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Quaternionic space forms and geodesic spheres and tubes (**)

1 - Introduction

In a previous article [1] we started with the study of the Ricci-semi-symmetry condition $\widetilde{R}_{XY}\cdot\widetilde{\varrho}=0$ for geodesic spheres and tubes in a Riemannian manifold. In view of the strong similarities between the intrinsic and extrinsic geometrical properties of geodesic spheres and tubes, determined respectively by the Ricci tensor $\widetilde{\varrho}$ and the second fundamental form σ (see [2], [4], [13]), also the semi-parallelism condition $\widetilde{R}_{XY}\cdot\sigma=0$ was investigated.

We proved that in a real space form the small geodesic spheres and tubes satisfy these two properties and that each one of them is sufficient for a connected Riemannian manifold to be of constant sectional curvature.

Next it was shown in [5] that these conditions can be used to characterize complex space forms in the sense that for a connected Kähler manifold of dimension $n\geqslant 4$ a necessary and sufficient condition to be of constant holomorphic sectional curvature is that all its small geodesic spheres satisfy $\widetilde{R}_{XY}\cdot\widetilde{\varrho}=0$ or $\widetilde{R}_{XY}\cdot\sigma=0$ for the so-called *horizontal* tangent vectors X, Y to the spheres. An analogous theorem is established for geodesic tubes by taking horizontal vectors only at *special* points.

In this paper quaternionic space forms are considered. First of all we sort out which class of tangent vectors X, Y, Z, W to the geodesic sphere or tube makes $(\widetilde{R}_{XY}\cdot\widetilde{\varrho})_{ZW}$ and $(\widetilde{R}_{XY}\cdot\sigma)_{ZW}$ vanish. This leads to an adapted notion of horizontal tangent vectors and special points. It turns out that in the case of geodesic spheres the tangent vectors X, Y need to be horizontal, whereas for geodesic

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tubes one has to restrict to horizontal vectors X, Y, Z in special points. Subsequently it is proved that the conditions obtained are sufficient for a quaternionic Kähler manifold of dimension $n \geqslant 8$ to be of constant Q-sectional curvature.

2 - Preliminaries

Let (M, g) be an n-dimensional, connected, smooth Riemannian manifold. Denote by ∇ the Levi Civita connection and by R and ϱ the corresponding Riemann curvature tensor and Ricci tensor, respectively. We use the sign convention

$$R_{XY} = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y]$$

for tangent vector fields X, Y on M.

Now, suppose that (M, g) is a *quaternionic Kähler manifold* [**9**], that is, there exists a three-dimensional bundle V of tensors of type $(1, 1)$ over M such that locally the bundle V has a basis of almost Hermitian structures $\{J_0, J_1, J_2\}$ satisfying

$$J_i J_j = -J_j J_i = J_k \tag{1}$$

$$\begin{aligned} \nabla_X J_0 &= r(X) J_1 - q(X) J_2 \\ \nabla_X J_1 &= -r(X) J_0 + p(X) J_2 \\ \nabla_X J_2 &= q(X) J_0 - p(X) J_1 \end{aligned} \tag{2}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and p, q, r are local one-forms. Such a basis is called *adapted*. It follows that $\dim M = n = 4m$. As is well-known, for $n \geqslant 8$, M is an Einstein manifold [**9**]. Let $X \in T_p M$ and denote by $Q(X)$ the four-dimensional subspace spanned by X, $J_0 X$, $J_1 X$, $J_2 X$, called the *Q-section* determined by X. If for any $Y, Z \in Q(X)$ the sectional curvature $K(Y, Z)$ is a constant $c(X, p)$, then it is called the *Q-sectional curvature with respect to X at p*. If this is also independent of X, then it is a global constant and in this case (M, g) is called a *space of constant Q-sectional curvature* or a *quaternionic space form*. Further, a quaternionic Kähler manifold of dimension $n \geqslant 8$ is of constant Q-sectional curvature c if and only if the curvature tensor has the form

$$\begin{aligned} R_{XY} Z = {} & \frac{c}{4} \{g(X, Z) Y - g(Y, Z) X\} \\ & + \frac{c}{4} \{\sum_{i=0}^{2} g(J_i X, Z) J_i Y - g(J_i Y, Z) J_i X + 2g(J_i X, Y) J_i Z)\} \end{aligned} \tag{3}$$

for any adapted basis $\{J_0, J_1, J_2\}$ of the tensor bundle V. For a proof, see [9], [14]. In the sequel we will need another characterization.

