Riv. Mat. Univ. Parma (5) 4 (1995), 283-306

ALESSANDRA COSCIA (*)

Relaxation results and minimum problems in *BV* and *SBV* (**)

1 - Introduction

The variational formulation of many problems in mathematical physics, computer vision, and mechanical engineering takes into account an energy functional depending on a function and a hypersurface, both a priori unknown (see [10]). Typically these functionals consist of two parts: the first one represents the *bulk energy* and is the integral of a potential, depending on the gradient of an unknown function u; the second represents the *surface energy* and is the integral of some function computed on a hypersurface, a priori unknown, where the function u is discontinuous.

These energies account for several phenomena such as crack growth and crack initiation in the theory of brittle fracture, interface formation between different phases of Cahn-Hilliard fluids, surface tension between small drops of liquid cristal, and are utilized for pattern recognition in computer vision. In particular in the one dimensional case energies of this form arise in the perception of speech, which requires segmenting time (the domain of the speech signal) into intervals during which a single phoneme is being pronounced.

This paper presents some one-dimensional results related to the variational formulation of static phenomena, which can be described by introducing an integral functional of the form

$$(1.1)\qquad \mathcal{F}(u, S) = \int_{I\setminus S} W(u')\,dt + \int_S \Theta(t, u(t+), u(t-))\,d\#(t).$$

(*) Dip. di Matem., Univ. Parma, Via D'Azeglio 85, 43100 Parma, Italia.
(**) Received December 21, 1995. AMS classification 49 J 45.

Here $\#$ denotes the counting measure on $\boldsymbol{R}$, the bounded open interval $I \subset \boldsymbol{R}$ is the reference configuration, the function u represents the displacement, which is differentiable outside the *discontinuity set* S, and $u(t+)$, $u(t-)$ are the right and left limits of u at the point t. The functions W and Θ represent the bulk and surface energy densities respectively.

We discuss the problem from the viewpoint of the so-called direct method of the Calculus of Variations. In order to give the problem a sufficiently weak formulation we make use of the theory of *functions of bounded variation* ($u \in BV(I)$) and in particular of the space $SBV(I)$ of *special functions of bounded variation*. For every $u \in BV(I)$ we denote by S_u the jump set of u, i.e. the set of points where u admits different right and left limits $u(t+) \neq u(t-)$. It is well known that S_u is at most a countable set of points. Moreover, for a function $u \in BV(I)$ we have the Lebesgue decomposition

$$u' = \dot{u}\,\mathrm{d}t + u'_s = \dot{u}\,\mathrm{d}t + \sum_{h=1}^{\infty} (u(t_h +) - u(t_h -))\,\delta_{t_h} + Cu$$

where $\dot{u}$ is the density of the absolutely continuous part of u' and Cu is the Cantor part of u', which vanishes on every finite subset of I. We say that u is a special function of bounded variation ($u \in SBV(I)$), if $u \in BV(I)$ and $Cu = 0$.

In this paper we deal with the case where the surface energy depends actually on the position of the jump points, considering functions Θ of the form

$$(1.2)\qquad \Theta(t, u(t+), u(t-)) = \begin{cases} 1 + a(t)\,|u(t+) - u(t-)| & \text{if } u(t+) \neq u(t-) \\ 0 & \text{otherwise} \end{cases}$$

where $a : \bar{I} \to]0, +\infty[$ is a continuous function. Then we are led to introducing the following two functionals defined on $BV(I)$:

$$(1.3)\qquad \mathcal{F}(u) = \int_I W(\dot{u})\,\mathrm{d}t + \#(S_u) + \int_I a(t)\,|u'_s|$$

$$(1.4)\qquad \mathcal{F}_S(u) = \begin{cases} \int_I W(\dot{u})\,\mathrm{d}t + \#(S_u) + \sum_{t \in S_u} a(t)\,|u(t+) - u(t-)| & \text{if } u \in SBV(I) \\ +\infty & \text{elsewhere on } BV(I). \end{cases}$$

The main results of this paper are a relaxation and integral representation theorem for the functional $\mathcal{F}$ on $BV(I)$ (Sec. 3), and the lower semicontinuity of the functional $\mathcal{F}_S$ on $SBV(I)$ (Sec. 4). Let us remark that we prove the lower semicontinuity of $\mathcal{F}_S$ with the only assumptions that $\Theta : I \times \boldsymbol{R} \times \boldsymbol{R} \to [0, +\infty[$ is lower semicontinuous and subadditive with respect to the last two variables.

In the literature the functionals $\mathcal{F}$ and $\mathcal{F}_S$ have been studied in [1] under the assumption $\Theta(u(t+), u(t-)) = \varphi(|u(t+) - u(t-)|)$ with φ concave, nondecreasing and superlinear at 0 (in particular we can take $\varphi = c$), while the *linear case* $\Theta(u(t+), u(t-)) = |u(t+) - u(t-)|$ is considered in [6]. For related results see also [5].

Finally in Section 5 we consider some minimization problems on $BV(I)$, with $I =]0, 1[$, with generalized Dirichlet boundary data $u(0+) = 0$, $u(1-) = \alpha$, $\alpha \in \boldsymbol{R}$, associated to the functionals $\mathcal{F}$ and $\mathcal{F}_S$ in the cases $a(t) = 1$ and $a(t) = (t - \frac{1}{2})^2 + 1$. We determine all the minimum values corresponding to the different α and we describe all the minimum points.

In particular we point out that we obtain examples of minimization problems where the minimum is achieved on $SBV(I)$, but not on $BV(I)$, or, both are achieved, but the minimum values are different.

2 - Notation and preliminaries

2.1. Notations

Let I be a bounded open interval of $\boldsymbol{R}$; we use the standard notation for the Sobolev and Lebesgue spaces $W^{k,p}(I)$ and $L^p(I)$. In particular we will denote by $\|u\|_p$ the L^p-norm of a function u in $L^p(I)$. We use the notation dt for the Lebesgue measure, and $\#$ for the counting measure; $|J|$ indicates the Lebesgue measure of a measurable set $J \subset \boldsymbol{R}$.

If $W: I \times \boldsymbol{R} \to [0, \infty[$ is a Borel function, we shall denote by $W^{**}(t, x)$ the greatest function less than or equal to W, which is convex with respect to the variable x. If $W: I \times \boldsymbol{R} \to [0, +\infty[$ is convex with respect to the second variable, we define the *recession function* $W^\infty: I \times \boldsymbol{R} \to [0, +\infty]$ by

$$W^\infty(t, x) = \lim_{s \to +\infty} \frac{W(t, sx)}{s}.$$

We remark that W^∞ is a Borel function, which is convex and positively homogeneous of degree one with respect to the variable x.

We say that a function $W: \boldsymbol{R} \to [0, +\infty]$ is *superlinear* at $+\infty$ (resp. $-\infty$) if

$$\lim_{x \to +\infty} \frac{W(x)}{x} = +\infty \qquad \left(\text{resp.} \lim_{x \to -\infty} \frac{W(x)}{|x|} = +\infty\right). \tag{2.1}$$

Given a Radon measure μ on I, we adopt the notation $|\mu|$ for its total variation (see [13], 2.2.5). We denote by $\mathcal{M}(I)$ the set of the scalar Radon measures on

I with bounded total variation and by $\mathcal{M}_+(I)$ the space of the positive Radon measures on I with bounded total variation. Let $\mu \in \mathcal{M}(I)$; we denote by μ_a and μ_s the absolutely continuous and the singular parts of μ with respect to the Lebesgue measure, and by $\frac{\mu}{|\mu|}$ the *Radon-Nikodym derivative* of μ with respect to its total variation. The integral on I of a function ψ with respect to the measure $|\mu|$ will be denoted simply by $\int_I \psi\,|\mu|$. The usual *weak* topology* on $\mathcal{M}(I)$ is defined as the weakest topology on $\mathcal{M}(I)$ for which the maps $\mu \mapsto \int_I \psi \,\mathrm{d}\mu$ are continuous for every $\psi \in \mathcal{C}^0(\bar{I})$ such that $\psi|_{\partial I} = 0$.

2.2. Functions of Bounded Variation

Let I be a bounded open interval of $\boldsymbol{R}$. We say that $u \in L^1(I)$ is a *function of bounded variation* ($u \in BV(I)$) if its distributional derivative $u' = Du$ is a Radon measure with bounded total variation on I. We have that $BV(I)$ is a Banach space, if endowed with the norm $\|u\|_{BV} = \|u\|_1 + |Du|(I)$. We recall that for every sequence $\{u_h\}_h$ in $BV(I)$ with $\|u_h\|_{BV(I)} \leq c$ there exist a subsequence $\{u_{h_k}\}_k$ and a function $u \in BV(I)$ such that $u_{h_k} \to u$ in $L^1(I)$. For the general theory of functions of bounded variation we refer to **[13]**, **[14]**, **[16]**, **[17]**.

