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Oscillation and asymptotic behaviour of certain nonlinear difference equations (**)

1 - Introduction

In this paper we are concerned with a class of *nonlinear difference equations* of the form

$$\Delta^2(u_n + p_n u_{n-k}) + q_n f(u_{n-l}) = 0\,, \qquad n = 0, 1, 2, \ldots \tag{1}$$

where Δ is the forward difference operator, i.e. $\Delta v_n = v_{n+1} - v_n$ and $\Delta^2 v_n = \Delta(\Delta v_n)$, (p_n) and (q_n) are sequences of real numbers with $q_n \geq 0$ eventually, k and l are nonnegative integers. The function f is a real valued function satisfying $uf(u) > 0$ for $u \neq 0$.

By a solution of (1) we mean a sequence (u_n) defined for $n \geq -\max\{k, l\}$, which satisfies (1) for $n = 0, 1, 2, 3 \ldots$. A nontrivial solution (u_n) of (1) is said to be *oscillatory* if for every positive integer N there exists $n \geq N$ such that $u_n u_{n+1} \leq 0$. Otherwise it is called nonoscillatory.

In recent years there has been considerable interest in the study of oscillation and asymptotic behaviour of solutions of difference equations; see for example [2], [5], [7], [9]-[19] and the references cited therein. For the general theory of difference equations one can refer to [1] and [8].

Our purpose in this paper is to study the *asymptotic and oscillatory behaviour of solutions of equation* (1) *in the case* $q_n \geq 0$. The obtained results supplement those contained in [18]. For related results for differential equations we refer the reader to [3], [4], [6].

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2 - Main results

Here we will establish some oscillatory and asymptotic properties of the solution of (1). In doing so, we will ask:

i. $f(u)$ is bounded away from zero, if u is bounded away from zero

ii. $\sum_{0}^{\infty} q_n = \infty$.

Let (u_n) be a solution of (1). Set $z_n = u_n + p_n u_{n-k}$. The following lemma describes some asymptotic properties of the sequence (z_n) when (u_n) is a nonoscillatory solution of (1).

Lemma. *Assume that* **i** *and* **ii** *hold and that there exists a constant* $P_1 < 0$ *such that* $P_1 \leqslant p_n \leqslant 0$. *Then:*

a. *If* (u_n) *is an eventually positive solution of* (1), *then the sequences* (z_n) *and* (Δz_n) *are monotonic and either*

$$\lim_{n \to \infty} z_n = \lim_{n \to \infty} \Delta z_n = -\infty \tag{2}$$

or

$$\lim_{n \to \infty} z_n = \lim_{n \to \infty} \Delta z_n = 0 \qquad \Delta z_n > 0 \text{ and } z_n < 0 . \tag{3}$$

In addition, if $P_1 \geqslant -1$, *then* (3) *holds.*

b. *If* (u_n) *is an eventually negative solution of* (1), *then the sequences* (z_n) *and* (Δz_n) *are monotonic and either*

$$\lim_{n \to \infty} z_n = \lim_{n \to \infty} \Delta z_n = \infty \tag{4}$$

or

$$\lim_{n \to \infty} z_n = \lim_{n \to \infty} \Delta z_n = 0 \qquad \Delta z_n < 0 \text{ and } z_n > 0 . \tag{5}$$

In addition, if $P_1 \geqslant -1$, *then* (5) *holds.*

Proof. Let (u_n) be an eventually positive solution of (1). From (1) we have that there exists a positive integer n_1 such that

$$\Delta^2 z_n = -q_n f(u_{n-i}) \leqslant 0 \qquad \text{for } n \geqslant n_1 \tag{6}$$

so (Δz_n) is nonincreasing, which implies that (z_n) is monotonic.