Proposition 1 [10]. *A quaternionic Kähler manifold of dimension $n \geqslant 8$ is a quaternionic space form if and only if $g(R_{XY}X, Z) = 0$ for all X, all $Y \in Q(X)$ and $Z \in Q(X)^{\perp}$, or equivalently, $R_{XJ_iXXZ} = 0$ $(i = 0, 1, 2)$ for all X and Z as above.*

Now, let m be a point in an arbitrary Riemannian manifold M and γ a geodesic parametrized by arc length r such that $\gamma(0) = m$. Put $u = \gamma'(0)$. Next, let $\{E_1, \ldots,, E_n\}$ be a parallel orthonormal frame field along γ with $E_1(0) = u$. Let $G_m(r)$ denote the *geodesic sphere* centered at m and with radius $r < i(m)$, the injectivity radius at m. For a point $p = \gamma(r) = \exp_m(ru) \in G_m(r)$ we have the following expansions for the curvature tensor $\widetilde{R}$, the Ricci-tensor $\widetilde{\varrho}$ and the second fundamental form σ of $G_m(r)$ with respect to $\{E_1, \ldots, E_n\}$:

$$\begin{aligned}\widetilde{R}_{abcd}(p) &= \frac{1}{r^2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \\ &\quad + \{R_{abcd} - \frac{1}{3}(R_{ubud}\delta_{ac} + R_{uauc}\delta_{bd} - R_{ubuc}\delta_{ad} - R_{uaud}\delta_{bc})\}(m) + O(r),\end{aligned} \tag{4}$$

$$\begin{aligned}\widetilde{\varrho}_{ab}(p) &= \frac{n-2}{r^2}\delta_{ab} + (\varrho_{ab} - \frac{1}{3}\varrho_{uu}\delta_{ab} - \frac{n}{3}R_{uaub})(m) \\ &\quad + r(\nabla_u\varrho_{ab} - \frac{1}{4}\nabla_u\varrho_{uu}\delta_{ab} - \frac{n+1}{4}\nabla_u R_{uaub})(m) \\ &\quad + r^2(\frac{1}{2}\nabla^2_{uu}\varrho_{ab} - \frac{1}{10}\nabla^2_{uu}\varrho_{uu}\delta_{ab} - \frac{n+2}{10}\nabla^2_{uu}R_{uaub} \\ &\quad + \frac{1}{9}R_{uaub}\varrho_{uu} - \frac{1}{45}\sum_{\lambda,\mu=2}^{n}R^2_{u\lambda u\mu}\delta_{ab} - \frac{n+2}{45}\sum_{\lambda=2}^{n}R_{uau\lambda}R_{ubu\lambda})(m) + O(r^3),\end{aligned} \tag{5}$$

$$\sigma_{ab}(p) = \frac{1}{r}\delta_{ab} - \frac{r}{3}R_{uaub}(m) + O(r^2) \tag{6}$$

for $a, b, c, d = 2, \ldots, n$, where $R_{abcd} = g(R_{E_aE_b}E_c, E_d)$ and similarly for the other tensors. We refer to [2], [6], [7], [12] for more details.

It is easy to see that along a geodesic γ one can choose an adapted basis $\{J_0, J_1, J_2\}$ of parallel tensor fields along γ, that is, $\nabla_{\gamma'} J_i = 0$. So, in a quaternionic Kähler manifold we make a more specific choice for the frame field $\{E_i;\, i = 1, \ldots, n\}$, taking as initial conditions $E_{i+2}(0) = J_i u$ $(i = 0, 1, 2)$. Hence, by virtue of the parallelism, $E_{i+2} = J_i\gamma' = J_i E_1$. Then the technique of Jacobi vector fields makes possible to write down complete formulas for the second fundamental form of a geodesic sphere [11], [12]

$$\sigma = \lambda g + \mu \sum_{i=0}^{2} \eta_i \otimes \eta_i . \tag{7}$$

This together with (3) and the Gauss equation yields an expression for the curvature tensor, which by contraction results in

$$\tilde{\varrho} = \bar{\lambda} g + \bar{\mu} \sum_{i=0}^{2} \eta_i \otimes \eta_i \tag{8}$$

where g denotes the induced metric,

$$\lambda = \frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}}{2} r \qquad \mu + \lambda = \sqrt{c} \cot \sqrt{c}\, r$$

$$\bar{\lambda} = (n+7)\frac{c}{4} + (n-2)\lambda^2 + 3\mu\lambda \qquad \bar{\mu} = -\frac{3c}{4} + (n-3)\mu\lambda + 2\mu^2$$

for $c > 0$ and $\eta_i(X) = g(X, J_i \gamma') = g(X, E_{i+2})$. When $c < 0$ one has to replace the trigonometric functions by their corresponding hyperbolic functions and the formulas for $c = 0$ are obtained by taking the limit as $c \to 0$.

