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Asymptotic properties of solutions to semilinear damped equations for elastic beams (**)

1 - Introduction

In this work we want to study some questions related to the asymptotic behaviour of nonlinear equations arising in the theory of elastic rods. In a previous paper, Taboada and You [2] proved the existence of an inertial manifold (that is, a smooth invariant finite-dimensional manifold for the flow generated by the problem, which attracts exponentially the orbits) for a model problem proposed in the 50s by Woinowsky-Krieger [5] in order to describe the transversal vibrations of an extensible beam subject to an axial internal force.

The concept of inertial manifold is very important for two main reasons: one is that its existence gives a theoretical foundation to the series expansions of the solution in terms of eigenfunctions that is customarily performed by many physicians, and the other is that the so-called inertial system, that is, the equations set in the manifold, is a finite-dimensional system of ODEs which approximates the given system with an exponentially decreasing error.

Our study generalizes the results of Taboada and You in two different directions; in one direction we consider a more general nonlinear term which can be used in connection with the problem of finite deformations, and in the other we

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investigate the nonhomogeneous problem, i.e. the case when an external driving term is present, which was not studied by Taboada and You.

Our main results are the following: in the homogeneous *generally nonlinear* case (Sec. 2) we obtain a result of existence of a *flat* inertial manifold which includes the result of [2]; in the nonhomogeneous case we prove the existence (Sec. 3) of an *approximate inertial manifold* (see for instance [4]), that is, a finite-dimensional invariant manifold which has a neighbourhood attracting exponentially the orbits. The advantages for numerical analysis are in this way not lost, since an inertial manifold must be approximated when the inertial system is coded into a computer. In that Section we also prove the existence of a global attractor for the flow, and the exponential attractivity of the origin in the special case of a force vanishing as $t \to +\infty$.

More precisely, we consider the nonlinear hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} + \delta \frac{\partial u}{\partial t} - g\left(\int_0^1 u_x^2(\xi, t)\,\mathrm{d}\xi\right) \frac{\partial^2 u}{\partial x^2} = h(x) \tag{1.1}$$

in the Hilbert space $H = (L^2(0, 1), \langle \cdot, \cdot \rangle)$ with norm denoted by $|\cdot|$, where $\alpha, \delta > 0$, $h(x) \in L^2(0, 1)$ and $g: \boldsymbol{R} \to \boldsymbol{R}$ is a C^1 function whose primitive vanishing in 0 is denoted by G. The case $g(s) = \beta + s$ is studied in [2]. The following hypotheses on g will be made:

$$\liminf_{s \to +\infty} (2s\, g(s) - G(s)) \geq 0 \tag{1.2}$$

$$\forall k > 0,\ \exists \tilde{\delta} > 0 : s \geq 0 \Rightarrow G(s) \geq ks - \tilde{\delta}. \tag{1.3}$$

Equation (1.1) becomes an initial-boundary value problem when endowed with the following conditions for $x \in [0, 1]$, $t \geq 0$:

$$\begin{gathered} u(x, 0) = u_0(x) \qquad u_t(x, 0) = u_1(x) \\ u(0, t) = u_x(0, t) = u_{xx}(1, t) = u_{xxx}(1, t) = 0, \end{gathered} \tag{1.4}$$

this means that the left endpoint $x = 0$ is fixed at rest and that the right endpoint $x = 1$ is not subject to any transversal force and any bending torque.

The system (1.1)-(1.4) can be put in a different form as shown below [2]. We define a linear operator $A: D(A) \to H$ as $A\varphi = \dfrac{\mathrm{d}^4 \varphi}{\mathrm{d}x^4}$ and set

$$D(A) = \hat{H}^4(0, 1) = \{\varphi \in H^4(0, 1) \mid \varphi(0) = \varphi'(0) = \varphi''(1) = \varphi'''(1) = 0\}.$$

A is self-adjoint, positive definite, with dense domain in H, a compact resolvent

A^{-1}, and its spectrum is $\sigma(A) = \{\lambda_j\}_{j \in N}$, $\lambda_j = \mu_j^4$, where $\{\mu_j\}_{j \in N}$ are the increasing ordered positive roots of the equation

$$\cos\mu + \operatorname{sech}\mu = 0 \qquad \mu > 0 . \tag{1.5}$$

Every λ_j has multiplicity one; μ_j are distributed pair by pair in the intervals

$$((2i + \frac{1}{2})\pi, (2i + \frac{3}{2})\pi) \qquad i \in N .$$

As a consequence

$$A^{\frac{1}{2}}\varphi = -\frac{d^2\varphi}{dx^2} \qquad D(A^{\frac{1}{2}}) = \overline{D(A)} \quad \text{in } H^2(0, 1)$$
$$A^{\frac{1}{4}}\varphi = |\frac{d\varphi}{dx}| \qquad D(A^{\frac{1}{4}}) = \overline{D(A)} \quad \text{in } H^1(0, 1).$$

In this way equation (1.1) becomes

$$\frac{d^2u}{dt^2} + \alpha Au + \delta\frac{du}{dt} + g(|A^{\frac{1}{4}}u|^2)A^{\frac{1}{2}}u = h \qquad t \geq 0 . \tag{1.6}$$

We set $u(t) = u(\cdot, t)$, $V = D(A^{\frac{1}{2}})$ and $E = V \times H$; the norm $\|v\| = |A^{\frac{1}{2}}v|$ is equivalent to the norm of $H^2(0, 1)$.

We underline that equation (1.1) does not satisfy the spectral gap condition, that is

$$\lambda_{N+1} - \lambda_N > K(\lambda_{N+1}^{\frac{1}{2}} + \lambda_N^{\frac{1}{2}})$$

by which the existence of an inertial manifold cannot be obtained by the standard theory (see [3], p. 423).