Now suppose that there exists $n_2 \geqslant n_1$ such that $\Delta z_{n_2} \leqslant 0$, then, since $q_n \not\equiv 0$

eventually, there exists $n_3 > n_2$ such that $\Delta z_n \leqslant \Delta z_{n_3} < 0$ for $n \geqslant n_3$ and a summation shows that $z_n \to -\infty$ as $n \to \infty$. Since (Δz_n) is nonincreasing, so $\Delta z_n \to L \geqslant -\infty$. If $L = -\infty$ clearly (2) holds. If $L > -\infty$, summing (6) we have

$$\Delta z_{n+1} = \Delta z_{n_3} - \sum_{i=n_3}^{n} q_i f(u_{i-l})$$

and then let $n \to \infty$ to obtain

$$\sum_{i=n_3}^{\infty} q_i f(u_{i-l}) = \Delta z_{n_3} - L < \infty .$$

The last inequality, together with **i** and **ii** implies $\lim_{n\to\infty} \inf u_n = 0$. Since $L < 0$, a summation shows that (z_n) is eventually negative. Therefore we can choose $n_4 > n_3$ such that $\Delta z_n < \frac{L}{2}$ for $n \geqslant n_4$ and $z_{n_4} < 0$. Summing the above inequality we have

$$z_n - z_{n_4} < \frac{L}{2}(n - n_4) \qquad n > n_4$$

thus

$$z_n < \frac{L}{2}(n - n_4) < \frac{L}{4} n \qquad \text{for } n > 2n_4 .$$

By the assumptions, we obtain

$$P_1 y_{n-k} \leqslant p_n u_{n-k} < z_n < \frac{Ln}{4} \qquad \text{and} \qquad y_{n-k} > \frac{Ln}{4P_1} \to \infty \quad \text{as } n \to \infty$$

which contradicts $\lim_{n\to\infty} \inf u_n = 0$. Hence $\lim_{n\to\infty} \Delta z_n = -\infty$. Now, if $\Delta z_n > 0$ for $n \geqslant n_1$, then $\Delta z_n \to L_1 \geqslant 0$ as $n \to \infty$. As before, summing (6) from $n \geqslant n_1$ to m and then letting $m \to \infty$, we get

$$\Delta z_n = L_1 + \sum_{i=n}^{\infty} q_i f(u_{i-k})$$

which again implies that $\lim_{n\to\infty} \inf u_n = 0$.

Suppose that $L_1 > 0$. Then we have $\Delta z_n \geqslant L_1 > 0$, and so $z_n \to \infty$ as $n \to \infty$ and since $u_n \geqslant z_n$ hence $u_n \to \infty$ as $n \to \infty$, a contradiction. Therefore $L_1 = 0$. Furthermore, if there exists $n_2 \geqslant n_1$ such that $z_{n_2} \geqslant 0$, then $\Delta z_n > 0$ implies that $z_n \geqslant z_{n_3} > 0$ for $n \geqslant n_3 > n_2$, which again contradicts $\lim_{n\to\infty} \inf u_n = 0$. Therefore, we have $z_n < 0$ for $n \geqslant n_1$.

Thus $z_n \to L_2 \leqslant 0$. If $L_2 < 0$, then

$$P_1 u_{n-k} \leqslant u_n + P_1 u_{n-k} \leqslant u_n + p_n u_{n-k} = z_n \leqslant L_2 < 0 \quad \text{for } n \geqslant n_1 \text{ and}$$

$$u_{n-k} \geqslant \frac{L_2}{P_1} > 0 \quad n \geqslant n_1 . \tag{7}$$

But $\lim\limits_{n\to\infty} \inf u_n = 0$ implies there exists and increasing sequence of natural numbers (n_i) such that $u_{n_i-k} \to 0$ as $i \to \infty$, contraditcting (7). Thus, we conclude that $\lim\limits_{n\to\infty} z_n = 0$.

In addition, we assume that $P_1 \geqslant -1$. Suppose that (3) does not hold. Then (2) holds, so $z_n < 0$ for all large n. We have

$$u_n < -p_n u_{n-k} \leqslant -P_1 u_{n-k} \leqslant u_{n-k}$$

for all large n. But the last inequality implies that (u_n) is bounded, which contradicts (2). Therefore (3) holds.

The proof of **b** is similar to that of **a** and hence will be omitted.

Using the asymptotic properties of the sequence (z_n), we now prove the following result about the asymptotic behaviour of the nonoscillatory solutions of (1).

Theorem 1. *Let* **i** *and* **ii** *hold. If there exists a constant P_1 such that*

$$-1 < P_1 \leqslant p_n \leqslant 0 \tag{8}$$

then every nonoscillatory solution (u_n) of (1) tends to zero as $n \to \infty$.

Proof. If (u_n) is eventually positive, then by part **a** of Lemma we have that (3) holds. Thus $z_n = u_n + p_n u_{n-k} < 0$ for all large n. Then (8) implies $u_n < -p_n u_{n-k} < u_{n-k}$ and hence (u_n) is bounded.