Let $u \in BV(I)$; we denote by S_u the complement of the Lebesgue set of u; i.e. $t \notin S_u$ if and only if

$$\lim_{\varrho \to 0^+} \frac{1}{2\varrho} \int_{t-\varrho}^{t+\varrho} |u(s) - z| \,\mathrm{d}s = 0$$

for some $z \in \boldsymbol{R}$. If such a z exists it is unique, and we denote it by $\tilde{u}(t)$, the *approximate limit* of u at t. It is well known that S_u is at most a countable set of points and $\tilde{u}$ coincides with u almost everywhere in I. Furthermore the function u admits right-hand and left-hand limits $u(t+) \neq u(t-)$ at every $t \in S_u$ in an approximate sense, which means that

$$\lim_{\varrho \to 0^+} \frac{1}{\varrho} \int_{t-\varrho}^{t} |u(s) - u(t-)| \,\mathrm{d}s = 0 \qquad \lim_{\varrho \to 0^+} \frac{1}{\varrho} \int_{t}^{t+\varrho} |u(s) - u(t+)| \,\mathrm{d}s = 0\,.$$

Therefore the set S_u is also called the *jump set* of u.

In general, for a function $u \in BV(I)$, we have the Lebesgue decomposition

$$u' = u_a' + u_s' = \dot{u}\,\mathrm{d}t + u_s'$$

where we denote by $\dot{u}$ the density of the *absolutely continuous part* of u'. The *singular part* of u' can be further decomposed into two mutually singular measures as

$$u_s' = Ju + Cu = \sum_{t \in S_u} (u(t+) - u(t-))\,\delta_t + Cu$$

where $Ju = \sum_{t \in S_u} (u(t+) - u(t-))\,\delta_t$ is the *jump part* (δ_t denotes the Dirac measure at t) and Cu is the *Cantor part* of u', which vanishes on every finite subset of I:

$$J \subset I,\ \#(J) < +\infty \Rightarrow Cu(J) = 0\,. \tag{2.2}$$

In this paper we will frequently use Cantor functions and we will describe any one of them simply by saying that it increases (or decreases) by α on the interval J.

Let us remark that, if $I =]a, b[$, $t_0 \in [a, b[$, and $\alpha \in \boldsymbol{R}$ are fixed, then there is a $1-1$ correspondence between $\mathcal{M}(I)$ and the subspace

$$\{u \in BV(I): u(t_0 +) = \alpha\}\,,$$

given by $\mu \mapsto u_\mu$, where

$$u_\mu(t) = \alpha + \mu(]t_0, t]) \quad \text{if } t \geq t_0 \qquad\qquad u_\mu(t) = \alpha - \mu(]t, t_0]) \quad \text{otherwise}\,.$$

Therefore in the sequel we will sometimes define functions in $BV(I)$ by describing their measure derivative and the value $u(t_0 +)$ at some point $t_0 \in [a, b[$.

We say that a function $u \in BV(I)$ is a *special function of bounded variation*, and we write $u \in SBV(I)$, if $Cu = 0$. For the properties of a function $u \in SBV(I)$ we refer to **[11]**, **[1]**, **[2]**, **[3]**.

2.3. Relaxation

We recall the notion of *relaxed functional*. Let $F: X \to \boldsymbol{R} \cup \{+\infty\}$ be a functional on a metric space (X, τ). The relaxed functional $\overline{F}$ of F, or *relaxation* of F in the topology τ, is the greatest τ-lower semicontinuous functional less than or equal to F; i.e. the greatest functional such that $\overline{F} \leq F$ and $\overline{F}(u) \leq \liminf_h \overline{F}(u_h)$ for every sequence $\{u_h\}_h$ converging to u in the topology τ. Then

$$\overline{F}(u) = \inf\{\liminf_{h \to \infty} F(u_h): u_h \to u,\ u_h \in X\}\,.$$

For a general treatment of this subject we refer to the books by Buttazzo [7] and by Dal Maso [9]. We point out here that the relaxed functional $\overline{F}$ allows to describe the behaviour of minimizing sequences for F; indeed minimizing sequences for problems involving F converge, up to a subsequence, to solutions for the corresponding problems for $\overline{F}$. Throughout the paper we shall consider relaxation in the L^1 topology.

3 - Lower semicontinuity and relaxation results in BV

In this section we shall state and prove some lower semicontinuity and relaxation theorems concerning the functionals (1.3) and (1.4).

In the sequel we shall deal with functionals defined on the space $BV(I)$, where I is a bounded open interval of $\boldsymbol{R}$, and we assume that $W: \boldsymbol{R} \to [0, +\infty[$ is a convex function such that $W(0) = 0$; we also define a convex function with linear growth $\overline{W}: \boldsymbol{R} \to [0, +\infty[$ associated to W by setting

$$\overline{W}(x) = (W(x) \wedge |x|)^{**} = (\min\{W(x), |x|\})^{**} .$$

Let us consider the functionals $F, F_S : BV(I) \to [0, +\infty]$ defined by

$$F(u) = \int_I W(\dot{u})\, dt + |u_s'|(I)$$

$$F_S(u) = \begin{cases} \int_I W(\dot{u})\, dt + \sum_{t \in S_u} |u(t+) - u(t-)| & \text{if } u \in SBV(I) \\ +\infty & \text{elsewhere on } BV(I) . \end{cases}$$

Note that $F(u) = F_S(u)$ for every $u \in SBV(I)$.

It is easy to see that F_S is not lower semicontinuous with respect to the topology of $L^1(I)$ for any choice of the function W, while in general the functional F is not lower semicontinuous (take for instance $W(x) = x^2$). The following theorem concerns the relaxation of the functionals F and F_S.

Theorem 1. *The relaxation of the functionals F and F_S with respect to the L^1 convergence is given by the functional L defined on $BV(I)$ by*

$$L(u) = \int_I \overline{W}(\dot{u})\, dt + \int_I \overline{W}^\infty \left(\frac{u_s'}{|u_s'|}\right) |u_s'| . \tag{3.1}$$

In order to prove Theorem 1 we shall need two results about relaxation in BV and $W^{1,1}$, proved respectively in [8] and [15].

Proposition 1. *Let $V: \boldsymbol{R} \to [0, +\infty[$ be a Borel function such that $V(z) \leq c(1 + |z|)$; then the relaxation of the functional*

$$E(u) = \begin{cases} \int_I V(u')\,dt & \text{if } u \in \mathcal{C}^1(I) \cap W^{1,1}(I) \\ +\infty & \text{elsewhere on } W^{1,1}(I) \end{cases}$$

with respect to the L^1 topology is given by $\overline{E}(u) = \int_I V^{**}(u')\,dt$ for every $u \in W^{1,1}(I)$.

Proposition 2. *Let $V: \boldsymbol{R} \to [0, +\infty[$ be a convex function; then the relaxation of the functional*

$$E(u) = \begin{cases} \int_I V(u')\,dt & \text{if } u \in W^{1,1}(I) \\ +\infty & \text{elsewhere on } BV(I) \end{cases}$$

with respect to the L^1 topology is given by $\overline{E}(u) = \int_I V(\dot{u})\,dt + \int_I V^\infty\left(\frac{u_s'}{|u_s'|}\right)|u_s'|$ for every $u \in BV(I)$.

Proof of Theorem 1. Since $L \leq F \leq F_S$, we have $\overline{F} = L$, once we prove $\overline{F}_S = L$.

By Propositions 1 and 2 it is enough to prove that

$$\overline{F}_S(u) \leq \begin{cases} \int_I \widetilde{W}(u')\,dt & \text{if } u \in \mathcal{C}^1(I) \cap W^{1,1}(I) \\ +\infty & \text{elsewhere on } W^{1,1}(I) \end{cases}$$

where $\widetilde{W}(x) = W(x) \wedge |x|$.

If $u \in \mathcal{C}^1(I) \cap W^{1,1}(I)$ we set $I_u = \{t \in I: W(u'(t)) > |u'(t)|\}$. We approximate u on I_u with piecewise constant functions, thus replacing the quick growth of u by a sequence of jumps. The set I_u is open, hence, considering its connected components, it can be split in an at most countable union of open intervals of $I: I_u = \bigcup_{i=1}^{+\infty} I_u^i$.