2 - Inertial manifolds in the homogeneous case

2.1 - Existence and uniqueness

In order to prove existence and uniqueness for the problem, we use the procedure illustrated in [2]; that is, we set $\mathcal{A} : D(\mathcal{A}) = D(A) \times V \to E$ the linear operator

$$\mathcal{A} = \begin{pmatrix} 0 & I_V \\ -\alpha A & -\delta I_H \end{pmatrix}$$

and $f\colon D(\mathcal{A}) \to D(\mathcal{A})$ the nonlinear mapping

$$f(\varphi, \Psi) = \begin{pmatrix} 0 \\ g(|\varphi_x|^2)\varphi_{xx} \end{pmatrix} = \begin{pmatrix} 0 \\ -g(|A^{\frac{1}{4}}\varphi|^2)A^{\frac{1}{2}}\varphi \end{pmatrix}.$$

In this way we obtain the following system for the unknown $w = \begin{pmatrix} u \\ v \end{pmatrix}$

$$\text{(2.1)} \qquad \begin{aligned} &\frac{\mathrm{d}w}{\mathrm{d}t} = \mathcal{A}\, w(t) + f(w(t)) \qquad t \geq 0 \\ &w(0) = w_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in E\,. \end{aligned}$$

Theorem 1. *For every initial condition* $w_0 \in E$, *there is an instant* $\tau = \tau(w_0) > 0$ *such that the regular solution of* (2.1)

$$w(t) = W(t)\, w_0 + \int_0^t W(t-s) f(w(s))\, \mathrm{d}s \qquad 0 \leq t < \tau$$

exists and is unique on the interval $[0, \tau)$. *Moreover, if* $w_0 \in D(\mathcal{A})$, *then the solution in* $C([0, \tau); E)$ *is a strong solution for* (2.1).

The assertion follows from the following facts: f is a locally Lipschitz mapping and $\mathcal{A}$ generates a C_0-semigroup of contractions denoted by $\{W(t)\}_{t \geq 0}$. The method of proof is described in [1], p. 185-189.

We'll show later that problem (2.1) has a unique global solution for every initial condition.

2.2 - Uniform estimates

We suppose that for every initial condition in E the mild solution of (2.1) exists and is unique in $C([0, +\infty); E)$.

We set

$$S(t)\colon \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \mapsto \begin{pmatrix} u(t;\, u_0, u_1) \\ v(t;\, u_0, u_1) \end{pmatrix} \in E \qquad \forall t \geq 0$$

the nonlinear mapping that originates from the semigroup $\{S(t)\}_{t \geq 0}$, solution of (2.1).

Theorem 2. *For every initial condition in E, there exists an unique global* *solution in* $C([0, +\infty); E)$ *for the problem* (2.1). *Moreover the solution semigroup* $\{S(t)\}_{t \geq 0}$ *has an absorbing set* $\mathscr{B}_0$ *in* E.

Proof. Let $w_0 \in D(\mathscr{A})$. We suppose that the mild solution of (2.1) is a strong solution on $(0, t_{max})$. Let ε be a positive constant.

If we take the inner product of the homogeneous (1.6) with $2u_t + \varepsilon u$ in H, for $t \in (0, t_{max})$, we get

$$\begin{aligned} &\frac{d}{dt}\{|u_t|^2 + \alpha|A^{\frac{1}{2}}u|^2 + G(|A^{\frac{1}{4}}u|^2) + \varepsilon\langle u_t, u\rangle\} \\ &+\{(2\delta - \varepsilon)|u_t|^2 + \varepsilon\alpha|A^{\frac{1}{2}}u|^2 + \varepsilon\delta\langle u_t, u\rangle + \varepsilon|A^{\frac{1}{4}}u|^2 \cdot g(|A^{\frac{1}{4}}u|^2)\} = 0 \end{aligned} \tag{2.2}$$

where $\quad \frac{d}{dt} G(|A^{\frac{1}{4}}u|^2) = g(|A^{\frac{1}{4}}u|^2) \cdot \frac{d}{dt}|A^{\frac{1}{4}}u|^2 = g(|A^{\frac{1}{4}}u|^2) \cdot \langle A^{\frac{1}{2}}u, 2u_t\rangle.$

Consider now the expression

$$N(t) = (2\delta - \varepsilon)|u_t|^2 + \varepsilon\alpha|A^{\frac{1}{2}}u|^2 + \varepsilon\delta\langle u_t, u\rangle + \varepsilon g(|A^{\frac{1}{4}}u|^2) \cdot |A^{\frac{1}{4}}u|^2. \tag{2.3}$$

It is easy to show that, if we choose

$$\eta = \alpha\mu_1^4\delta^{-1} \qquad 0 < \varepsilon < \min(1, \alpha\mu_1^4, 4\delta(3 + \frac{\delta^2}{\alpha\mu_1^4})^{-1}) \tag{2.4}$$

where μ_1 is the least positive root of equation (1.5), then the following inequalities hold in $(0, t_{max})$:

$$\begin{aligned} N(t) &\geq \varepsilon[(\frac{2\delta - \varepsilon}{\varepsilon} - \frac{\delta - \frac{\varepsilon}{2}}{2\eta})|u_t|^2 + \frac{\alpha}{2}|A^{\frac{1}{2}}u|^2] \\ &+\varepsilon[(\frac{\alpha}{2}|A^{\frac{1}{2}}u|^2 - \frac{\eta}{2}(\delta - \frac{\varepsilon}{2})|u|^2) + \frac{\varepsilon}{2}\langle u_t, u\rangle + \varepsilon g(|A^{\frac{1}{4}}u|^2) \cdot |A^{\frac{1}{4}}u|^2] \\ &\geq \frac{\varepsilon}{2}(|u_t|^2 + \alpha|A^{\frac{1}{2}}u|^2 + \varepsilon\langle u_t, u\rangle + 2g(|A^{\frac{1}{4}}u|^2) \cdot |A^{\frac{1}{4}}u|^2). \end{aligned} \tag{2.5}$$

By (1.2), there exists $D > 0$ such that

$$2g(|A^{\frac{1}{4}}u|^2) \cdot |A^{\frac{1}{4}}u|^2 \geq G(|A^{\frac{1}{4}}u|^2) - D$$

and then, if we put

$$F(t) = |u_t|^2 + \alpha|A^{\frac{1}{2}}u|^2 + \varepsilon\langle u_t, u\rangle + G(|A^{\frac{1}{4}}u|^2) \tag{2.6}$$

from (2.2)-(2.6) we have

$$\frac{d}{dt}F(t)+\frac{\varepsilon}{2}F(t)-\frac{\varepsilon}{2}D-\varepsilon D\leqslant\frac{d}{dt}F(t)+N(t)-\varepsilon D\leqslant 0$$

that is
$$\frac{d}{dt}F(t)+\frac{\varepsilon}{2}F(t)\leqslant\frac{3}{2}\varepsilon D\qquad t\in(0,\,t_{\max}).$$

By the Gronwall lemma, in $t\in(0,\,t_{\max})$, we obtain

$$F(t)\leqslant e^{-\frac{\varepsilon}{2}t}F(0)+3D(1-e^{\frac{\varepsilon}{2}t})\leqslant e^{-\frac{\varepsilon}{2}t}F(0)+3D$$

where $\varepsilon>0$ satisfies (2.4).