Now, we wil consider *geodesic tubes*, that is, tubes about a geodesic curve. We refer to [4], [6], [8], [12], [13] for more details.

Let $\sigma: [a, b] \to M$ be a smooth embedded geodesic curve and let P_r denote the tube of radius r about σ, where r is supposed to be smaller than the distance from σ to its nearest focal point. In that case, P_r is a hypersurface of M.

Let σ be parametrized by the arc length and denote by $\{e_1, e_2, \ldots, e_n\}$ an orthonormal basis of $T_{\sigma(a)}M$ such that $e_1 = \dot{\sigma}(a)$. Further, let $E_1, \ldots, E_n$ be the vector fields alongs σ obtained by parallel translation of $e_1, \ldots, e_n$. Then $E_1 = \dot{\sigma}$ and $\{E_1, \ldots, E_n\}$ is a parallel orthonormal frame field along the geodesic σ.

Next, let $p \in P_r$ and denote by γ the geodesic through p which cuts σ orthogonally at $m = \sigma(t)$. We parametrize γ by arc length r such that $\gamma(0) = m$ and take $(E_2, \ldots, E_n)$ such that $E_2(t) = \gamma'(0) = u$. Finally, let $\{F_1, \ldots, F_n\}$ be the orthonormal frame field along γ obtained by parallel translation of $\{E_1(t), \ldots, E_n(t)\}$ along γ.

For the hypersurface P_r one has the following expansions with respect to this parallel frame field [4], [13]:

$$\begin{aligned}\tilde{R}_{1abc}(p) &= (R_{1abc} - \frac{1}{2} R_{1ubu}\delta_{ac} + \frac{1}{2} R_{1ucu}\delta_{ab})(m)\\ &+r(\nabla_u R_{1abc} - \frac{1}{3}\nabla_u R_{1ubu}\delta_{ac} + \frac{1}{3}\nabla_u R_{1ucu}\delta_{ab})(m)\\ &+r^2(\frac{1}{2}\nabla^2_{uu}R_{1abc} + \frac{1}{6}R_{1ubu}R_{aucu} - \frac{1}{6}R_{1ucu}R_{aubu})(m)\\ &-\frac{r^2}{8}(\nabla^2_{uu}R_{1ubu}\delta_{ac} - \nabla^2_{uu}R_{1ucu}\delta_{ab} + R_{1u1u}R_{1ubu}\delta_{ac} - R_{1u1u}R_{1ucu}\delta_{ab})(m)\\ &-\frac{r^2}{24}(\sum_{\lambda=3}^{n} R_{1u\lambda u}R_{bu\lambda u}\delta_{ac} - \sum_{\lambda=3}^{n} R_{1u\lambda u}R_{cu\lambda u}\delta_{ab})(m) + O(r^3)\end{aligned} \tag{9}$$

$$\begin{aligned}\tilde{R}_{abcd}(p) &= \frac{1}{r^2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + R_{abcd}(m)\\ &-\frac{1}{3}(R_{budu}\delta_{ac} - R_{bucu}\delta_{ad} + R_{aucu}\delta_{bd} - R_{audu}\delta_{bc})(m) + O(r)\end{aligned} \tag{10}$$

$$\tilde{\varrho}_{11}(p) = \varrho_{11}(m) - (n-1)R_{1u1u}(m) + O(r) \tag{11}$$

$$\begin{aligned}\tilde{\varrho}_{1a}(p) &= \varrho_{1a}(m) - \frac{n-1}{2}R_{1uau}(m) + r(\nabla_u\varrho_{1a} - \frac{n}{3}\nabla_u R_{1uau})(m)\\ &+r^2(\frac{1}{2}\nabla^2_{uu}\varrho_{1a} - \frac{n+1}{8}\nabla^2_{uu}R_{1uau} + \frac{1}{6}\varrho_{uu}R_{1uau})(m)\\ &-\frac{r^2}{24}((3n-5)R_{1u1u}R_{1uau} + (n+1)\sum_{\lambda=3}^{n}R_{1u\lambda u}R_{au\lambda u})(m) + O(r^3)\end{aligned} \tag{12}$$