We construct a sequence $\{u_h\}_h$ piecewise constant on I_u as follows. For every $h \in \boldsymbol{N}$, we subdivide every interval I_u^i in a finite number of contiguous subintervals of lenght less than $\frac{1}{h}$:

$$I_u^i = \bigcup_{k=1}^{n_{i,h}} I_k^{i,h}, \qquad |I_k^{i,h}| < \frac{1}{h}.$$

Set $a_k^{i,h} = \inf I_k^{i,h}$ for $k \leqslant n_{i,h}$, and $a_{n_{i,h}+1}^{i,h} = \sup I_u^i$, so that we may define

$$u_h(t) = u(a_k^{i,h}) \qquad \text{if } t \in I_k^{i,h} \qquad \text{and} \qquad u_h(t) = u(t) \qquad \text{if } t \in I \backslash I_u$$

(remark that $u \in AC(\bar{I})$, hence the values of u at $a_k^{i,h}$ make sense). It is obvious that the sequence $\{u_h\}_h$ converges to u in $L^1(I)$. Moreover, the absolute continuity of u on $\bar{I}$ gives

$$|(u_h)'_s|(I) = \sum_{i=1}^{+\infty} \sum_{k=1}^{n_{i,h}} |u(a_{k+1}^{i,h}) - u(a_k^{i,h})| \leqslant \sum_{i=1}^{+\infty} \sum_{k=1}^{n_{i,h}} \int_{a_k^{i,h}}^{a_{k+1}^{i,h}} |u'|\,\mathrm{d}t = \int_{I_u} |u'|\,\mathrm{d}t < +\infty .$$

Therefore $u_h \in SBV(I)$ for every h and

$$F_S(u_h) = \int_{I \backslash I_u} W(u'(t))\,\mathrm{d}t + |(u_h)'_s|(I) \leqslant \int_I \widetilde{W}(u'(t))\,\mathrm{d}t$$

which concludes the proof.

Example 1. For example if $W(x) = x^2$, then

$$(3.2) \qquad \overline{W}(x) = \begin{cases} x^2 & \text{if } |x| \leqslant \dfrac{1}{2} \\ |x| - \dfrac{1}{4} & \text{otherwise} \end{cases} \qquad \text{and } \overline{W}^\infty(z) = |z| .$$

Remark 1. In [6], Prop. 4.1, it is proved that if we define on $BV(I)$ the functional F_0 by setting

$$F_0(u) = \begin{cases} \int_I W(\dot{u})\,\mathrm{d}t + \sum_{t \in S_u} |u(t+) - u(t-)| & \text{if } u \in SBV(I) \text{ and } \#(S_u) < +\infty \\ +\infty & \text{elsewhere on } BV(I) \end{cases}$$

we still have $\overline{F}_0 = L$ (see (3.1)).

Remark 2. With the same arguments of the proof of Theorem 1 we can prove that if we substitute in the definition of the functional F the term $|u_s'|$ with $\gamma |u_s'|$, where γ is a positive constant, then the relaxation of F has the same expression with $\overline{W}(x) = (W(x) \wedge \gamma |x|)^{**}$. Obviously the same is true for F_S.

Let us consider now the functionals $G, H: BV(I) \to [0, +\infty]$ defined by

$$G(u) = \int_I W(\dot{u})\,dt + \#(S_u) + |u_s'|(I)$$

$$H(u) = \int_I W(\dot{u})\,dt + \#(S_u) + \int_I a(t)\,|u_s'|$$

where $a: \bar{I} \to \boldsymbol{R}$ is a strictly positive continuous function.

Theorem 2. *The functional G is not lower semicontinuous with respect to the L^1 convergence and its relaxation is given by the functional L defined in* (3.1).

Proof. As $G \geq F$, by Theorem 1 and Remark 1 it is enough to prove that $\bar{G} \leq F_0$. Let $u \in SBV(I)$ such that $F_0(u) < +\infty$. Then $\#(S_u) = m$ and

$$S_u = \{t_i : i = 1, \dots, m;\ 0 < t_1 < \dots < t_m < 1\}.$$

For h sufficiently large let us consider the sequence

$$u_h(t) = \begin{cases} \tilde{u}(t) & \text{on } I \setminus \bigcup_{i=1}^{m}]t_i - \frac{1}{h}, t_i + \frac{1}{h}[\\ f_C^{i,h}(t) & \text{on }]t_i - \frac{1}{h}, t_i + \frac{1}{h}[\qquad i = 1, \dots, m \end{cases}$$

where $f_C^{i,h}$ is the Cantor function increasing (or decreasing) by $\tilde{u}(t_i + \frac{1}{h}) - \tilde{u}(t_i - \frac{1}{h})$ on the interval $]t_i - \frac{1}{h},\ t_i + \frac{1}{h}[$.

Then $u_h \to u$ in $L^1(I)$, $\#(S_{u_h}) = 0$ for every h, and

$$\begin{aligned} \bar{G}(u) &\leq \liminf_{h \to \infty} G(u_h) \\ &= \liminf_{h \to \infty} \int_{I \setminus \bigcup_{i=1}^{m}]t_i - \frac{1}{h}, t_i + \frac{1}{h}[} W(\dot{u})\,dt + \sum_{i=1}^{m} |\tilde{u}(t_i + \tfrac{1}{h}) - \tilde{u}(t_i - \tfrac{1}{h})| \\ &\leq \int_I W(\dot{u})\,dt + \lim_{h \to +\infty} \sum_{i=1}^{m} |\tilde{u}(t_i + \tfrac{1}{h}) - \tilde{u}(t_i - \tfrac{1}{h})| \\ &= \int_I W(\dot{u})\,dt + \sum_{i=1}^{m} |u(t_i +) - u(t_i -)| = F_0(u) \end{aligned}$$

and the proof is complete.

Let us define now the functions $\widetilde{W}: I \times \boldsymbol{R} \to [0, +\infty[$ by setting

$$\widetilde{W}(t, x) = (W(x) \wedge a(t)\,|x|)^{**}$$

(recall that convexification only applies to the variable x).

Theorem 3. *The functional H is not lower semicontinuous with respect to the L^1 convergence and its relaxation is given by the functional*

$$\overline{H}(u) = \int_I \widetilde{W}(t, \dot{u}(t))\,\mathrm{d}t + \int_I \widetilde{W}^\infty \left(t, \frac{u_s'}{|u_s'|}\right) |u_s'|\,.$$

Proof. First we claim that $\overline{H} = \overline{E}$, where E is the functional defined on $BV(I)$ by

$$E(u) = \int_I W(\dot{u})\,\mathrm{d}t + \int_I a(t)\,|u_s'|\,.$$

To this purpose let us fix $u \in BV(I)$. If $\#(S_u) < +\infty$, the same arguments of the proof of Theorem 2 show that $\overline{H}(u) \leq E(u)$. Therefore let us consider the case where S_u is a countable set: $S_u = \{t_i\}_{i \in N}$, $t_i \in I$. We can construct a sequence $\{u_h\}_{h \in N}$, $u_h \in BV(I)$, $\#(S_{u_h}) = h$, by setting

$$u_h(0+) = u(0+) \qquad \text{and} \qquad u_h' = \dot{u}\,\mathrm{d}t + Cu + \sum_{i=1}^{h} (u(t_i+) - u(t_i-))\,\delta_{t_i}\,.$$

It is clear that $u_h \to u$ in $L^1(I)$. Moreover, using again the arguments of the proof of Theorem 2, for every h there exists a sequence $\{u_h^n\}_n$, $u_h^n \in BV(I)$, $\#(S_{u_h^n}) = 0$ for every n, $u_h^n \to u_h$ in $L^1(I)$, and $E(u_h^n) \leq E(u_h) + n^{-1}$. Then by a diagonal argument we find a sequence $\{u_h^{n_h}\}_h$ converging to u in $L^1(I)$ and such that

$$\overline{H}(u) \leq \liminf_{h \to \infty} H(u_h^{n_h}) = \liminf_{h \to \infty} E(u_h^{n_h}) \leq \liminf_{h \to \infty} (E(u_h) + \frac{1}{n_h}) = E(u)\,.$$

Hence $\overline{H} \leq E$ and the claim is proved.

We prove that $\overline{E}(u) = \int_I \widetilde{W}(t, \dot{u}(t))\,\mathrm{d}t + \int_I \widetilde{W}^\infty \left(t, \frac{u_s'}{|u_s'|}\right) |u_s'|\,.$

Set

$$m = \min_I a \qquad M = \max_I a\,. \tag{3.3}$$

We localise the functional E by defining for every open subset A of I and $u \in BV(I)$

$$E(u, A) = \int_A W(\dot{u})\, dt + \int_A a(t)\, |u_s'| \,.$$

In the same way we define

$$\overline{E}(u, A) = \inf \{\liminf_{h \to \infty} E(u_h, A): u_h \to u \text{ in } L^1(A),\ u_h \in BV(I)\}\,.$$

We need the following result.