This inequality holds for every initial condition in E, because of the denseness of $D(\mathscr{A})$ in E and of the continuous dependence of the mild solution on the initial data.

Moreover $F(t)\geqslant\frac{1}{2}\,[\,|u_t|^2+\alpha\,|A^{\frac{1}{2}}u|^2+2G(|A^{\frac{1}{4}}u|^2)]$ and for (1.3)

$$|u_t|^2+\alpha\,|A^{\frac{1}{4}}u|^2+2G(|A^{\frac{1}{4}}u|^2)$$

$$\geqslant|u_t|^2+\alpha\,|A^{\frac{1}{4}}u|^2+k\,|A^{\frac{1}{4}}u|^2-\widetilde{\delta}\geqslant|u_t|^2+\alpha\,|A^{\frac{1}{4}}u|^2-\widetilde{\delta}$$

then $$\frac{\min\{1,\,\alpha\}}{2}(|u_t|^2+|A^{\frac{1}{2}}u|^2)\leqslant\frac{1}{2}\,|u_t|^2+\frac{1}{2}\,\alpha\,|A^{\frac{1}{2}}u|^2\leqslant F(t)+\widetilde{\delta}$$

that is

$$\frac{\min\{1,\,\alpha\}}{2}\,\|S(t)\,w_0\|_E^2\leqslant e^{-\frac{\varepsilon}{2}t}F(0)+3D+\widetilde{\delta}\qquad\forall t\in[0,\,t_{\max}).\tag{2.7}$$

Inequality (2.7) implies that there exists an unique global solution for (2.1) on $[0,\,+\infty)$ (see [**1**], p. 185), and also that

$$\limsup_{t\to+\infty}\|S(t)\,w_0\|_E^2\leqslant\frac{6D+2\widetilde{\delta}}{\min\{1,\,\alpha\}}\,.\tag{2.8}$$

If we put

$$\varrho_0^2=\frac{6D+2\widetilde{\delta}}{\min\{1,\,\alpha\}}\tag{2.9}$$

then the ball $\mathscr{B}_0=\{e\in E\colon\|e\|<\sqrt{2}\,\varrho_0\}$ is an *absorbing set* for the solution semigroup $\{S(t)\}_{t\geqslant0}$.

2.3 - Global attractor and inertial manifold

First we recall some important results stated in [2]. We consider a decomposition of the semigroup $\{S(t)\}_{t \geq 0}$ as follows:

$$S(t) = T(t) + U(t) \qquad t \geq 0$$

where $\{T(t)\}_{t \geq 0}$ is the semigroup of linear operators generated by A.

Lemma 1. *The operator $T(t)$ is continuous, and for any bounded set $B \subseteq E$*

$$\lim_{t \to +\infty} \{\sup_{w_0 \in B} \|T(t) w_0\|_E\} = 0 .$$

Lemma 2. *Let $B \subseteq E$ be a bounded set. Then for any $w_0 \in D(\mathcal{A}) \cap B$ the mild solution of* (2.1) *satisfies*

$$\sup_{t \geq 0} |A^{\frac{3}{4}} u(t)| \leq K(B) < +\infty$$

where the constant $K(B)$ depends only on B.

The proof of these results is essentially the same as in [2].

Lemma 3. *The operators of the family $\{U(t)\}_{t \geq 0}$ are uniformly compact operators for t large; this amounts to say that for every bounded set $\mathcal{B}$ there exists $t_0 > 0$, which may depend on $\mathcal{B}$, such that $\bigcup_{t \geq t_0} S(t) \mathcal{B}$ is relatively compact in H* (see [3], p. 23).

Proof. Let

$$\widetilde{w}(t) = \begin{pmatrix} \widetilde{u}(t) \\ \widetilde{v}(t) \end{pmatrix} = U(t) w_0$$

be the mild solution of the problem

$$\frac{d\widetilde{w}}{dt} = \mathcal{A} \widetilde{w}(t) + f(w(t)) \qquad t \geq 0 \qquad \widetilde{w}(0) = 0 .$$

Then

$$\widetilde{u}_{tt} + \alpha A \widetilde{u} + \delta \widetilde{u}_t + g(|A^{\frac{1}{4}} u|^2) \cdot A^{\frac{1}{2}} u = 0 \qquad t \geq 0 , \qquad \widetilde{u}(0) = \widetilde{u}_t(0) = 0 .$$

Taking the inner product with $2A^{\frac{1}{2}}\,\tilde{u}_t + \varepsilon A^{\frac{1}{2}}\,\tilde{u}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\{\,|A^{\frac{1}{4}}\,\tilde{u}_t|^2 + \alpha\,|A^{\frac{3}{4}}\,\tilde{u}|^2 + \varepsilon\langle A^{\frac{1}{4}}\,\tilde{u}_t,\,A^{\frac{1}{4}}\,\tilde{u}\rangle\}$$

$$+\{(2\delta-\varepsilon)\,|A^{\frac{1}{4}}\,\tilde{u}_t|^2 + \varepsilon\alpha\,|A^{\frac{3}{4}}\,\tilde{u}|^2 + \varepsilon\delta\langle A^{\frac{1}{4}}\,\tilde{u}_t,\,A^{\frac{1}{4}}\,\tilde{u}\rangle\}$$

$$+g(|A^{\frac{1}{4}}\,u|^2)\cdot(2\langle A^{\frac{3}{4}}\,u,\,A^{\frac{1}{4}}\,\tilde{u}_t\rangle + \varepsilon\langle A^{\frac{1}{2}}\,u,\,A^{\frac{1}{2}}\,\tilde{u}\rangle) = 0\,.$$

From Lemma 2 and from the fact that g is a C^1 function and that $|A^{\frac{1}{4}}\,u|^2 \leqslant \frac{1}{\mu_1^{\frac{2}{1}}}\,|A^{\frac{1}{2}}\,u|^2$, where $|A^{\frac{1}{2}}\,u|^2$ is uniformly bounded for (2.8), arises that for every bounded set $B \subseteq E$ and for every initial condition in $B \cap D(\mathscr{A})$, there exist $K_1(B)$, $K_2(B) \geqslant 0$ such that

$$|g(|A^{\frac{1}{4}}\,u|^2)\cdot|A^{\frac{3}{4}}\,u|\,| \leqslant K_1(B) \quad \text{and} \quad |g(|A^{\frac{1}{4}}\,u|^2)\cdot|A^{\frac{1}{2}}\,u|\,| \leqslant K_2(B)\,.$$

