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Complex interpolation and (n, δ)-convexity (**)

1 - Introduction and notation

In this paper we investigate the relations between the complex method of interpolation for infinite families of Banach spaces, introduced in [4], and the notion of (n, δ)-convexity, due to D. P. Giesy and R. C. James [6].

In particular, we prove that the intermediate spaces obtained by interpolation from a family $\{X(\theta)\}$, $0 \leqslant \theta < 2\pi$, are (n, δ)-convex provided that the boundary spaces satisfy the same property when θ ranges in a subset U of $[0, 2\pi)$ with positive measure.

Our result includes those in [2] and [3], about, respectively, the stability of (n, δ)-convexity for complex interpolation of pair of spaces, and the case $n = 2$, i.e. uniform non-squareness.

For $n \geqslant 2$ and $\delta > 0$, a Banach space X is (n, δ)-*convex* if for any $x_1, \ldots, x_n$ of X such that $\|x_j\| \leqslant 1$ for every j, there exists a choice of signs $\varepsilon_1, \ldots, \varepsilon_n$, with $\varepsilon_j = \pm 1$ such that

$$\left\| \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j x_j \right\| \leqslant 1 - \delta .$$

A Banach space X is said to be *uniformly non-*l_n^1 if it is (n, δ)-convex for some $\delta > 0$, and X is *B-convex* if it is uniformly non-l_n^1 for some $n \geqslant 2$. Uniformly non-l_2^1 spaces are known as *uniformly non-square.* (For these definitions see [7] and [6]).

The complex method of interpolation for infinite families of Banach spaces is a

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generalization of the well known method for pairs of spaces introduced by Calderón, [1]. For sake of completeness we give a brief introduction to this interpolation method.

Let D be the open unit disk in the complex plane and denote by ∂D its boundary. We identify the points $e^{i\theta} \in \partial D$ and $\theta \in T = [0, 2\pi)$. For $z \in D$, the Poisson kernel at z is $P_z(\theta) = \frac{1}{2\pi}(1 - |z|^2)|z - e^{i\theta}|^{-2}$, and for any measurable subset U of T we denote by $|U|_z = \int_U P_z(\theta)\, d\theta$ its harmonic measure.

The family of complex Banach spaces $\{X(\theta),\ \theta \in T\}$ is an *interpolation family* if:

i. all the spaces are continuously embedded in a common Banach space $\mathcal{U}$

ii. for every $x \in \bigcap_\theta X(\theta)$ the function $\theta \to \|x\|_{X(\theta)}$ is measurable with respect to the Lebesgue measure $d\theta$ on T

iii. there exists a measurable function k on T satisfying the inequality $\int_T \log^+ k(\theta)\, P_z(\theta)\, d\theta < +\infty$ for some (and hence any) $z \in D$, and such that $\|x\|_{\mathcal{U}} \leq k(\theta) \|x\|_{X(\theta)}$ for every x belonging to the set:

$$\mathcal{A} = \{x \in \bigcap_\theta X(\theta) : \int_T \log^+ \|x\|_{X(\theta)} P_z(\theta)\, d\theta < +\infty\}.$$

Let $\mathcal{F}$ be the completion of the space of functions $f: D \to \mathcal{A}$ of the form $f(z) = \sum_{j=1}^{n} x_j \phi_j(z)$, where $x_j \in \mathcal{A}$ and $\phi_j \in N^+(D)$ (see [5]), with respect to the norm $\|f\|_\infty = \operatorname{Ess\,Sup}_\theta \|f(\theta)\|_{X(\theta)}$. (Here, $f(\theta)$ is the non-tangential limit of $f(z)$ as $z \to e^{i\theta}$.)

For $z \in D$, the intermediate spaces $X(z)$ are the images at z of the functions in the class $\mathcal{F}$, and the norm in $X(z)$ is $\|x\|_z = \operatorname{Inf}\{\|f\|_\infty : f \in \mathcal{F},\ f(z) = x\}$.

In the proof of our result we shall make use of the following inequality (Proposition 2.4 [4]). For every $f \in \mathcal{F}$ and for every $z \in D$

$$\|f(z)\|_z \leq \exp \int_T \log \|f(\theta)\|_{X(\theta)} P_z(\theta)\, d\theta\,. \tag{1}$$

2 - The main result

Theorem. *Let $\{X(\theta),\ \theta \in T\}$ be an interpolation family of complex Banach spaces, such that $X(\theta)$ is (n, δ_θ)-convex when θ belongs to a measurable subset U of T. If $|U|_z > 0$ for some (hence any) $z \in D$ and if the function $\theta \to \delta_\theta$ is mea-*

surable, then for every $z \in D$ there exists $\delta_z > 0$ such that $X(z)$ is (n, δ_z)-convex.