Proposition 3. *For every $u \in BV(I)$ the set function $\overline{E}(u, \cdot)$ is (the restriction to the family of the open subsets of I of) a finite Radon measure on I.*

Proof. Step i: $\overline{E}(u, \cdot)$ is *inner regular*, i.e. for every open set $A \subset I$ we have

$$\overline{E}(u, A) = \sup \{\overline{E}(u, A'): A' \text{ open}, A' \subset\subset A\}\,. \tag{3.4}$$

Remark that $\overline{E}(u, \cdot)$ is an increasing set function, i.e. $\overline{E}(u, A') \leqslant \overline{E}(u, A)$ if $A' \subset A$, hence the inequality $\geqslant$ in (3.4) is trivial. Let us prove now the opposite inequality.

Fix a compact subset K of A and define $\delta = \frac{1}{2}\,\mathrm{dist}(\partial A, K)$. Then put $d_K(t) = \mathrm{dist}(t, K)$, $B(\varrho) = \{t \in A : \mathrm{d}_K(t) < \varrho\}$ if $\varrho \in]0, \delta[$ and define $B = B(\delta) = \{t \in A: \mathrm{d}_K(t) < \delta\}$. Choose any two sequences $\{u_h\}_h$, $\{v_h\}_h$ in $BV(I)$ such that $u_h \to u$ in $L^1(B)$, $v_h \to u$ in $L^1(A \backslash K)$, and

$$\overline{E}(u, B) = \lim_{h \to +\infty} E(u_h, B) \qquad \overline{E}(u, A\backslash K) = \lim_{h \to +\infty} E(v_h, A\backslash K)\,.$$

For every h, using the fact that $\tilde{u}_h = u_h$, $\tilde{v}_h = v_h$ almost everywhere in I, by the mean value theorem we can choose $\varrho_h \in]0, \delta[$ such that $\tilde{u}_h(t) = u_h(t)$, $\tilde{v}_h(t) = v_h(t)$ for every $t \in \partial B(\varrho_h)$, and

$$\sum_{t \in \partial B(\varrho_h)} |\tilde{u}_h(t) - \tilde{v}_h(t)| \leqslant \frac{1}{\delta} \int_{B\backslash K} |u_h - v_h|\, dt\,. \tag{3.5}$$

Therefore we can define the sequence $\{w_h\}_h$ in $BV(I)$ by setting

$$w_h(t) = \begin{matrix} u_h(t) & \text{if } t \in B(\varrho_h) \\ v_h(t) & \text{otherwise}\,. \end{matrix}$$

Using (3.3) and (3.5), we have

$$E(w_h, A) \leqslant E(u_h, B) + E(v_h, A\backslash K) + \sum_{t \in \partial B(\varrho_h)} a(t)\, |\tilde{u}_h(t) - \tilde{v}_h(t)|$$

$$\leqslant E(u_h, B) + E(v_h, A\backslash K) + M\, \frac{1}{\delta} \int_{B\backslash K} |u_h - v_h|\, dt\,.$$

Since $w_h \to u$ in $L^1(A)$ and $(u_h - v_h) \to 0$ in $L^1(B\backslash K)$, taking the limit as $h \to +\infty$ we obtain $\overline{E}(u, A) \leqslant \overline{E}(u, B) + \overline{E}(u, A\backslash K)$.

As $B \subset\subset A$, in order to conclude the proof of Step **i** we have to show that we can choose K in such a way that $\overline{E}(u, A\backslash K)$ is arbitrarily small. To this purpose let us remark that for every open subset A of I we have $E(u, A) \leqslant \int_A W(\dot{u})\, dt + M\, |u_s'|\, (A)$, hence by Remark 2

$$\overline{E}(u, A) \leqslant \int_A \overline{W}(\dot{u})\, dt + \int_A \overline{W}^\infty\Big(\frac{u_s'}{|u_s'|}\Big)\, |u_s'| \tag{3.6}$$

with $\overline{W}(x) = (W(x) \wedge M\,|x|)^{**}$. For every $u \in BV(I)$ the right-hand side in (3.6) is a finite Radon measure on I (remark that $\overline{W}(x) \leqslant M\,|x|$), hence

$$\int_{A\backslash K} \overline{W}(\dot{u})\, dt + \int_{A\backslash K} \overline{W}^\infty\Big(\frac{u_s'}{|u_s'|}\Big)\, |u_s'|$$

vanishes for K invading A and the proof is complete.

Step **ii**: $\overline{E}(u, \cdot)$ is a *subadditive set function*, i.e. we have

$$\overline{E}(u, A_1 \cup A_2) \leqslant \overline{E}(u, A_1) + \overline{E}(u, A_2)$$

for every pair of open subsets A_1, A_2 of I.

By the regularity of $\overline{E}(u, \cdot)$ (Step **i**) it is enough to prove that

$$\overline{E}(u, A) \leqslant \overline{E}(u, A_1) + \overline{E}(u, A_2)$$

for every open set $A \subset\subset A_1 \cup A_2$. This inequality can be proved by arguing as in Step **i**, choosing $K = \overline{A}\backslash A_2$ and

$$B = \{t \in A : d_K(t) < \frac{1}{2}\, \mathrm{dist}(K, \overline{A}\backslash A_1)\}\,.$$

Moreover it is clear that $\overline{E}(u, \cdot)$ is *additive on disjoint sets*, i.e.

$$\overline{E}(u, A_1 \cup A_2) = \overline{E}(u, A_1) + \overline{E}(u, A_2)$$

if $A_1 \cap A_2 = \emptyset$.

Step **iii**: $\overline{E}(u, \cdot)$ is the restriction to the open subsets of I of a finite Radon measure.

It suffices to remark that the set function $\overline{E}(u, \cdot)$ verifies:

a. $\overline{E}(u, \cdot)$ is a positive and increasing set function

b. $\overline{E}(u, \cdot)$ is inner regular (Step **i**)

c. $\overline{E}(u, \cdot)$ is subadditive and it is additive on disjoint sets (Step **ii**)

d. $\overline{E}(u, \cdot) < +\infty$ (see (3.6))

and to apply [12], Theorem 5.6.

Proof of Theorem 3 (Continuation). The function a is uniformly continuous on I and $\min_I a > 0$, hence for every $n \in N$ if we consider $\varepsilon_n = \dfrac{M-m}{n}$ there exists $\delta_n > 0$ such that $|t-s| < \delta_n \Rightarrow |a(t) - a(s)| < \varepsilon_n$.

Take $u \in BV(I)$; by Proposition 1 $\overline{E}(u, \cdot)$ is the restriction to the open subsets of I of a finite Radon measure. Then for every $n \in N$ we can find a finite open cover with small overlap $\{I_{i,n}\}_{i=1,\dots,k_n}$ of I such that every interval satisfies $|I_{i,n}| < \delta_n$, $I_{i,n} \subset I$, and by (3.6), $\overline{E}(u, I_{i,n} \cap I_{j,n})$ is arbitrarily small if $i \neq j$ (remark that the support of $|u_s'|$ has zero measure). So we can assume

$$\sum_{i=1}^{k_n} \overline{E}(u, I_{i,n}) - \overline{E}(u, I) \leq \frac{1}{n}. \tag{3.7}$$

The same argument allows us to suppose also that:

$$\begin{aligned}&\sum_{i=1}^{k_n} \int_{I_{i,n}} \widetilde{W}(t, \dot{u})\, dt + \int_{I_{i,n}} \widetilde{W}^\infty\Big(t, \frac{u_s'}{|u_s'|}\Big)|u_s'| \\ &\leq \int_I \widetilde{W}(t, \dot{u}(t))\, dt + \int_I \widetilde{W}^\infty\Big(t, \frac{u_s'}{|u_s'|}\Big)|u_s'| + \frac{1}{n}\end{aligned} \tag{3.8}$$

$$\sum_{i=1}^{k_n} |u'|(I_{i,n}) \leq c\,|u'|(I) \qquad c > 1. \tag{3.9}$$

Remark that, if $m_{i,n} = \inf_{I_{i,n}} a$ and $M_{i,n} = \sup_{I_{i,n}} a$, then we have $M_{i,n} \leq m_{i,n} + \varepsilon_n$ for every $i = 1, \ldots, k_n$.