Then

$$g(|A^{\frac{1}{4}}\,u|^2)(2\langle A^{\frac{3}{4}}\,u,\,A^{\frac{1}{4}}\,\tilde{u}_t\} + \varepsilon\langle A^{\frac{1}{2}}\,\tilde{u},\,A^{\frac{1}{2}}\,u\rangle)$$

$$\geqslant -\frac{|K_1(B)|^2}{\eta} - \varepsilon^2\,\frac{|K_2(B)|^2}{\eta} - \eta\,|A^{\frac{1}{4}}\,\tilde{u}_t\,|^2 - \varepsilon\eta\,|A^{\frac{1}{2}}\,\tilde{u}|^2$$

and for $t \geqslant 0$; ε, δ, η, $\gamma > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\{\,|A^{\frac{1}{4}}\,\tilde{u}_t\,|^2 + \alpha\,|A^{\frac{3}{4}}\,\tilde{u}|^2 + \varepsilon\langle A^{\frac{1}{4}}\,\tilde{u}_t,\,A^{\frac{1}{4}}\,\tilde{u}\rangle\}$$

$$+\varepsilon\{[\frac{\delta-\varepsilon}{\varepsilon} - \frac{\delta-\frac{\varepsilon}{2}}{2\gamma}]\,|A^{\frac{1}{4}}\,\tilde{u}_t|^2 + \frac{\alpha}{2}\,|A^{\frac{3}{4}}\,\tilde{u}|^2 + \frac{\delta}{2}\,\langle A^{\frac{1}{4}}\,\tilde{u}_t,\,A^{\frac{1}{4}}\,\tilde{u}\rangle\}$$

$$+\varepsilon[\frac{\alpha}{4}\,|A^{\frac{3}{4}}\,\tilde{u}|^2 - \frac{\gamma}{2}(\delta-\frac{\varepsilon}{2})|A^{\frac{1}{4}}\,\tilde{u}|^2] + \{(\delta-\eta)|A^{\frac{1}{4}}\,\tilde{u}_t|^2 + \varepsilon[\frac{\alpha}{4}\,|A^{\frac{3}{4}}\,\tilde{u}|^2 - \eta\,|A^{\frac{1}{2}}\,\tilde{u}|^2]\}$$

$$\leqslant \eta^{-1}(|K_1(B)|^2 + \varepsilon^2\,|K_2(B)|^2)\,.$$

If we choose

$$0 < \eta < \min\{\delta,\,\frac{\alpha}{4}\,\mu_1^{\frac{2}{1}}\} \quad \gamma = \frac{1}{2}\,\delta^{-1}\alpha\mu_1^{\frac{4}{1}} \quad 0 < \varepsilon \leqslant \min\{1,\,\alpha\mu_1^{\frac{4}{1}},\,\delta[\frac{3}{2} + \frac{\delta^2}{\alpha\mu_1^{\frac{4}{1}}}]^{-1}\}$$

then, for $t \geqslant 0$, we have

$$\frac{d}{dt}\{|A^{\frac{1}{4}}\tilde{u}_t|^2 + \alpha|A^{\frac{3}{4}}\tilde{u}|^2 + \varepsilon\langle A^{\frac{1}{4}}\tilde{u}_t, A^{\frac{1}{4}}\tilde{u}\rangle\}$$

$$+\frac{\varepsilon}{2}\{|A^{\frac{1}{4}}\tilde{u}_t|^2 + \alpha|A^{\frac{3}{4}}\tilde{u}|^2 + \varepsilon\langle A^{\frac{1}{4}}\tilde{u}_t, A^{\frac{1}{4}}\tilde{u}\rangle\} \leqslant \eta^{-1}[|K_1(B)|^2 + \varepsilon^2|K_2(B)|^2]$$

and by the Gronwall lemma

$$\frac{1}{2}|A^{\frac{1}{4}}\tilde{u}_t|^2 + \frac{\alpha}{2}|A^{\frac{3}{4}}\tilde{u}|^2 \leqslant 2\varepsilon^{-1}\eta^{-1}[|K_1(B)|^2 + \varepsilon^2|K_2(B)|^2].$$

By denseness of $D(\mathscr{A})$ in $D(A^{\frac{3}{4}}) \times D(A^{\frac{1}{4}})$ we have

$$\left\|\begin{pmatrix}\tilde{u}\\ \tilde{u}_t\end{pmatrix}\right\|^2_{D(A^{\frac{3}{4}})\times D(A^{\frac{1}{4}})} \leqslant K_3(B) \qquad \forall\begin{pmatrix}u_0\\ u_1\end{pmatrix} \in B.$$

For the Rellich theorem $D(A^{\frac{3}{4}}) \times D(A^{\frac{1}{4}})$ is compactly imbedded in E, and $\bigcup_{t>0} U(t)B$ is relatively compact in E.

Finally we can state that the attractor is the ω-limit set of $\mathscr{B}_0$.

Theorem 3. *The ω-limit set $\omega(\mathscr{B}_0) = \bigcap_{s\geqslant 0}\overline{\bigcup_{t\leqslant s} S(t)\mathscr{B}_0}$ is a compact and maximal global attractor in E for the semigroup $\{S(t)\}_{t\geqslant 0}$ associated with the initial value problem* (2.1).

Proof. See [3], Theorem I.1.1.

We now show the existence of an inertial manifold for the problem (2.1).

We denote by $\{w_j\}_{j\in N}$ the eigenfunctions of A, which constitute an orthonormal basis for H and for $V = D(A^{\frac{1}{2}})$. Let

$$H_m = \text{span}\{w_1, \dots, w_m\}$$

$P_m : H \to H_m$ an orthogonal projector, $Q_m = I_H - P_m$ so that

$$H = P_m H \oplus Q_m H = H_m \oplus Q_m H.$$

In $E = V \times H$ consider

$$\Phi_m = \begin{pmatrix}P_m & 0\\ 0 & P_m\end{pmatrix} : E \to H_m \times H_m \qquad \Psi_m = I_E - \Phi_m.$$

In this way we have $E = (H_m \times H_m) + \Psi_m E$.