Now suppose that $\lim\limits_{n\to\infty} \sup u_n = a > 0$. Then there exists a subsequence of (u_n), say (u_{n_i}), such that $u_{n_i} \to a$ as $i \to \infty$. Then for all large i we have

$$0 > z_{n_i} \geqslant u_{n_i} + P_1 u_{n_i-k} , \qquad \text{so} \qquad u_{n_i-k} > -\frac{u_{n_i}}{P_1} .$$

But this implies that $\lim\limits_{i\to\infty} u_{n_i-k} \geqslant -\dfrac{a}{P_1} > a$ contradicting the choice of a. Hence, we conclude that $u_n \to 0$ as $n \to \infty$. The proof when (u_n) is eventually negative is similar.

An example to which Theorem 1 applies is the equation

$$\Delta^2 (u_n - \frac{1}{2} u_{n-2}) + 2^{2n-5} (u_{n-1})^3 = 0$$

which has the nonoscillatory solution $u_n = 2^{-n}$.

Now we obtain results regarding the oscillatory behaviour of solutions of (1).

Theorem 2. *If* **i**, **ii** *hold and* $-1 \leqslant p_n \leqslant 0$, *then every unbounded solution of* (1) *is oscillatory.*

Proof. We need only to observe that under the present hypotheses Lemma implies that all nonoscillatory solutions of (1) are bounded.

In the next theorem we obtain the conclusion of Theorem 2 without requiring **ii** but with more restrictive conditions of f.

Theorem 3. *Let* $-1 \leqslant p_n \leqslant 0$ *and* f *be a nondecreasing continuous function, such that*

$$\int_{\varepsilon}^{\infty} \frac{ds}{f(s)} < \infty \qquad \int_{-\varepsilon}^{-\infty} \frac{ds}{f(s)} < \infty \qquad \varepsilon > 0 .$$

If we have

$$\sum_{n=n_0}^{\infty} \sum_{i=n+l+1}^{\infty} q_i = \infty \tag{9}$$

then every unbounded solution of (1) *is oscillatory.*

Proof. Assume that (1) has an unbounded nonoscillatory solution and let this solution be eventually positive. Then from (1) we have $\Delta^2 z_n \leqslant 0$, which implies that (Δz_n) is nonincreasing and (z_n) is monotonic. Now if (z_n) is eventually nonpositive, then by assumption $u_n \leqslant -p_n u_{n-k} \leqslant u_{n-k}$, contradicting the assumption that (u_n) is unbounded. Therefore $z_n > 0$ eventually. Now, if (Δz_n) is eventually negative, then clearly z_n is eventually negative which is a contradiction. Thus we have $z_n > 0$ and $\Delta z_n > 0$ eventually, say for $n \geqslant n_1$.

Since $0 < z_n \leqslant u_n$ and f is nondecreasing, we have

$$\Delta^2 z_n + q_n f(z_{n-l}) \leqslant 0 \qquad\qquad n \geqslant n_2 = n_1 + l .$$

Summing the above inequality from $n \geqslant n_2$ to $m \geqslant n$ we get

$$\Delta z_{m+1} - \Delta z_n + \sum_{i=n}^{m} q_i f(z_{i-l}) \leqslant 0 .$$

Letting $m \to \infty$ we see that $\sum_{i=n}^{\infty} q_i f(z_{i-l}) \leqslant \Delta z_n$, so that we can write $\sum_{i=n+l+1}^{\infty} q_i f(z_{i-l}) \leqslant \Delta z_n$, from which, by monotonicity of f, we obtain

$$f(z_{n+1}) \sum_{i=n+l+1}^{\infty} q_i \leqslant \Delta z_n \qquad \text{for } n \geqslant n_2 .$$

Thus

$$\sum_{i=n+l+1}^{\infty} q_i \leqslant \frac{\Delta z_n}{f(z_{n+1})} \leqslant \int_{z_n}^{z_{n+1}} \frac{ds}{f(s)} .$$

Summing the last inequality from n_2 to n we have

$$\sum_{j=n_2}^{n} \sum_{i=j+l+1}^{\infty} q_i \leqslant \int_{z_{n_2}}^{z_{n+1}} \frac{ds}{f(s)} < \int_{z_{n_2}}^{\infty} \frac{ds}{f(s)} < \infty$$

which contradicts (9). The proof is similar when (u_n) is eventually negative.

