n + q_{bn})\mathbf{v}, \nabla\eta) + ((\beta_n + q_{bn})\,\nabla\phi_n, \nabla\eta) = 0 \tag{28}$$

for any $\eta \in H_0^1(\Omega)$.

Setting $\eta = \beta_n$ we get

$$D\|\beta_n\|^2 \leq D\|\beta_n\|\,|\nabla q_{bn}| + \|q_{bn}\|_{L^\infty(\Omega)}\|\mathbf{v}\|\,\|\beta_n\| + \|\beta_n + q_{bn}\|_{L^\infty(\Omega)}\,|\nabla\phi_n|\,|\nabla\beta_n|\,.$$

By (27) we obtain $|\nabla\phi_n| \leq \|q_n\| \leq C\|q_n\|_{L^\infty(\Omega)} \leq C$. Hence $\|\beta_n\| \leq C$.

Therefore we can extract subsequences (not relabelled)

$$\beta_n \rightharpoonup \beta \quad \text{in } H^1(\Omega) \qquad \beta_n \to \beta \quad \text{in } L^2(\Omega) \text{ and a.e.} \qquad \nabla\phi_n \rightharpoonup \nabla\phi \quad \text{in } L^2(\Omega)\,.$$

Taking the limit in (28) we conclude that $q = \beta + q_b$ and ϕ give a solution to the problem.

We are now in a position to prove the main result of this section

Theorem 3. *There exists at least one weak solution to Problem* 3.

Proof. We use the Galerkin method in a standard way. Let $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ be a *basis* of V in the following sense: $\mathbf{w}_1, \dots, \mathbf{w}_m$ are linearly independent $\forall m \in N$, and finite combinations are dense in V. Such a basis exists since V is separable and it is possible to assume that $\mathbf{w}_j \in (\mathcal{C}_0^\infty(\Omega))^3$.

Let $\mathbf{v}_m \in \operatorname{span}\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$. We denote by (ϕ_m, q_m) the solution of problem

(29) $$q_m - q_b \in H_0^1(\Omega) \qquad -D\Delta q_m + (\mathbf{v}_m - \nabla\phi_m)\cdot\nabla q_m + q_m^2 = 0 \qquad \text{in } \Omega$$

(30) $$\phi_m - \phi_b \in H_0^1(\Omega) \qquad -\Delta\phi_m = q_m \qquad \text{in } \Omega\,.$$

By Theorem 2 (29), (30) have at least one solution such that

(31) $$0 \leqslant q_m(x) \leqslant q_0\,.$$

Let $\mathbf{E}_m = -\nabla\phi_m$. From (30) we have $\|\mathbf{E}_m\|_{H^{2,p}(\Omega)} \leqslant C_1$ and by Sobolev's embedding theorem

(32) $$\|\mathbf{E}_m\|_{(\mathcal{C}^{0,\alpha}(\Omega))^3} \leqslant C_2$$

where the constants C_1, C_2 depend only on Ω, q_0, ϕ_b.

We seek a solution $\mathbf{v}_m \in \operatorname{span}\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ of the finite-dimensional problem

(33) $$a(\mathbf{E}_m; \mathbf{v}_m, \mathbf{w}_j) + b(\mathbf{v}_m; \mathbf{v}_m, \mathbf{w}_j) + d(\mathbf{E}_m; \mathbf{v}_m, \mathbf{w}_j) + g(\mathbf{E}_m; \mathbf{v}_m, \mathbf{w}_j) = (q_m\mathbf{E}_m, \mathbf{w}_j)$$

for $1 \leqslant j \leqslant m$.

From (33) we get, recalling that $b(\mathbf{v}_m; \mathbf{v}_m, \mathbf{v}_m) = 0$,

$$a(\mathbf{E}_m; \mathbf{v}_m, \mathbf{v}_m) + d(\mathbf{E}_m; \mathbf{v}_m, \mathbf{v}_m) + g(\mathbf{E}_m; \mathbf{v}_m, \mathbf{v}_m) = (q_m\mathbf{E}_m, \mathbf{v}_m)\,.$$

By Lemma 1 we have $\alpha_0\|\mathbf{v}_m\|_V^2 \leqslant C_1\|q_m\mathbf{E}_m\|_{V'}\|\mathbf{v}_m\|_V$ and recalling (31), (32)

(34) $$\|\mathbf{v}_m\|_V \leqslant C_2$$

where the constant does not depend on m.