Let
$$g_u(t) = \vartheta_{\varrho_0}(\|w\|) \quad g(|A^{\frac{1}{4}} u|^2)$$

where ϱ_0 is given by (2.9), and $\vartheta_{\varrho_0} : [0, +\infty[\to [0, 1]$ is a C^∞-mollifier, which vanishes outside some compact set in $\boldsymbol{R}$ and is equal to unity in the interval $[0, 1]$. As said before for $g(|A^{\frac{1}{4}} u|^2)$, g_u is uniformly bounded for any orbit of (2.1) and in $\mathscr{B}_0$ there exists a positive constant $d = d(\varrho_0, B)$ such that $|g_u(t)| \leqslant d$ for any $t \geqslant 0$. This procedure leads to the *prepared equation* ([3], p. 416)

$$\frac{d^2u}{dt^2} + \alpha A u + \delta \frac{du}{dt} + \vartheta_{\varrho_0}(\|w\|^2)\; g(|A^{\frac{1}{2}} u|^2)\cdot A^{\frac{1}{2}} u = 0 \qquad t \geqslant 0. \tag{2.10}$$

If we decompose u in $H = H_m + Q_m H$ as

$$u(t) = P_m u(t) + Q_m u(t) = p(t) + q(t)$$

equation (2.10) becomes

$$\begin{aligned} &\frac{d^2p}{dt^2} + \alpha A p + \delta \frac{dp}{dt} + g_u(t) A^{\frac{1}{2}} p = 0 && \text{in } H_m \\ &\frac{d^2q}{dt^2} + \alpha A q + \delta \frac{dq}{dt} + g_u(t) A^{\frac{1}{2}} q = 0 && \text{in } Q_m H. \end{aligned} \tag{2.11}$$

Theorem 4. *If m is a suitably large positive integer such that the $(m+1)$-th root of equation* (1.5) *satisfies*

$$\mu_{m+1}^2 \geqslant \max\{\frac{4d}{\alpha}, \frac{8C_2\varrho_0^2}{\varepsilon\alpha}, \frac{2(d+\varepsilon)}{\alpha}\}$$

where C_2 is a specific positive constant and

$$\varepsilon = \min\{\frac{1}{2}, 2\delta[3 + \frac{2\delta^2}{\alpha\mu_1^4}]^{-1}\}$$

then the flat manifold $M_m = H_m \times H_m$ is an inertial manifold in E for the semigroup $\{S(t)\}_{t\geqslant 0}$ related to (2.1).

Proof.

i. It is obvious that M_m is a finite-dimensional subspace of E

ii. M_m is positively invariant [2]

iii. We now prove the exponential attraction of M_m showing that all the elements of $\Psi_m E$ tend to zero with an exponential rate.

From now on we refer only to the absorbing set $\mathcal{B}_0$, because Theorem 2 shows that all the orbits eventually enter in $\mathcal{B}_0$.

Let $w_0 \in D(\mathcal{A})$. Taking the inner product of (2.11) with $2q_t + 2\varepsilon q$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[|q_t|^2 + \alpha|A^{\frac{1}{2}}q|^2 + 2\varepsilon\langle q_t, q\rangle + g_u(t)|A^{\frac{1}{4}}q|^2] + (2\delta - 2\varepsilon)|q_t|^2$$

$$+2\varepsilon\alpha|A^{\frac{1}{2}}q|^2 + 2\varepsilon\delta\langle q_t, q\rangle + 2\varepsilon g_u(t)\cdot|A^{\frac{1}{2}}q|^2 - (\frac{\mathrm{d}}{\mathrm{d}t}g_u(t))\cdot|A^{\frac{1}{4}}q|^2 = 0.$$

Moreover, in $\mathcal{B}_0$, $g_u(t) = g \in C^1$; therefore there exists a positive constant C_2 such that $|g'(|A^{\frac{1}{4}}u|^2)| \leqslant C_2$, and

$$|\frac{\mathrm{d}}{\mathrm{d}t}g_u(t)| = |g'(|A^{\frac{1}{4}}u|^2)|\cdot|\frac{\mathrm{d}}{\mathrm{d}t}|A^{\frac{1}{4}}u|^2| \leqslant 2\varrho_0^2 C_2.$$

Then $$\frac{\mathrm{d}}{\mathrm{d}t}[|q_t|^2 + \alpha|A^{\frac{1}{2}}q|^2 + 2\varepsilon\langle q_t, q\rangle + g(|A^{\frac{1}{4}}u|^2)\cdot|A^{\frac{1}{4}}q|^2]$$

$$+\{(2\delta - 2\varepsilon)|q_t|^2 + \frac{3}{2}\varepsilon\alpha|A^{\frac{1}{2}}q|^2 + 2\varepsilon\delta\langle q_t, q\rangle + \varepsilon g(|A^{\frac{1}{4}}u|^2)\cdot|A^{\frac{1}{4}}q|^2\}$$

$$+[\frac{\varepsilon\alpha}{2} - \frac{\varepsilon d}{\mu_{m+1}^2} - \frac{2C_2\varrho_0^2}{\mu_{m+1}^2}]|A^{\frac{1}{4}}u|^2 \leqslant 0.$$

Following the procedure of Theorem 2, if

$$\eta = \frac{\alpha\mu_1^4}{2\delta} \qquad \varepsilon = \min\{\frac{1}{2}, 2\delta[3 + \frac{2\delta^2}{\alpha\mu_1^4}]^{-1}\} \qquad \mu_{m+1}^2 \geqslant \max\{\frac{4d}{\alpha}, \frac{8C_2\varrho_0^2}{\varepsilon\alpha}\}$$

$$L(t) = |q_t|^2 + \alpha|A^{\frac{1}{2}}q|^2 + 2\varepsilon\langle q_t, q\} + g(|A^{\frac{1}{4}}u|^2)\cdot|A^{\frac{1}{4}}q|^2$$

then $\frac{\mathrm{d}}{\mathrm{d}t}L(t) + \varepsilon L(t) \leqslant 0$, $t \geqslant 0$ and for the Gronwall lemma $L(t) \leqslant L(0)e^{-\varepsilon t}$, $t \geqslant 0$.