We estimate $\overline{E}(u, I_{i,n})$:

$$\int_{I_{i,n}} W(\dot{u})\,dt + (M_{i,n} - \varepsilon_n)\,|u_s'|(I_{i,n}) \leq E(u, I_{i,n}) \leq \int_{I_{i,n}} W(\dot{u})\,dt + (m_{i,n} + \varepsilon_n)\,|u_s'|(I_{i,n}).$$

Hence, taking the relaxations and using Remark 2, we obtain

$$\begin{aligned} &\overline{E}(u, I_{i,n}) \geq \int_{I_{i,n}} \overline{W}_{i,n}^{-}(\dot{u})\,dt + \int_{I_{i,n}} (\overline{W}_{i,n}^{-})^{\infty}\left(\frac{u_s'}{|u_s'|}\right)|u_s'| \\ &\overline{E}(u, I_{i,n}) \leq \int_{I_{i,n}} \overline{W}_{i,n}^{+}(\dot{u})\,dt + \int_{I_{i,n}} (\overline{W}_{i,n}^{+})^{\infty}\left(\frac{u_s'}{|u_s'|}\right)|u_s'| \end{aligned} \tag{3.10}$$

where $\overline{W}_{i,n}^{-}(x) = (W(x) \wedge (M_{i,n} - \varepsilon_n)\,|x|)^{**}$ and $\overline{W}_{i,n}^{+}(x) = (W(x) \wedge (m_{i,n} + \varepsilon_n)\,|x|)^{**}$.

If we set:

$$\widetilde{W}_{i,n}^{-}(x) = (W(x) \wedge m_{i,n}\,|x|)^{**} \qquad \widetilde{W}_{i,n}^{+}(x) = (W(x) \wedge M_{i,n}\,|x|)^{**}$$

Lemma 1 below implies that

$$\overline{W}_{i,n}^{+}(x) \leq \widetilde{W}_{i,n}^{-}(x) + \varepsilon_n\,|x| \qquad \overline{W}_{i,n}^{-}(x) \geq \widetilde{W}_{i,n}^{+}(x) - \varepsilon_n\,|x|\,.$$

These inequalities can be extented to the recession functions:

$$(\overline{W}_{i,n}^{+})^{\infty}(z) \leq (\widetilde{W}_{i,n}^{-})^{\infty}(z) + \varepsilon_n\,|z| \qquad (\overline{W}_{i,n}^{-})^{\infty}(z) \geq (\widetilde{W}_{i,n}^{+})^{\infty}(z) - \varepsilon_n\,|z|\,.$$

Therefore, recalling that $|\dot{u}|\,dt + |u_s'| = |u'|$, the inequalities (3.10) become

$$\overline{E}(u, I_{i,n}) \geq \int_{I_{i,n}} \widetilde{W}_{i,n}^{+}(\dot{u})\,dt + \int_{I_{i,n}} (\widetilde{W}_{i,n}^{+})^{\infty}\left(\frac{u_s'}{|u_s'|}\right)|u_s'| - \varepsilon_n\,|u'|\,(I_{i,n})$$

$$\overline{E}(u, I_{i,n}) \leq \int_{I_{i,n}} \widetilde{W}_{i,n}^{-}(\dot{u})\,dt + \int_{I_{i,n}} (\widetilde{W}_{i,n}^{-})^{\infty}\left(\frac{u_s'}{|u_s'|}\right)|u_s'| + \varepsilon_n\,|u'|\,(I_{i,n})\,.$$

Finally, using (3.7), (3.8), the fact that $\widetilde{W}_{i,n}^{-}(x) \leq \widetilde{W}(t, x) \leq \widetilde{W}_{i,n}^{+}(x)$, $\forall t \in I_{i,n}$, and summing on i, we obtain that for every n

$$\left|\overline{E}(u, I) - \left(\int_I \widetilde{W}(t, \dot{u}(t))\,dt + \int_I \widetilde{W}^{\infty}\left(t, \frac{u_s'}{|u_s'|}\right)|u_s'|\right)\right| \leq \varepsilon_n c\,|u'|\,(I) + \frac{1}{n}\,.$$

Taking the limit as $n \to +\infty$, we conclude the proof.

Example 2. For example if $W(x) = x^2$, then

$$(x^2 \wedge a(t)\,|x|)^{**} = \begin{cases} x^2 & \text{if } |x| \leq \frac{1}{2}\,a(t) \\ a(t)\,x - \dfrac{(a(t))^2}{4} & \text{if } x > \frac{1}{2}\,a(t) \\ -a(t)\,x - \dfrac{(a(t))^2}{4} & \text{if } x < -\frac{1}{2}\,a(t). \end{cases}$$

Lemma 1. *Let $W: \mathbf{R} \to [0, +\infty[$ be a convex function such that $W(0) = 0$. Then for every $b > 0$, $\varepsilon > 0$, we have*

$$(W(x) \wedge (b+\varepsilon)\,|x|)^{**} \leq (W(x) \wedge b\,|x|)^{**} + \varepsilon\,|x|\,.$$

Proof. Let us recall that for every convex function $V: \mathbf{R} \to \mathbf{R}$ such that $V(0) = 0$ we have

$$V(x) = \int_0^x V'(t)\,dt = \int_0^x D^+ V(t)\,dt \tag{3.11}$$

where $D^+ V$ denotes the *right-hand derivative* of V, which is defined everywhere in $\mathbf{R}$ by the convexity of V. Indeed V is absolutely continuous and V' exists and coincides with $D^+ V$ almost everywhere in $\mathbf{R}$.

It is enough to prove the inequality for $x \geq 0$. Let us set

$$x_b = \inf\{x \geq 0 : D^+ W(x) \geq b\} \qquad x_{b+\varepsilon} = \inf\{x \geq 0 : D^+ W(x) \geq b + \varepsilon\}\,.$$

By the convexity of W the function $D^+ W$ increases on $\mathbf{R}$, hence $x_b \leq x_{b+\varepsilon}$.

The functions $(W(x) \wedge (b+\varepsilon)\,|x|)^{**}$ and $(W(x) \wedge b\,|x|)^{**} + \varepsilon|x|$ are convex and take the value 0 at $x = 0$. Therefore, by (3.11) it is enough to show that

$$D^+((W(x) \wedge (b+\varepsilon)\,|x|)^{**}) \leq D^+((W(x) \wedge b\,|x|)^{**} + \varepsilon\,|x|)\,. \tag{3.12}$$

Since $$D^+((W(x) \wedge (b+\varepsilon)\,|x|)^{**}) = \begin{cases} D^+ W(x) & \text{if } x < x_{b+\varepsilon} \\ b+\varepsilon & \text{if } x \geq x_{b+\varepsilon} \end{cases}$$

and $$D^+((W(x) \wedge b\,|x|)^{**} + \varepsilon\,|x|) = \begin{cases} D^+ W(x) + \varepsilon & \text{if } x < x_b \\ b+\varepsilon & \text{if } x \geq x_b \end{cases}$$

the inequality (3.12) is proved.

4 - Lower semicontinuity results in SBV

Let us consider the functionals G_S, $H_S: BV(I) \to [0, +\infty]$ defined by

$$G_S(u) = \begin{cases} \int_I W(\dot u)\,dt + \#(S_u) + \sum_{t \in S_u} |u(t+) - u(t-)| & \text{if } u \in SBV(I) \\ +\infty & \text{elsewhere on } BV(I) \end{cases}$$

$$H_S(u) = \begin{cases} \int_I W(\dot u)\,dt + \#(S_u) + \sum_{t \in S_u} a(t)\,|u(t+) - u(t-)| & \text{if } u \in SBV(I) \\ +\infty & \text{elsewhere on } BV(I) \end{cases}$$

where the function $a: \bar I \to \boldsymbol{R}$ is continuous and strictly positive, as in Section 3.