Proof. Let $x_1, \ldots, x_n$ belong to the unit ball of $X(z)$. For $\eta > 0$ fixed, we can find $f_1, \ldots, f_n \in \mathscr{F}$ with $\|f_j\|_\infty \leq 1$, and $(1+\eta) f_j(z) = x_j$.

For every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, $(\varepsilon_j = \pm 1)$ we define the sets

$$E_\varepsilon = E_\varepsilon(\{f_j\}) = \{\theta \in U : \|\frac{1}{n} \sum_{j=1}^{n} \varepsilon_j f_j(\theta)\|_{X(\theta)} < 1 - \delta_\theta\}.$$

Since $X(\theta)$ is (n, δ_θ)-convex for every $\theta \in U$, it is $\bigcup_\varepsilon E_\varepsilon = U$. This implies that there exists $\bar\varepsilon = (\bar\varepsilon_1, \ldots, \bar\varepsilon_n)$ such that $|E_{\bar\varepsilon}|_z \geq \frac{|U|_z}{2^{n-1}}$.

By (1) and since we have $\|\frac{1}{n} \sum_{j=1}^{n} \bar\varepsilon_j f_j(\theta)\|_{X(\theta)} \leq 1$ for every $\theta \in T$, it follows

$$\begin{aligned}\frac{1}{1+\eta} \|\frac{1}{n} \sum_{j=1}^{n} \bar\varepsilon_j x_j\|_z &\leq \exp \int_T \log \|\frac{1}{n} \sum_{j=1}^{n} \bar\varepsilon_j f_j(\theta)\|_{X(\theta)} P_z(\theta)\, d\theta \\ &\leq \exp \int_{E_{\bar\varepsilon}} \log \|\frac{1}{n} \sum_{j=1}^{n} \bar\varepsilon_j f_j(\theta)\|_{X(\theta)} P_z(\theta)\, d\theta \\ &\leq \exp \int_{E_{\bar\varepsilon}} \log (1 - \delta_\theta) P_z(\theta)\, d\theta .\end{aligned}$$

Moreover, Jensen's inequality yields

$$\begin{aligned}\exp \int_{E_{\bar\varepsilon}} \log (1 - \delta_\theta) P_z(\theta)\, d\theta &\leq [\int_{E_{\bar\varepsilon}} (1 - \delta_\theta) P_z(\theta) \frac{d\theta}{|E_{\bar\varepsilon}|_z}]^{|E_{\bar\varepsilon}|_z} \\ &= [1 - \int_{E_{\bar\varepsilon}} \delta_\theta P_z(\theta) \frac{d\theta}{|E_{\bar\varepsilon}|_z}]^{|E_{\bar\varepsilon}|_z}.\end{aligned}$$

Recalling that $\frac{|U|_z}{2^{n-1}} \leq |E_{\bar\varepsilon}|_z \leq |U|_z$, we get

$$\frac{1}{1+\eta} \|\frac{1}{n} \sum_{j=1}^{n} \bar\varepsilon_j x_j\|_z \leq [1 - \frac{1}{|U|_z} \int_{E_{\bar\varepsilon}} \delta_\theta P_z(\theta)\, d\theta]^{\frac{|U|_z}{2^{n-1}}}.$$

To evaluate the last term we observe that each harmonic measure is absolutely continuous with respect to the measure λ given by $\lambda(M) = \int_M \delta_\theta P_z(\theta)\, d\theta$,

$M \subseteq U$. Hence

$$\operatorname{Inf}\{\lambda(E) : |E|_z \geq \frac{|U|_z}{2^{n-1}}\} = \alpha_z > 0 .$$

This implies

$$\frac{1}{1+\eta} \| \frac{1}{n} \sum_{j=1}^{n} \bar{\varepsilon}_j x_j \|_z \leq [1 - \frac{\alpha_z}{|U|_z}]^{\frac{|U|_z}{2^{n-1}}} .$$

Since η is arbitrarily small the proof is complete if we take

$$\delta_z = 1 - [1 - \frac{\alpha_z}{|U|_z}]^{\frac{|U|_z}{2^{n-1}}} .$$

Remark. If we also assume that there exists a positive δ_0 such that $\delta_\theta \geq \delta_0$ for every $\theta \in U$, then it is easy to deduce that

$$\delta_z = 1 - [1 - \delta_0]^{\frac{|U|_z}{2^{n-1}}} \qquad \forall z \in D .$$

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Sommario

Si prova che gli spazi ottenuti mediante il metodo di interpolazione complessa per famiglie di spazi di Banach sono (n, δ)-convessi se gode di questa proprietà un numero «sufficiente» di spazi sul bordo.

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