$$\begin{aligned}\tilde{\varrho}_{ab}(p) &= \frac{n-3}{r^2}\delta_{ab} + (\varrho_{ab} - \frac{n-1}{3}R_{aubu})(m)\\ &-\frac{1}{3}(\varrho_{uu}\delta_{ab} + 2R_{1u1u}\delta_{ab})(m) + O(r)\end{aligned} \tag{13}$$

$$\sigma_{11}(p) = O(r) \tag{14}$$

$$\sigma_{1a}(p) = -\frac{r}{2}R_{1uau}(m) + O(r^2) \tag{15}$$

$$\sigma_{ab}(p) = \frac{1}{r}\delta_{ab} + O(r) \tag{16}$$

for $a, b, c, d \in \{3, 4, \dots, n\}$.

Next, suppose that (M, g, V) is a quaternionic Kähler manifold. Then a point $p = \exp_m(ru)$ on the geodesic tube will be called a *special point* when $u = J\dot{\sigma}(t)$, where J is a tensor of the three-dimensional tensor bundle V in the point m. So, $u = (a\bar{J}_0 + b\bar{J}_1 + c\bar{J}_2)(m)(\dot{\sigma}(t))$ for some adapted basis $\{\bar{J}_0, \bar{J}_1, \bar{J}_2\}$ and $a, b, c \in \boldsymbol{R}$ such that $a^2 + b^2 + c^2 = 1$.

As it is easy to see we can choose an adapted basis $\{J_0, J_1, J_2\}$ of parallel tensor fields along the geodesic γ such that $J_0(m) = (a\bar{J}_0 + b\bar{J}_1 + c\bar{J}_2)(m)$. Therefore, without loss of generality, a special point p can be obtained by taking $u = J_0\dot{\sigma}(t)$ for $\{J_0, J_1, J_2\}$ as above and it suffices to determine the second fundamental form with these assumptions.

Straightforward computations using the technique of Jacobi vector fields (see [3], [12]) give then an explicit expression for the second fundamental form *at these special points*

$$\sigma = \lambda g + \sum_{i=0}^{2} \nu_i \eta_i \otimes \eta_i . \tag{17}$$

Together with (3) and the Gauss equation we obtain

$$\tilde{\varrho} = \bar{\lambda} g + \sum_{i=0}^{2} \bar{\nu}_i \eta_i \otimes \eta_i \tag{18}$$

where g denotes the induced metric,

$$\nu_0 + \lambda = -\sqrt{c}\tan\sqrt{c}\,r \qquad \nu_1 + \lambda = \nu_2 + \lambda = \sqrt{c}\cot\sqrt{c}\,r$$

$$\lambda = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}\,r \qquad \bar{\lambda} = (n+7)\frac{c}{4} + (n-2)\lambda^2 + \lambda\sum_{i=0}^{2}\nu_i$$

$$\bar{\nu}_i = -\frac{3c}{4} + (n-3)\nu_i\lambda - \nu_i(\nu_i - \sum_{k=0}^{2}\nu_k)$$

for $c > 0$ and $\eta_i(X) = g(X, J_i\gamma')$. When $c < 0$ one has to replace the trigonometric functions by their corresponding hyperbolic functions and the case $c = 0$ can be obtained by taking the limit as $c \to 0$.

Finally, a tangent vector X at a point p of a geodesic sphere $G_m(r)$ or a geodesic tube P_r is called *horizontal* (with respect to this sphere or tube) if X is orthogonal to $Q(\gamma'(r))$ or equivalently if $\eta_i(X) = 0$ for $i = 0, 1, 2$.

3 - Geodesic spheres

First, we prove

Theorem 1. *Let (M^n, g, V), $n \geqslant 8$, be a quaternionic space form. Then for all small geodesic spheres in M it holds that*

$$\tilde{R}_{XY}\cdot\sigma = 0 = \tilde{R}_{XY}\cdot\tilde{\varrho}$$

for all horizontal tangent vectors X, Y to these spheres.