In order to study the lower semicontinuity of G_S and H_S with respect to the topology of $L^1(I)$, we shall need the following result about lower semicontinuity properties of functionals defined on $SBV(I)$. We say that a function $\varphi: \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ is *subadditive* if we have

$$\varphi(x, y) \le \varphi(x, z) + \varphi(z, y) \qquad \text{for all } x, y, z \in \boldsymbol{R}.$$

Proposition 4. *Let $\{u_h\}_h$ be a sequence in $SBV(I)$ such that $u_h \to u$ in $L^1(I)$, $\|u_h\|_{BV(I)} \le c$ and $\int_I W(\dot u_h)\,dt + \#(S_{u_h}) \le c$, where $W: \boldsymbol{R} \to [0, +\infty[$ is a convex function superlinear at $\pm\infty$. Then $u \in SBV(I)$, $\dot u_h \rightharpoonup \dot u$ weakly in $L^1(I)$, $(u_h)'_s = Ju_h \to u'_s = Ju$ in the weak* topology of measures, and*

$$\int_I W(\dot u)\,dt \le \liminf_{h \to \infty} \int_I W(\dot u_h)\,dt \qquad \#(S_u) \le \liminf_{h \to \infty} \#(S_{u_h}).$$

Moreover, if $\phi: I \times \boldsymbol{R} \times \boldsymbol{R} \to [0, +\infty[$ is lower semicontinuous on $I \times \boldsymbol{R} \times \boldsymbol{R}$ and subadditive in the last two variables, then

$$\sum_{t \in S_u} \phi(t, u(t+), u(t-)) \le \liminf_{h \to \infty} \sum_{t \in S_{u_h}} \phi(t, u_h(t+), u_h(t-)). \tag{4.1}$$

Proof. For the first part of the proposition see [1], Proposition 4.2. Let us prove (4.1). We can assume that the right-hand side is finite, otherwise the result is trivial. Let $\{u_k\}_k$ be a subsequence of $\{u_h\}_h$ such that

$$\lim_{k \to +\infty} \sum_{t \in S_{u_k}} \phi(t, u_k(t+), u_k(t-)) = \liminf_{h \to \infty} \sum_{t \in S_{u_h}} \phi(t, u_h(t+), u_h(t-)) < +\infty. \tag{4.2}$$

Moreover, since $\#(S_{u_h}) \le c$, we can assume $\#(S_{u_k}) = N \in \boldsymbol{N}$ (N independent of k), which means that $S_{u_k} = \{t_i^k : i = 1, \dots, N\}$ with $0 < t_1^k < \dots < t_N^k < 1$. Then

$$(u_k)'_s = \sum_{i=1}^{N} (u_k(t_i^k +) - u_k(t_i^k -))\, \delta_{t_i^k}\,.$$

Without loss of generality, we can suppose that $t_i^k \to t_i \in [0, 1]$ for every $i = 1, \dots, N$. If for some $i \in \{1, \dots, N\}$ we have $t_i^k \to 0$, then the sequence $(u_k(t_i^k +) - u_k(t_i^k -))\, \delta_{t_i^k}$ does not give any contribution to the limit measure u'. The same is true if $t_i^k \to 1$. If instead $\bar{t} \in S_u$, then there exist sequences $\{t_l^k\}_k$, $\{t_{l+1}^k\}_k, \dots, \{t_m^k\}_k$, $l \le m$, converging to $\bar{t}$ and we have

$$u(\bar{t} +) - u(\bar{t} -) = \lim_{k \to +\infty} \sum_{i=l}^{m} (u_k(t_i^k +) - u_k(t_i^k -))\,.$$

We can also suppose that $u_k(t_i^k +) \to a_i^+$, $u_k(t_i^k -) \to a_i^-$ for every $i = l, \dots, m$. Then $a_m^+ = u(\bar{t} +)$, $a_l^- = u(\bar{t} -)$, and $a_i^+ = a_{i+1}^-$ for $i = l, \dots, m-1$, so that

$$u(\bar{t} +) - u(\bar{t} -) = \sum_{i=l}^{m} (a_i^+ - a_i^-)\,.$$

By the subadditivity of ϕ in the last two variables, we have

$$\phi(\bar{t}, u(\bar{t} +), u(\bar{t} -)) \le \sum_{i=l}^{m} \phi(\bar{t}, a_i^+, a_i^-)\,.$$

Moreover, by the lower semicontinuity of ϕ, we obtain

$$\phi(\bar{t}, a_i^+, a_i^-) \le \liminf_{k \to \infty} \phi(t_i^k, u_k(t_i^k +), u_k(t_i^k -)) \qquad \forall i = l, \dots, m$$

which implies that for every $\bar{t} \in S_u$

$$\phi(\bar{t}, u(\bar{t} +), u(\bar{t} -)) \le \liminf_{k \to \infty} \sum_{i=l}^{m} \phi(t_i^k, u_k(t_i^k +), u_k(t_i^k -))\,.$$

Finally, summing on $t \in S_u$ and recalling (4.2), we conclude that

$$\sum_{t \in S_u} \phi(t, u(t +), u(t -)) \le \liminf_{h \to \infty} \sum_{t \in S_{u_h}} \phi(t, u_h(t +), u_h(t -))\,.$$

Remark 3. If $a : I \to [0, +\infty[$ is a lower semicontinuous function, then every function ϕ of the form $\phi(t, x, y) = a(t)\,|x - y|$ satisfies the hypotheses of Proposition 4.

Theorem 4. *Let us assume that the volume energy density W is superlinear at $\pm\infty$. Then the functionals G_S and H_S are lower semicontinuous with respect to the L^1 convergence.*

Proof. Let us prove that the functional G_S is lower semicontinuous (the proof for H_S is exactly the same, recalling that $\min_I a > 0$). Let us consider a sequence $\{u_h\}_h$ in $BV(I)$ such that $u_h \to u$ in $L^1(I)$. We have to show that $G_S(u) \leq \liminf_{h\to\infty} G_S(u_h)$.

Let us assume that the right-hand side is finite, otherwise the result is trivial. Then $u_h \in SBV(I)$ and $G_S(u_h) \leq c$. Using the superlinearity of W it is easy to see that the sequence $\{u_h\}_h$ is bounded in $BV(I)$, hence we can apply Proposition 4 to conclude the proof (see also Remark 3).

5 - Minimum problems

In this section we study some minimum problems corresponding to the functionals G, G_S, H, and H_S introduced in Sections 3 and 4. Without loss of generality we will take $I =]0, 1[$. As a model case we will consider $W(x) = x^2$ and $a(t) = (t - \frac{1}{2})^2 + 1$.

We can describe the behaviour of our functionals by examining some minimum problems with generalized Dirichlet boundary data.

Let us fix $\alpha \in \boldsymbol{R}$ and consider the boundary conditions $u(0) = 0$ and $u(1) = \alpha$. It is well known that these conditions are not well-posed for problems in $BV(I)$ (see [4]). We have to relax these conditions by penalizing jumps at $t = 0, 1$. Then we introduce the functionals

$$G^\alpha(u) = \int_0^1 |\dot{u}|^2 \mathrm{d}t + \#(S_{u_*}) + |(u_*)'_s|([0, 1])$$

$$G^\alpha_S(u) = \begin{cases} \int_0^1 |\dot{u}|^2 \mathrm{d}t + \#(S_{u_*}) + \sum_{t\in S_{u_*}} |u_*(t+) - u_*(t-)| & \text{if } u \in SBV(]0, 1[) \\ +\infty & \text{elsewhere on } BV(]0, 1[) \end{cases}$$

where the function $u_* \in BV_{\mathrm{loc}}(\boldsymbol{R})$ is obtained by extending u to 0 in $]-\infty, 0]$ and to α in $[1, +\infty[$.

In order to study the minimization problems associated to our functionals we will need the following proposition, describing all the minimum points of the functional

$$L^{a}(u) = \int_0^1 \overline{W}(\dot{u})\,dt + |(u_*)'_s|([0,1]) \tag{5.1}$$

which corresponds to the functional L defined in (3.1) in the particular case $W(x) = x^2$ (the function $\overline{W}$ is given by (3.2)). All the minimum values and the minimum points of the problem

$$\min\{L^{a}(u) : u \in BV(]0,1[)\}$$

are described in [6], Proposition 4.2.

Theorem 5. *Let us consider the minimization problem*

$$m_G^a = \inf\{G^a(u) : u \in BV(]0,1[)\}.$$

Then we have

i. *if* $|a| \leq \frac{1}{2}$, *then* $m_G^a = a^2$ *and the unique minimum point is* $u(t) = at$

ii. *if* $a > \frac{1}{2}$, *then* $m_G^a = a - \frac{1}{4}$ *and the minimum points are all* $u \in BV(]0,1[)$ *such that* $\#(S_{u_*}) = 0$, $u' \in \mathcal{M}_+(]0,1[)$, $\dot{u} = \frac{1}{2}$ *almost everywhere;*

iii. *if* $a < -\frac{1}{2}$, *then* $m_G^a = |a| - \frac{1}{4}$ *and the minimum points are all* $u \in BV(]0,1[)$ *such that* $\#(S_{u_*}) = 0$, $-u' \in \mathcal{M}_+(]0,1[)$, $\dot{u} = -\frac{1}{2}$ *almost everywhere.*

Proof. If is easy to see that the relaxation of the functional G^a is given by L^a. Therefore the minimum value of the functional L^a coincides with the infimum of G^a and a function u is a minimum point of G^a if and only if u is a minimum point of L^a such that $G^a(u) = L^a(u)$. Using Proposition 4.2 of [6] and computing G^a on the minimizers of L^a we obtain the assertion of the theorem.