But now if $\mu_{m+1}^2 \geqslant \frac{2(d+\varepsilon)}{\alpha}$, then $L(t) \geqslant \frac{1}{2}|q_t|^2 + \frac{\alpha}{2}|A^{\frac{1}{2}}q|^2$ and so

$$\frac{1}{2}[|q_t|^2 + \alpha|A^{\frac{1}{2}}q|^2] \leqslant L(0)e^{-\varepsilon t} \leqslant [2(1+\alpha) + d]\left\|\begin{pmatrix} q(0) \\ q_t(0)\end{pmatrix}\right\|_E^2 e^{-\varepsilon t}.$$

This inequality holds for every initial condition in E because $D(\mathscr{A})$ is dense in E. Then, if we put

$$\nu = \min\{\frac{1}{2}, \frac{1}{2}\alpha\mu_1^{\frac{1}{4}}, 2\delta[3+\frac{2\delta^2}{\alpha\mu_1^{\frac{1}{4}}}]^{-1}\} > 0$$

we find
$$\mathrm{dist}_E(S(t)\begin{pmatrix}u_0\\ u_1\end{pmatrix}, M_m) \leqslant \varrho(u_0, u_1)e^{-\nu t} \qquad t \geqslant 0$$

that is, a uniform exponential decay with constant ν.

3 - Approximate inertial manifold in the nonhomogeneous case

It is evident that the same results of existence and uniqueness hold also in this case.

3.1 - Uniform estimates

The hypotheses are the same of Section 2.2.

Theorem 5. *For every initial condition in E, there exists an unique global mild solution in $C([0, +\infty); E)$ for the problem* (2.1). *Moreover the solution semigroup $\{S(t)\}_{t\geqslant 0}$ has an absorbing set $\mathscr{B}_0$ in E. Obviously $\mathscr{B}_0$ depends on h.*

Proof. Let again $w_0 \in D(\mathscr{A})$ and get ε, γ_1, γ_2 positive constants as before. Taking the inner product of (1.6) with $2u_t + \varepsilon u$ and considering that we have

$$\langle h, 2u_t\rangle + \langle h, \varepsilon u\rangle \leqslant 2|h|\,|u_t| + \varepsilon|h|\,|u|$$

$$\leqslant \frac{1}{2}|h|^2(\frac{2}{\gamma_1^2}+\frac{\varepsilon}{\gamma_2^2}) + \frac{\varepsilon}{2}\gamma_2^2|u|^2 + \gamma_1^2|u_t|^2$$

for $t \in (0, t_{\max})$, then

$$\frac{1}{2}|h|^2(\frac{2}{\gamma_1^2}+\frac{\varepsilon}{\gamma_2^2}) + \frac{\varepsilon}{2}\gamma_2^2|u|^2 + \gamma_1^2|u_t|^2$$

$$\geqslant \frac{d}{dt}\{|u_t|^2 + \alpha|A^{\frac{1}{2}}u|^2 + G(|A^{\frac{1}{4}}u|^2) + \varepsilon\langle u_t, u\rangle\}$$

$$+\{(2\delta-\varepsilon)|u_t|^2 + \varepsilon\alpha|A^{\frac{1}{2}}u|^2 + \varepsilon\delta\langle u_t, u\rangle + \varepsilon|A^{\frac{1}{4}}u|^2\cdot g(|A^{\frac{1}{4}}u|^2)\}.$$

Adding and subtracting $2\gamma_2^2\langle u_t, u\rangle = -\frac{d}{dt}(-\gamma_2^2|u|^2)$, we have

$$\frac{1}{2}\left(\frac{\varepsilon}{\gamma_2^2}+\frac{2}{\gamma_1^2}\right)|h|^2 \geq \frac{d}{dt}\{|u_t|^2-\gamma_2^2|u|^2+\alpha|A^{\frac{1}{2}}u|^2+G(|A^{\frac{1}{4}}u|^2)+\varepsilon\langle u_t, u\rangle\}$$

$$+\{(2\delta-\varepsilon-\gamma_1^2)|u_t|^2+\varepsilon\alpha|A^{\frac{1}{2}}u|^2+(\varepsilon\delta+2\gamma_2^2)\langle u_t, u\rangle$$

$$-\frac{\varepsilon}{2}\gamma_2^2|u|^2+\varepsilon|A^{\frac{1}{4}}u|^2\cdot g(|A^{\frac{1}{4}}u|^2)\}.$$

Let

$$N(t) = (2\delta-\varepsilon-\gamma_1^2)|u_t|^2+\varepsilon\alpha|A^{\frac{1}{2}}u|^2+(\varepsilon\delta+2\gamma_2^2)\langle u_t, u\rangle$$

$$-\frac{\varepsilon}{2}\gamma_2^2|u|^2+\varepsilon g(|A^{\frac{1}{4}}u|^2)\cdot|A^{\frac{1}{4}}u|^2.$$

If we choose $\eta = \dfrac{\alpha\mu_1^4}{\delta+2}$, $\gamma_2 = \sqrt{\varepsilon}$, $\gamma_1 \in (0, \sqrt{2\delta})$

$$0<\varepsilon<\min\left(1, \frac{1}{2}\alpha\mu_1^4, 2(2\delta-\gamma_1^2)\left(3+\frac{(\delta+2)^2}{\alpha\mu_1^4}\right)^{-1}\right) \tag{3.1}$$

and put $F(t) = |u_t|^2-\gamma_2^2|u|^2+\alpha|A^{\frac{1}{2}}u|^2+G(|A^{\frac{1}{4}}u|^2)+\varepsilon\langle u_t, u\rangle$

we have, similarly to Theorem 2, that in $(0, t_{\max})$,

$$\frac{d}{dt}F(t)+\frac{\varepsilon}{2}F(t)-\frac{\varepsilon}{2}D-\varepsilon D \leq \frac{d}{dt}F(t)+N(t)-\varepsilon D \leq \frac{1}{2}\left(\frac{\varepsilon}{\gamma_2^2}+\frac{2}{\gamma_1^2}\right)|h|^2$$

that is $\dfrac{d}{dt}F(t)+\dfrac{\varepsilon}{2}F(t) \leq \dfrac{3}{2}\varepsilon D+\dfrac{1}{2}\left(\dfrac{\varepsilon}{\gamma_2^2}+\dfrac{2}{\gamma_1^2}\right)|h|^2.$

By the Gronwall lemma, for $t \in (0, t_{\max})$

$$F(t) \leq e^{-\frac{\varepsilon t}{2}}F(0)+3D+\frac{1}{\varepsilon}\left(\frac{\varepsilon}{\gamma_2^2}+\frac{2}{\gamma_1^2}\right)|h|^2 \tag{3.2}$$

where $\varepsilon > 0$ satisfies (3.1). Here also inequality (3.2) holds for every initial condition in E.