Proof. From (7) it is easy to see that

$$-(\widetilde{R}_{XY}\cdot\sigma)(W, W) = 2\mu\sum_{i=0}^{2}\eta_i(\widetilde{R}_{XY}W)\eta_i(W).$$

But, $\eta_i(\widetilde{R}_{XY}W) = -\widetilde{R}_{XYJ_i\gamma' W} = -\frac{c}{2}\sum_{k=0}^{2} g(J_kX, Y)g(J_kJ_i\gamma', W)$, where we used the Gauss equation together with (3) and the horizontality of X, Y. Switching the summation indices yields

$$(\widetilde{R}_{XY}\cdot\sigma)(W, W) = \mu c\sum_{k=0}^{2} g(J_kX, Y)\{\sum_{i=0}^{2}\eta_i(W)g(J_kJ_i\gamma', W)\}$$

in which the term between brackets vanishes. This proves the first result since $\widetilde{R}_{XY}\cdot\sigma$ is symmetric.

In the same way from (8) it follows that $\widetilde{R}_{XY}\cdot\widetilde{\varrho} = 0$.

Next, we prove the converse.

Theorem 2. *Let (M^n, g, V), $n \geqslant 8$, be a quaternionic Kähler manifold such that all its small geodesic spheres satisfy one of the conditions*

$$\widetilde{R}_{XY}\cdot\sigma = 0 \quad or \quad \widetilde{R}_{XY}\cdot\widetilde{\varrho} = 0$$

for all horizontal tangent vectors X, Y to these spheres. Then, (M, g, V) is a quaternionic space form.

Proof. For a point $p = \exp_m(ru)$ on a small geodesic sphere $G_m(r)$ we use the notations introduced in Section 2. In terms of the frame field $\{E_i;\ i = 1, \ldots, n\}$ along the geodesic ray γ between m and p, the space of horizontal tangent vectors to $G_m(r)$ at p is spanned by $\{E_5(r), \ldots, E_n(r)\}$.

By means of (4) and (6) we can compute the power series expansion of $(\widetilde{R}_{ab}\cdot\sigma)_{cd} = 0$ for $a, b = 5, \ldots, n$ and $c, d = 2, \ldots, n$. Considering the coefficient of r^{-1} we are led to

$$-\delta_{ac}R_{dubu} + \delta_{bc}R_{duau} - \delta_{ad}R_{cubu} + \delta_{bd}R_{cuau} = 0.$$

Taking $a = d \neq b$ and $c = i + 2$ for $i = 0, 1, 2$ (that is, c represents J_iu) yields $R_{uJ_iuub} = 0$. Since b stands for an arbitrary tangent vector at m, orthogonal to u, J_0u, J_1u, J_2u, the result follows from Proposition 1.

For the second case, we use (4), (5) and consider the coefficient of r^{-2} in the expansion of $(\widetilde{R}_{ab}\cdot\widetilde{\varrho})_{cd}=0$. This leads to a condition in which we take $b=d\neq a$ and $c=i+2$ for $i=0, 1, 2$. This yields $\varrho_{aJ_iu}=\frac{n}{3}R_{auJ_iuu}$. Since M is an Einstein space, it follows that $R_{uJ_iuua}=0$ for $i=0, 1, 2$, where a represents a vector of $Q(u)^{\perp}$. Again, Proposition 1 finishes the proof.

Note that the conditions in Theorem 2 may be replaced by the weaker conditions $(\widetilde{R}_{XY}\cdot\sigma)_{ZW}=0$ and $(\widetilde{R}_{XY}\cdot\widetilde{\varrho})_{ZW}=0$, where X, Y and Z are horizontal and W arbitrary.

4 - Geodesic tubes

We have

Theorem 3. *Let (M^n, g, V), $n\geq 8$, be a quaternionic space form. Then for all small geodesic tubes in M it holds that*

$$(\widetilde{R}_{XY}\cdot\sigma)_{ZW}=0=(\widetilde{R}_{XY}\cdot\widetilde{\varrho})_{ZW}$$

for all horizontal tangent vectors X, Y, Z and every tangent vector W to these tubes at the special points.

Proof. From (17) it follows that at the special points we have

$$-(\widetilde{R}_{XY}\cdot\sigma)(Z, W)=\sum_{i=0}^{2}\nu_i\{\eta_i(\widetilde{R}_{XY}Z)\eta_i(W)+\eta_i(\widetilde{R}_{XY}W)\eta_i(Z)\}.$$

As in the proof of Theorem 1, using the formula for $\eta_i(\widetilde{R}_{XY}W)$ we obtain for horizontal vectors X, Y

$$(\widetilde{R}_{XY}\cdot\sigma)(Z, W)=\frac{c}{2}\sum_{i=0}^{2}\nu_i\eta_i(W)\sum_{k=0}^{2}g(J_kX, Y)g(J_kJ_i\gamma', Z)$$
$$+\frac{c}{2}\sum_{i=0}^{2}\nu_i\eta_i(Z)\sum_{k=0}^{2}g(J_kX, Y)g(J_kJ_i\gamma', W)\}.$$

Taking Z horizontal obviously yields $(\widetilde{R}_{XY}\cdot\sigma)_{ZW}=0$.
In the same way from (18) it follows that $(\widetilde{R}_{XY}\cdot\widetilde{\varrho})_{ZW}=0$.