Theorem 4. *Assume* $0 \leqslant p_n \leqslant 1$. *If f is a nondecreasing function and*

$$\sum_{n=n_0}^{\infty} q_n f[(1-p_{n-l})c] = \infty \tag{10}$$

for every positive constant c, then all solutions of (1) *are oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution (u_n), say $u_{n-k-l} > 0$ for $n \geqslant n_1 \geqslant n_0$. Then $z_n = u_n + p_n u_{n-k} > 0$ and $\Delta^2 z_n \leqslant 0$ for $n \geqslant n_1$. It is easy to see that $\Delta z_n > 0$ for $n \geqslant n_1$. In fact there exists $n_2 \geqslant n_1$ such that $\Delta z_{n_2} \leqslant 0$, then there exists $n_3 > n_2$ such that $\Delta z_n \leqslant \Delta z_{n_3} < 0$ since (Δz_n) is nonincreasing and $q_n \not\equiv 0$ eventually. The last inequality yields $z_n \to -\infty$ as $n \to \infty$, which contradicts that $z_n > 0$.

Furthermore, since $z_n \geqslant u_n$, hence $z_{n-k} \leqslant z_n \leqslant u_n + p_n z_{n-k}$, so

$$(1-p_n) z_{n-k} \leqslant u_n . \tag{11}$$

From (1), by monotonicity of f and (11), we obtain

$$\Delta^2 z_n + q_n f[(1-p_{n-l}) z_{n-k-l}] \leqslant 0$$

and we see that there exists a constant $c > 0$ such that

$$\Delta^2 z_n + q_n f[(1 - p_{n-l})c] \leq 0 \qquad n \geq n_2 > n_1 . \tag{12}$$

Summing both sides of (12) from n_2 to n we have

$$\sum_{i=n_2}^{n} q_i f[(1 - p_{i-l})c] \leq \Delta z_{n_2}$$

which contradicts (10). The proof for (u_n) eventually negative is similar.

For the linear difference equation

$$\Delta^2 (u_n + p_n u_{n-k}) + q_n u_{n-l} = 0 \tag{13}$$

we obtain from Theorem 4 the following

Corollary. *If* $0 \leq p_n \leq 1$, $q_n \geq 0$ *and* $\sum_{n=n_0}^{\infty} q_n (1 - p_{n-i}) = \infty$, *then every solution of* (13) *is oscillatory.*

Theorem 5. *If* **i** *and* **ii** *hold and* (p_n) *is not eventually negative, then any solution* (u_n) *of* (1) *is either oscillatory or satisfies* $\lim_{n \to \infty} \inf |u_n| = 0$.

Proof. Let (u_n) be a solution of (1). If (u_n) is nonoscillatory, then $|u_n| > 0$ eventually. Suppose that $u_n > 0$. Then as before (6) implies that (Δz_n) is nonincreasing and also we have $z_n > 0$ eventually. We see as in the proof of Theorem 4 that $\Delta z_n > 0$ eventually. Therefore $\Delta z_n \to L \geq 0$ as $n \to \infty$. Summing (6) from n to $m > n$ with n sufficiently large and then letting $m \to \infty$ we get

$$\sum_{i=n}^{\infty} q_i f(u_{i-l}) = \Delta z_n - L < \infty \tag{14}$$

which, by **i** and **ii**, implies that $\lim_{n \to \infty} \inf u_n = 0$. The proof when $u_n < 0$ is similar.

Theorem 6. *If* $0 \leq p_n \leq p$, $q_n \geq q > 0$ *and there exists a constant* $A > 0$ *such that* $|f(u)| \geq A|u|$ *for all* u, *then all solutions of* (1) *are oscillatory.*

Proof. We observe that the assumptions of Theorem 6 imply the assumptions of Theorem 5. Therefore arguing as in the proof of Theorem 5 for an eventually positive solution (u_n) of (1) we obtain the equality (14).

Further, by assumptions, (14) gives $Aq \sum_{i=n}^{\infty} u_{i-l} \leq \Delta z_n - L < \infty$, which implies that $u_n \to 0$ as $n \to \infty$ and so $z_n \to 0$ as $n \to \infty$. But this is impossible, since $z_n > 0$ and $\Delta z_n > 0$ eventually. This remark completes the proof.