Let us consider the *minimum problem*

$$\begin{aligned} m_{G_S}^a &= \min\{G_S^a(u) : u \in BV(]0,1[)\} \\ &= \min\Big\{\int_0^1 |\dot{u}|^2\,dt + \#(S_{u_*}) + \sum_{t \in S_{u_*}} |u_*(t+) - u_*(t-)| : u \in SBV(]0,1[)\Big\}. \end{aligned} \tag{5.2}$$

Using Theorem 4 it is easy to see that for every a such a minimum value exists. Indeed let us consider the case $a > 0$ (the opposite case being analogous) and let $\{u_h\}_h$ be a minimizing sequence for G_S^a. We have just remarked in the proof of Theorem 4 that $\{u_h\}_h$ is bounded in $BV(]0, 1[)$, hence, up to a subsequence, we can suppose that $u_h \to u$ in $L^1(]0, 1[)$. By the lower semicontinuity of G_s^a we conclude that u is a minimum point.

Theorem 6. *We have:*

i. *if* $|a| < \frac{3}{2}$, *then* $m_{G_S}^a = a^2$ *and the unique minimum point is* $u(t) = at$

ii. *if* $a = \frac{3}{2}$, *then* $m_{G_S}^a = \frac{9}{4}$ *and the minimum points are* $u(t) = \frac{3}{2} t$ *and all* $u \in SBV(]0, 1[)$ *such that on* $[0, 1]$ *we have* $(u_*)' = \frac{1}{2}\,dt + \delta_{t_0}$ *for* $t_0 \in [0, 1]$

iii. *if* $a > \frac{3}{2}$, *then* $m_{G_S}^a = a + \frac{3}{4}$ *and the minimum points are all* $u \in SBV(]0, 1[)$ *such that on* $[0, 1]$ *we have* $(u_*)' = \frac{1}{2}\,dt + (a - \frac{1}{2})\,\delta_{t_0}$ *for* $t_0 \in [0, 1]$

iv. *if* $a = -\frac{3}{2}$, *then* $m_{G_S}^a = \frac{9}{4}$ *and the minimum points are* $u(t) = -\frac{3}{2} t$ *and all* $u \in SBV(]0, 1[)$ *such that on* $[0, 1]$ *we have* $(u_*)' = -\frac{1}{2}\,dt - \delta_{t_0}$ *for* $t_0 \in [0, 1]$

v. *if* $a < -\frac{3}{2}$, *then* $m_{G_S}^a = |a| + \frac{3}{4}$ *and the minimum points are all* $u \in SBV(]0, 1[)$ *such that on* $[0, 1]$ *we have* $(u_*)' = -\frac{1}{2}\,dt + (a + \frac{1}{2})\,\delta_{t_0}$ *for* $t_0 \in [0, 1]$.

Proof. It is not restrictive to suppose $a > 0$. Note also that the assertion of the theorem is trivial if $0 < a \leqslant \frac{1}{2}$, since $G_S^a \geqslant G^a$.

If $\frac{1}{2} < a \leqslant 1$ then a minimum point u satisfies $\#(S_{u_*}) = 0$, hence $u \in W^{1,2}(]0, 1[)$, $u(0+) = 0$, $u(1-) = a$, and it is well known that the minimum of $\int_0^1 |u'|^2 dt$ on $W^{1,2}(]0, 1[)$ with boundary conditions $u(0) = 0$, $u(1) = a$ is achieved on the affine function $u(t) = at$.

It remains to consider the case $a > 1$. First of all we can prove that, if u is a minimum point and $S_{u_*} \neq \emptyset$, then $\#(S_{u_*}) = 1$. Indeed if $\#(S_{u_*}) = k > 1$ and

$$(u_*)' = \dot{u}_*\,dt + \sum_{i=1}^{k} [u_*(t_i +) - u_*(t_i -)]\,\delta_{t_i}, \qquad \text{with} \qquad t_i \in [0, 1]$$

then the function $v \in SBV(]0, 1[)$ defined by the conditions $v(0+) = u(0+)$ and $(v_*)' = \dot{u}_*\,dt + (\sum_{i=1}^{k} [u_*(t_i +) - u_*(t_i -)])\,\delta_{t_1}$ is such that $G_S^a(v) < G_S^a(u)$ and this gives that u is not a minimizer.

Let us fix a minimum point u such that $\#(S_{u_*}) = 1$. We can prove that $\dot u$ is constant. Indeed we have that $(u_*)' = \dot u_*\, dt + (\alpha - \int_0^1 \dot u\, dt)\,\delta_{t_0}$ for some $t_0 \in [0, 1]$.

If we consider the function $v \in SBV(]0, 1[)$ defined by $v(0+) = u(0+)$ and $(v_*)' = (\int_0^1 \dot u\, dt)\, dt + (\alpha - \int_0^1 \dot u\, dt)\,\delta_{t_0}$ on $[0, 1]$, by Jensen's inequality we obtain that $G_S^a(v) < G_S^a(u)$ unless $\dot u$ is constant.

Therefore we have $(u_*)' = c\, dt + (\alpha - c)\,\delta_{t_0}$ on $[0, 1]$, for a constant $c \in \boldsymbol{R}$. We claim that $0 \leqslant c < \alpha$. If $c < 0$ it suffices to consider $v \in SBV(]0, 1[)$ such that $v(0+) = u(0+)$ and $(v_*)' = |c|\, dt + (\alpha - |c|)\,\delta_{t_0}$, while if $c > \alpha$ we can take $v = u \wedge \alpha$.

On such a function u we can compute $G_S^a(u) = c^2 - c + \alpha + 1$ and the minimum value on $[0, \alpha[$ is achieved at $c = \frac{1}{2}$.

To sum up, if for every $t_0 \in [0, 1]$ the function u_{t_0} is the function satisfying $(u_*)' = \frac{1}{2}\, dt + (\alpha - \frac{1}{2})\,\delta_{t_0}$ on $[0, 1]$, then $G_S^a(u_{t_0}) = \alpha + \frac{3}{4}$, while $G_S^a(\alpha t) = \alpha^2$. By comparing $\alpha + \frac{3}{4}$ with α^2 we obtain the assertion of the theorem.

Let us introduce now the functionals

$$H^\alpha(u) = \int_0^1 |\dot u|^2 dt + \#(S_{u_*}) + \int_0^1 ((t - \tfrac{1}{2})^2 + 1)\, |u_s'| + |u(0+)| + |\alpha - u(1-)|$$

$$H_S^a(u) = \int_0^1 |\dot u|^2 dt + \#(S_{u_*}) + \sum_{t \in S_{u_*}} ((t - \tfrac{1}{2})^2 + 1)\, |u_*(t+) - u_*(t-)|$$

if $u \in SBV(]0, 1[)$, while $H^\alpha(u)$ and $H_s^\alpha(u)$ are sets equal to $+\infty$ if $u \in BV(]0, 1[) \setminus SBV(]0, 1[)$.

Theorem 7. *Let us consider the minimization problem*

$$m_H^a = \inf\{H^\alpha(u) : u \in BV(]0, 1[)\}.$$

We have:

i. *if* $|\alpha| \leqslant \frac{1}{2}$, *then* $m_H^a = \alpha^2$ *and the unique minimum point is* $u(t) = \alpha t$

ii. *if* $|\alpha| > \frac{1}{2}$, *then* $m_H^a = |\alpha| - \frac{1}{4}$, *but the infimum is not achieved.*

Proof. As $a(t) \geqslant 1$ on $[0, 1]$, we have that $H^\alpha \geqslant G^\alpha$, hence assertion **i** is trivial.

Let us prove **ii.** As above, it is not restrictive to assume $\alpha > 0$.