Proceeding again as in Theorem 2 we conclude, saying that there exists an unique global solution of (2.1) in $[0, +\infty)$ and that

$$\text{(3.3)} \quad \limsup_{t \to +\infty} \|S(t)\,w_0\|_E^2 \leq \frac{2}{\min\{1, \alpha\}} \cdot [(3D + \frac{1}{\varepsilon}(\frac{\varepsilon}{\gamma_2^2} + \frac{2}{\gamma_1^2})\,|h|^2) + \tilde{\delta}]\,.$$

Then the ball in E of radius $\sqrt{2}\,\varrho_0$, where

$$\varrho_0^2 = \frac{2}{\min\{1, \alpha\}} \cdot [(3D + \frac{1}{\varepsilon}(\frac{\varepsilon}{\gamma_2^2} + \frac{2}{\gamma_1^2})\,|h|^2) + \tilde{\delta}]$$

is an absorbing set for $\{S(t)\}_{t \geq 0}$.

3.2. - Global attractor and approximate inertial manifold

Theorem 6. *The ω-limit set $\omega(\mathcal{B}_0)$ is a compact and maximal global attractor in E for the semigroup $\{S(t)\}_{t \geq 0}$ associated with the initial value problem* (2.1).

The proof is very similar to the proof of Section 2.3.

Now we consider the solution v of the stationary problem

$$\text{(3.5)} \qquad \alpha A v + g(|A^{\frac{1}{4}}\,u|^2)\,A^{\frac{1}{2}}\,u = h \qquad t \geq 0\,.$$

Be $w = u - v$, so that $w_t = u_t$ and $w_{tt} = u_{tt}$. Taking the difference between (1.6) and (3.5) we have te homogeneous equation

$$\frac{d^2 w}{dt^2} + \alpha A w + \delta\,\frac{dw}{dt} - g(|A^{\frac{1}{4}}\,u|^2)\,A^{\frac{1}{2}}\,u + g(|A^{\frac{1}{4}}\,v|^2)\,A^{\frac{1}{2}}\,v = 0$$

that is

$$\frac{d^2 w}{dt^2} + \alpha A w + \delta\,\frac{dw}{dt} - \tilde{g}(t)\,A^{\frac{1}{2}}\,w + g(|A^{\frac{1}{4}}\,u|^2)\,A^{\frac{1}{2}}\,v - g(|A^{\frac{1}{4}}\,v|^2)\,A^{\frac{1}{2}}\,u = 0\,.$$

where $\tilde{g}(t) = g(|A^{\frac{1}{4}}\,u|^2) + g(|A^{\frac{1}{4}}\,v|^2)$.

We proceed with an orthogonal decomposition of the space H and E as in Section 2.3. Now $g_u(t) = \vartheta_{\varrho_0}\tilde{g}$ and in $\mathcal{B}_0$ we have $d = C + \tilde{C}$. In fact, since $|A^{\frac{1}{4}}\,u|^2$ is uniformly bounded by ϱ_0, then there exists C such for $g \in C^1$ in $\mathcal{B}_0$ it holds $|g(|A^{\frac{1}{4}}\,u|^2)| \leq C$. Moreover v is continuous on the bounded sets of $D(A^{\frac{1}{2}})$, and so there exist ϱ_1, $\tilde{C}$ such that $|A^{\frac{1}{2}}\,v|^2 \leq \varrho_1^2$ and $|g(|A^{\frac{1}{4}}\,v|^2)| \leq \tilde{C}$.

If we put $q = Q_m w(t)$, $r = Q_m u(t)$, $s = Q_m v$, the projection of the *prepared* *equation* on $Q_m H$ becomes

$$\frac{d^2 q}{dt^2} + \alpha A q + \delta \frac{dq}{dt} + g_u(t) A^{\frac{1}{2}} q + g(|A^{\frac{1}{4}} u|^2) A^{\frac{1}{2}} s - g(|A^{\frac{1}{4}} v|^2) A^{\frac{1}{2}} r = 0 . \tag{3.6}$$

Theorem 7. *If m is a suitably large positive integer such that the $(m+1)$-th root of equation* (1.5) *satisfies*

$$\mu_{m+1}^2 \geqslant \max \{ \frac{4\tilde{d}}{\alpha}, \frac{8 C_2 \varrho_0^2 \gamma_3^2}{\varepsilon \alpha \gamma_3^2 - 4\tilde{C}}, \frac{2(2\tilde{d} + \varepsilon)}{\alpha} \}$$

where C, $\tilde{C}$, C_2 are suitable positive constant, $\gamma_3 \in (0, \sqrt{\frac{2\delta}{\tilde{C}}})$ and

$$\varepsilon = \min \{ \frac{1}{2}, (2\delta - \tilde{C}\gamma_3^2)[3 + \frac{2\delta^2}{\alpha \mu_1^4}]^{-1}, \frac{\tilde{C}}{2} \gamma_3^2 + \delta \}$$

then the manifold $M_m = H_m \times H_m + \xi$ is an approximate inertial manifold in E for (2.1), *where ξ tends to zero as $m \to \infty$.*

Proof. **i** and **ii** are the same of Theorem 4.

iii. Exponential attraction of M_m. We refer to the absorbing set $\mathscr{B}_0$.

Let $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in D(\mathscr{A})$ and let ε a positive constant. Taking the inner product of (3.6) with $2q_t + 2\varepsilon q$, we get

$$\frac{d}{dt} [|q_t|^2 + \alpha |A^{\frac{1}{2}} q|^2 + 2\varepsilon \langle q_t, q \rangle + g_u(t) |A^{\frac{1}{4}} q|^2]$$

$$+ (2\delta - 2\varepsilon) |q_t|^2 + 2\varepsilon\alpha |A^{\frac{1}{2}} q|^2 + 2\varepsilon\delta \langle q_t, q \rangle + 2\varepsilon g_u(t) \cdot |A^{\frac{1}{2}} q|^2 - (\frac{d}{dt} g_u(t)) \cdot |A^{\frac{1}{4}} q|^2$$

$$+ 2g(|A^{\frac{1}{4}} u|^2)[\langle q_t, A^{\frac{1}{2}} s \rangle + \varepsilon \langle q, A^{\frac{1}{2}} s \rangle] - 2g(|A^{\frac{1}{4}} v|^2)[\langle q_t, A^{\frac{1}{2}} r \rangle + \varepsilon \langle q, A^{\frac{1}{2}} r \rangle] = 0 .$$