The existence of a solution $\mathbf{v}_m$ to (33) follows from Brouwer's fixed point theorem (see J. L. Lions [6], 53). By (32) we infer from (29) that

$$D|\nabla q_m|^2 \leqslant q_0(\|\mathbf{v}_m\||\nabla q_m| + \|\phi_m\||\nabla q_m| + \|q_m\|)$$

and, by using the boundedness of $\|\mathbf{v}_m\|_V$ and $\|\phi_m\|_{H^1(\Omega)}$,

$$|\nabla q_m| \leqslant C_3. \tag{35}$$

By (32), (34) and (35) we can extract from $\{\mathbf{v}_m\}$, $\{q_m\}$, $\{\mathbf{E}_m\}$ subsequences (not relabelled) such that

$$\mathbf{v}_m \rightharpoonup \mathbf{v} \quad \text{in } V \qquad \mathbf{v}_m \to \mathbf{v} \quad \text{in } H \quad \text{and a.e. in } \Omega \tag{36}$$

$$q_m \rightharpoonup q \quad \text{in } H^1(\Omega) \qquad q_m \to q \quad \text{in } L^2(\Omega) \quad \text{and a.e. in } \Omega \tag{37}$$

$$\mathbf{E}_m \to \mathbf{E} \quad \text{uniformly in } (\mathcal{C}^{0,\alpha}(\overline{\Omega}))^3 . \tag{38}$$

Using $(36)_2$ and (38) in (33) we obtain, keeping j fixed,

$$a(\mathbf{E}_m; \mathbf{v}_m, \mathbf{w}_j) \to a(\mathbf{E}; \mathbf{v}, \mathbf{w}_j).$$

As in the Navier-Stokes equations (see [7] for details) we have

$$b(\mathbf{v}_m; \mathbf{v}_m, \mathbf{w}_j) \to b(\mathbf{v}; \mathbf{v}, \mathbf{w}_j).$$

Taking the limit in $d(\mathbf{E}_m; \mathbf{v}_m, \mathbf{w}_j)$ and $g(\mathbf{E}_m; \mathbf{v}_m, \mathbf{w}_j)$ is also immediate, since by (38) we have:

$$(E_m)_i(E_m)_k \to E_i E_k \qquad \text{in } (\mathcal{C}^{0,\alpha}(\overline{\Omega}))^{3^2}$$

$$(E_m)_i(E_m)_k(E_l)_l(E_m)_l \to E_i E_k E_l E_m \qquad \text{in } (\mathcal{C}^{0,\alpha}(\overline{\Omega}))^{3^4} .$$

By (22) we also have

$$\beta(|\mathbf{E}_m|) \to \beta(|\mathbf{E}|) \qquad \gamma(|\mathbf{E}_m|) \to \gamma(|\mathbf{E}|).$$

Therefore, letting $m \to +\infty$ in (33), we have

$$a(\mathbf{E}; \mathbf{v}, \mathbf{w}_j) + b(\mathbf{v}; \mathbf{v}, \mathbf{w}_j) + d(\mathbf{E}; \mathbf{v}, \mathbf{w}_j) + g(\mathbf{E}; \mathbf{v}, \mathbf{w}_j) = (q\mathbf{E}, \mathbf{w}_j)$$

where $1 \leqslant j \leqslant m$.

By density we conclude that (25) holds. Finally we can pass to the limit in (29) and (30). Hence $(\mathbf{v}, q, \phi)$ is a weak solution to Problem 3.

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Sommario

In questo lavoro vengono ricavate le equazioni costitutive per i fluidi elettroreologici. Le ipotesi assunte sono legate alla dipendenza tensoriale della viscosità dal campo elettrico. Mediante l'approssimazione di Galerkin si prova poi l'esistenza di soluzioni deboli per tali equazioni.

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