It is clear that $m_H^\alpha \geqslant \alpha - \frac{1}{4}$. First let us construct a sequence $\{u_h\}_h$ such that $H^\alpha(u_h) \to (\alpha - \frac{1}{4})$. For every $h \in N$, $h > 2$, we set

$$u_h(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \leqslant t < \frac{1}{2} - \frac{1}{h}, \\ f_C^h(t) & \text{if } \frac{1}{2} - \frac{1}{h} \leqslant t \leqslant \frac{1}{2} + \frac{1}{h}, \\ \frac{1}{2}t + (\alpha - \frac{1}{2}) & \text{if } \frac{1}{2} + \frac{1}{h} < t \leqslant 1, \end{cases}$$

where f_C^h is the Cantor function increasing by $\alpha - \frac{1}{2} + \frac{1}{h}$ on the interval $[\frac{1}{2} - \frac{1}{h}, \frac{1}{2} + \frac{1}{h}]$. Then

$$H^\alpha(u_h) = \frac{1}{2}(\frac{1}{2} - \frac{1}{h}) + \int_{\frac{1}{2}-\frac{1}{h}}^{\frac{1}{2}+\frac{1}{h}} ((t - \frac{1}{2})^2 + 1)\, |Df_C^h|$$

$$\leqslant \frac{1}{4} - \frac{1}{2h} + (1 + \frac{1}{h^2})(\alpha - \frac{1}{2} + \frac{1}{h}) = \alpha - \frac{1}{4} + O(\frac{1}{h}).$$

This implies that $H^\alpha(u_h) \to \alpha - \frac{1}{4}$ for $h \to +\infty$, hence $m_H^\alpha = \alpha - \frac{1}{4}$.

In order to conclude the proof it is enough to show that $H^\alpha(u) > \alpha - \frac{1}{4}$ for every $u \in BV(I)$. By contradiction, if $H^\alpha(u) = \alpha - \frac{1}{4}$, then u is a minimizer of G^α. Using Theorem 5 and comparing $G^\alpha(u)$ and $H^\alpha(u)$, we obtain $\#(S_{u_*}) = 0$, $\dot{u} = \frac{1}{2}$, and

$$\int_0^1 (t - \frac{1}{2})^2\, |Cu| = 0.$$

By the properties of the Cantor part of the derivative of a BV function (Sec. 2), we conclude that $Cu = 0$. Hence u_* is continuous and $u' = \frac{1}{2}$ on $]0, 1[$, which is impossible since $\alpha > \frac{1}{2}$.

Let us consider the *minimum problem*

$$m_{H_S}^\alpha = \min\{H_S^\alpha(u) : u \in BV(]0, 1[)\}$$

$$= \min\{\int_0^1 |\dot{u}|^2 dt + \#(S_{u_*}) + \sum_{t \in S_{u_*}} (1 + (t - \frac{1}{2})^2)\, |u_*(t+) - u_*(t-)| : u \in SBV(]0, 1[)\}.$$

As we have done for (5.2), using Theorem 4, we can prove that for every α such a minimum value exists.

Theorem 8. *We have*

i. *if* $|\alpha| < \frac{3}{2}$, *then* $m^{\alpha}_{H_S} = \alpha^2$ *and the unique minimum point is* $u(t) = \alpha t$

ii. *if* $\alpha = \frac{3}{2}$, *then* $m^{\alpha}_{H_S} = \frac{9}{4}$ *and the minimum points are* $u(t) = \frac{3}{2}\, t$ *and the function* $u \in SBV(]0, 1[)$ *such that on* $[0, 1]$ *we have* $(u_*)' = \frac{1}{2}\,\mathrm{d}t + \delta_{\frac{1}{2}}$

iii. *if* $\alpha > \frac{3}{2}$, *then* $m^{\alpha}_{H_S} = \alpha + \frac{3}{4}$ *and the minimum point is the function* $u \in SBV(]0, 1[)$ *such that on* $[0, 1]$ *we have* $(u_*)' = \frac{1}{2}\,\mathrm{d}t + (\alpha - \frac{1}{2})\,\delta_{\frac{1}{2}}$

iv. *if* $\alpha = -\frac{3}{2}$, *then* $m^{\alpha}_{H_S} = \frac{9}{4}$ *and the minimum points are* $u(t) = -\frac{3}{2}\, t$ *and the function* $u \in SBV(]0, 1[)$ *such that we have* $(u_*)' = -\frac{1}{2}\,\mathrm{d}t - \delta_{\frac{1}{2}}$ *on* $[0, 1]$

v. *if* $\alpha < -\frac{3}{2}$, *then* $m^{\alpha}_{H_S} = |\alpha| + \frac{3}{4}$ *and the minimum point is the function* $u \in SBV(]0, 1[)$ *such that on* $[0, 1]$ *we have* $(u_*)' = -\frac{1}{2}\,\mathrm{d}t + (\alpha + \frac{1}{2})\,\delta_{\frac{1}{2}}$.

Proof. The theorem can be easily deduced from Theorem 6, comparing G^g_S and H^g_S, and remarking that a minimum point u of H^g_S cannot jump in t_0 if $t_0 \neq \frac{1}{2}$.

Remark 4. By considering the functional G^{α} for $|\alpha| > \frac{1}{2}$ we obtain examples of minimization problems where the minimum values on $BV(I)$ and $SBV(I)$ exist and are different. On the other hand, by considering the functional H^{α} for $|\alpha| > \frac{1}{2}$ we obtain examples of minimization problems which can be solved on $SBV(I)$ but not on $BV(I)$.

References

[1] L. Ambrosio, *A compactness theorem for a special class of functions of bounded variation*, Boll. Un. Mat. Ital. **3-B** (1989), 857-881.

[2] L. Ambrosio, *Variational problems in SBV*, Acta Appl. Math. **17** (1989), 1-40.

[3] L. Ambrosio, *Existence theory for a new class of variational problem*, Arch. Rational Mech. Anal. **111** (1990), 291-322.

[4] G. Anzellotti e M. Giaquinta, *Funzioni BV e tracce*, Rend. Sem. Mat. Univ. Padova **60** (1978), 1-21.

[5] G. Bouchitté and G. Buttazzo, *Integral representation of nonconvex functionals defined on measures*, Ann. Inst. H. Poincaré *Anal. Non Linéaire*, to appear.

[6] A. BRAIDES and A. COSCIA, *A singular perturbation approach to variational problems in fracture mechanics*, Math. Mod. Meth. Appl. Sci. 3 (1993), 303-340.

[7] G. BUTTAZZO, *Semicontinuity, relaxation and integral representation in the calculus of variations*, Pitman Res. Notes Math. Ser. 207, Longman, Harlow 1989.

[8] G. BUTTAZZO and G. DAL MASO, *Integral representation and relaxation of local functionals*, Nonlinear Anal. 9 (1985), 515-532.

[9] G. DAL MASO, *An introduction to Γ-convergence*, Birkhäuser, Basel 1993.

[10] E. DE GIORGI, *Free discontinuity problems in calculus of variations*, Analyse mathématique et applications, Gauthier-Villars, Paris 1988.

[11] E. DE GIORGI e L. AMBROSIO, *Un nuovo tipo di funzionale del calcolo delle variazioni*, Atti Accad. Naz. Lincei Rend. 82 (1988), 199-210.

[12] E. DE GIORGI et G. LETTA, *Une notion générale de convergence faible pour des fonctions croissantes d'ensemble*, Ann. Scuola Norm. Sup. Pisa 4 (1977), 61-69.

[13] H. FEDERER, *Geometric measure theory*, Springer, Berlin 1969.

[14] E. GIUSTI, *Minimal surfaces and functions of bounded variation*, Birkhäuser, Basel 1983.

[15] C. GOFFMAN and J. SERRIN, *Sublinear functions of measures and variational integrals*, Duke Math. J. 31 (1964), 159-178.

[16] A. I. VOL'PERT, *The space BV and quasilinear equations*, Math. USSR Sb. 2 (1967), 225-267.

[17] W. P. ZIEMER, *Weakly differentiable functions*, Springer, Berlin 1990.

Sommario

Su un intervallo aperto e limitato $I \subset \mathbf{R}$ *si considerano funzionali della forma* (1.1), *che sono legati alla formulazione variazionale di molti problemi in fisica matematica, ricostruzione del linguaggio e ingegneria meccanica. Nell'espressione del funzionale* $\#$ *denota la misura che conta i punti su* $\mathbf{R}$, I *è la configurazione di riferimento, la funzione* u *rappresenta lo spostamento, che è differenziabile al di fuori dell'insieme di «discontinuità»* S, *e* $u(t+)$, $u(t-)$ *sono i limiti destro e sinitro di* u *nel punto* t. *Le funzioni* W *e* Θ *rappresentano le densità di energia rispettivamente di volume e di superficie. Si considerano energie di superficie* Θ *della forma* (1.2), *dipendenti anche dalla posizione dei punti di salto, per una generica funzione continua* $a: \bar{I} \to]0, +\infty[$.

Si studiano le proprietà di semicontinuità inferiore dei funzionali corrispondenti e alcuni problemi di minimo con condizioni ai limiti di tipo Dirichlet.

* * *