Moreover $\quad \frac{d}{dt} \langle q, A^{\frac{1}{2}} s \rangle = \langle q_t, A^{\frac{1}{2}} s \rangle, \quad \frac{d}{dt} \langle q, A^{\frac{1}{2}} r \rangle = \langle q_t, A^{\frac{1}{2}} r \rangle + \langle A^{\frac{1}{2}} q, q_t \rangle$

where $\quad \langle A^{\frac{1}{2}} q, q_t \rangle \geqslant - \frac{\gamma_3^2}{2} |q_t|^2 - \frac{1}{2\gamma_3^2} |A^{\frac{1}{2}} q|^2$

and, similarly to Theorem 4, in $\mathscr{B}_0$ there exists a positive constant C_2 such that

$$\left|\frac{d}{dt} g_u(t)\right| = |g'(|A^{\frac{1}{4}} u|^2)| \cdot \left|\frac{d}{dt} |A^{\frac{1}{4}} u|^2\right| \leqslant 2\varrho_0^2 C_2$$

then
$$\frac{d}{dt}[|q_t|^2 + \alpha |A^{\frac{1}{2}} q|^2 + 2\varepsilon\langle q_t, q\rangle + g(|A^{\frac{1}{4}} u|^2)\cdot |A^{\frac{1}{4}} q|^2]$$

$$+\frac{d}{dt}[2g(|A^{\frac{1}{4}} u|^2)\langle q, A^{\frac{1}{2}} s\rangle - 2g(|A^{\frac{1}{4}} v|^2)\langle q, A^{\frac{1}{2}} r\rangle]$$

$$+\{(2\delta - 2\varepsilon - \widetilde{C}\gamma_3^2)|q_t|^2 + \frac{3}{2}\varepsilon\alpha |A^{\frac{1}{2}} q|^2 + 2\varepsilon\delta\langle q_t, q\rangle\}$$

$$+\{\varepsilon g(|A^{\frac{1}{4}} u|^2)\cdot |A^{\frac{1}{4}} q|^2 + 2\varepsilon g(|A^{\frac{1}{4}} u|^2)\langle q, A^{\frac{1}{2}} s\rangle - 2g(|A^{\frac{1}{4}} v|^2)\langle q, A^{\frac{1}{2}} r\rangle\}$$

$$+[\frac{\varepsilon\alpha}{2} - \frac{\varepsilon d}{\mu_{m+1}^2} - \frac{2C_2\varrho_0^2}{\mu_{m+1}^2} - \frac{\widetilde{C}}{\gamma_3^2}]|A^{\frac{1}{4}} u|^2 \leqslant 0.$$

If we choose ε, γ_3 as before and

$$\eta = \frac{\alpha\mu_1^4}{2\delta} \qquad \mu_{m+1}^2 \geqslant \max\{\frac{4d}{\alpha}, \frac{8C_2\varrho_0^2\gamma_3^2}{\varepsilon\alpha\gamma_3^2 - 4\widetilde{C}}\}$$

then, following the same procedure of Theorem 2 and putting

$$L(t) = |q_t|^2 + \alpha |A^{\frac{1}{2}} q|^2 + 2\varepsilon\langle q_t, q\rangle + g(|A^{\frac{1}{4}} u|^2)\cdot |A^{\frac{1}{4}} q|^2$$

$$+2g(|A^{\frac{1}{4}} u|^2)\langle q, A^{\frac{1}{2}} r\rangle - 2g(|A^{\frac{1}{4}} v|^2)\langle q, A^{\frac{1}{2}} s\rangle$$

we have $\frac{d}{dt} L(t) + \varepsilon L(t) \leqslant 0$, $t \geqslant 0$ and by the Gronwall lemma $L(t) \leqslant L(0)e^{-\varepsilon t}$, $t \geqslant 0$.

But if
$$\mu_{m+1}^2 \geqslant \frac{2(2d+\varepsilon)}{\alpha}$$

then
$$L(t) \geqslant \frac{1}{2}|q_t|^2 + \frac{\alpha}{2}|A^{\frac{1}{2}} q|^2 - C|A^{\frac{1}{4}} s|^2 - \widetilde{C}|A^{\frac{1}{4}} r|^2$$

where
$$C|A^{\frac{1}{4}} s|^2 + \widetilde{C}|A^{\frac{1}{4}} r|^2 \leqslant \frac{1}{\mu_{m+1}^2}(C\varrho_1^2 + \widetilde{C}\varrho_0^2).$$

If we put
$$\xi = \frac{1}{\mu^2_{m+1}} (C\varrho_1^2 + \widetilde{C}\varrho_0^2)$$

then
$$L(t) \geqslant \frac{1}{2} |q_t|^2 + \frac{\alpha}{2} |A^{\frac{1}{2}} q|^2 - \xi$$

and so, for every $t \geqslant 0$ and for every initial condition in E, we have

$$\frac{1}{2} [|q_t|^2 + \alpha |A^{\frac{1}{2}} q|^2] \leqslant L(0) e^{-\varepsilon t} + \xi \leqslant [2(1+\alpha) + d] \left\| \begin{pmatrix} q(0) \\ q_t(0) \end{pmatrix} \right\|_E^2 e^{-\varepsilon t}$$

$$+ [2g(\|A^{\frac{1}{4}} u_0\|^2) \langle q(0), A^{\frac{1}{2}} s\rangle - 2g(|A^{\frac{1}{4}} v|^2) \langle q(0), A^{\frac{1}{2}} r(0)\rangle] e^{-\varepsilon t} + \xi .$$

Remark. In the case $h = h(x, t)$, it is not difficult to show in the usual way that, using (3.2), if there exist four positive constants K_1, K_2, k_1, k_2 such that

$$\limsup_{t \to +\infty} e^{k_1 t} |h|^2_{L^2}(t) \leqslant K_1 \qquad \limsup_{t \to +\infty} e^{k_2 t} \left| \frac{\mathrm{d}}{\mathrm{d}t} f \right|^2_{L^2}(t) \leqslant K_2$$

then the problem (2.1) has an inertial manifold.

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Sommario

Si considera una classe di equazioni semilineari nell'ambito della teoria delle travi elastiche. Si prova, nel caso omogeneo, l'esistenza di uno spazio vettoriale di dimensione finita, esponenzialmente attrattivo nello spazio di Hilbert delle soluzioni (una cosiddetta «varietà inerziale»). Una forma più debole di questo risultato viene ottenuta nel caso non omogeneo.

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