Finally, we consider the converse.

Theorem 4. *Let (M^n, g, V), $n \geqslant 8$, be a quaternionic Kähler manifold such that all its small geodesic tubes satisfy one of the conditions*

$$(\widetilde{R}_{XY} \cdot \sigma)_{ZW} = 0 \qquad or \qquad (\widetilde{R}_{XY} \cdot \widetilde{\varrho})_{ZW} = 0$$

for all horizontal tangent vectors X, Y, Z and every tangent vector W to these tubes at arbitrary special points. Then, (M, g, V) is a quaternionic space form.

Proof. For an arbitrary point m on M and an arbitrary unit tangent vector u to M in m we choose a geodesic σ through $m = \sigma(t)$ such that $u = J_0 \dot{\sigma}(t)$. This vector u determines a special point $p = \exp_m(ru)$ on the geodesic tube P_r about the axial curve σ. In terms of the notation introduced in Section 2, this means that $F_2 = J_0 F_1$. Additionally we can choose F_3, F_4 such that $F_3 = J_1 F_1$ and $F_4 = J_2 F_1$. Then the space of horizontal tangent vectors to P_r at p is spanned by $\{F_5, \dots, F_n\}$.

So, the first condition gives $(\widetilde{R}_{ab} \cdot \sigma)_{c1} = 0$ with $a, b, c = 5, \dots, n$. Calculating the power series expansion of this expression and considering the coefficient of r^{-1} yields $R_{1cab} = 0$. Next, we take $b = c = x$ and $a = J_k x$. Since $F_1(0) = -J_0 u$, this yields $R_{J_0 ux J_k xx} = 0$, where x represents a tangent vector at m, orthogonal to u and $J_i u (i = 0, 1, 2)$. We may replace u by $J_0 u$. Then it follows $R_{ux J_k xx} = 0$ for all x and all $u \in Q(x)^{\perp}$, which is what we need in view of Proposition 1.

For the Ricci-condition we can calculate the power series expansion of $(\widetilde{R}_{ab} \cdot \widetilde{\varrho})_{a1} = 0$ for $a, b = 5, \dots, n$. Considering the coefficient of r^{-2} we get $\varrho_{1b} = R_{1ubu} + (n-3) R_{1aba}$. Since M is an Einstein manifold and through the special choice of the point p, it follows that $R_{bu J_0 uu} + (n-3) R_{J_0 uaba} = 0$. Taking $b = J_0 a$ gives

$$R_{J_0 au J_0 uu} + (n-3) R_{J_0 ua J_0 aa} = 0$$

for a orthogonal to u and $J_i u (i = 0, 1, 2)$. Switching a and u and subtracting the equations obtained, we have $R_{J_0 ua J_0 aa} = 0$. Again we may replace u by $J_0 u$, which results in $R_{ua J_0 aa} = 0$ for all a and all $u \in Q(a)^{\perp}$.

Finally, applying the same procedure for the special points p determined by $u = J_1 \dot{\sigma}(a)$ and $u = J_2 \dot{\sigma}(a)$, we get respectively $R_{ua J_1 aa} = 0$ and $R_{ua J_2 aa} = 0$ for the same choice of a and u. Then Proposition 1 finishes the proof.

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Sommario

Si dimostra che una varietà M a curvatura sezionale quaternionale costante (quaternionic space form), connessa e di dimensione almeno 8, può essere caratterizzata da una condizione di semisimmetria della forma $(\tilde{R}_{XY}\cdot\tilde{\varrho})_{ZW}=0$ *o da una condizione di semiparallelismo della forma* $(\tilde{R}_{XY}\cdot\sigma)_{ZW}=0$, *con W arbitrario ed X, Y, Z speciali.*

$\tilde{R}$, $\tilde{\varrho}$, σ *indicano rispettivamente il tensore di curvartura di Riemann, il tensore di Ricci e la seconda forma fondamentale di piccole sfere geodetiche o di tubi geodetici ed* $\tilde{R}_{XY}$ *opera come derivazione.*

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