Our final theorem shows that, if (p_n) is bounded with upper bound less than -1, then **i** and **ii** are sufficient to ensure that bounded nonoscillatory solutions of (1) tend to zero as $n \to \infty$.

Theorem 7. *If, in addition to* **i** *and* **ii**, *there exist constants* P_1 *and* P_2 *such that*

$$P_1 \leqslant p_n \leqslant P_2 < -1 \tag{15}$$

then every bounded solution (u_n) *of* (1) *is either oscillatory or satisfies* $u_n \to 0$ *as* $n \to \infty$.

Proof. Suppose that (1) has a bounded nonoscillatory solution (u_n) and let (u_n) be eventually positive. By part **a** of Lemma either (2) or (3) holds. Clearly (2) cannot hold in view of (15) and the fact that (u_n) is bounded. From (3) we have $z_n < 0$ and $z_n \to 0$ as $n \to \infty$. Therefore, for any number $\varepsilon > 0$ there exists n_1 so that for $n \geqslant n_1$ we have $-\varepsilon < z_n \leqslant u_n + P_2 u_{n-k}$ or $u_{n-k} < -\dfrac{(u_n + \varepsilon)}{P_2}$ and consequently

$$u_n < -\frac{1}{P_2} u_{n+k} - \frac{1}{P_2}\varepsilon \tag{16}$$

and hence

$$u_{n+k} > -P_2 u_n - \varepsilon . \tag{17}$$

From (16) $u_{n+k} < -\dfrac{1}{P_2} u_{n+2k} - \dfrac{1}{P_2}\varepsilon$ and, by (17), we get

$$u_n < (-\frac{1}{P_2})^2 u_{n+2k} + (-\frac{1}{P_2})^2 \varepsilon + (-\frac{1}{P_2})\varepsilon .$$

After m iterations, we obtain

$$u_n < (-\frac{1}{P_2})^m u_{n+km} + \varepsilon \sum_{i=1}^{m} (-\frac{1}{P_2})^i .$$

Let $\lambda = 1 + \dfrac{1}{P_2} > 0$ and $u_n < M$. Now choose m large enough so that $(-\dfrac{1}{P_2})^m < \dfrac{\varepsilon}{\lambda M}$. Thus for every $\varepsilon > 0$ there exists $n_2 \geqslant n_1$ such that for

$n \geq n_2$ we have

$$u_n < \frac{\varepsilon}{\lambda} + \varepsilon\left(-\frac{1}{P_2}\right)\frac{1-\left(-\frac{1}{P_2}\right)^m}{1+\frac{1}{P_2}} < 2\frac{\varepsilon}{\lambda}.$$

That is $u_n \to 0$ as $n \to \infty$. The proof for (u_n) eventually negative is similar.

References

[1] R. P. AGARWAL, *Difference equations and inequalities. Theory, methods and applications*, Dekker, New York, 1992.

[2] D. A. GEORGIOU, E. A. GROVE and G. LADAS, *Oscillations of neutral difference equations*, Appl. Anal. **33** (1989), 234-253.

[3] J. R. GRAEF, M. K. GRAMMATIKOPOULOS and P. W. SPIKES, *Asymptotic properties of solutions of nonlinear neutral delay differential equations of the second order*, Rad. Mat. 4 (1988), 133-149.

[4] J. R. GRAEF, M. K. GRAMMATIKOPOULOS and P. W. SPIKES, *On the asymptotic behavior of solutions of a second order nonlinear neutral delay differential equation*, J. Math. Anal. Appl. **156** (1991), 23-39.

[5] J. R. GRAEF and P. W. SPIKES, *Asymptotic decay of oscillatory solutions of forced nonlinear difference equations*, Dynam. Syst. Appl. 3 (1994), 95-102.

[6] M. K. GRAMMATIKOPOULOS, G. LADAS and A. MEIMARIDOU, *Oscillation and asymptotic behavior of second order neutral differential equations*, Ann. Mat. Pura Appl. **148** (1987), 29-40.

[7] J. W. HOOKER and W. T. PATULA, *A second order nonlinear difference equation: Oscillation and asymptotic behavior*, J. Math. Anal. Appl. **91** (1983), 9-29.

[8] V. LAKSHMIKANTHAM and D. TRIGIANTE, *Theory of difference equations, numerical methods and applications*, Acad. Press, New York 1988.

[9] B. S. LALLI and S. R. GRACE, *Oscillation theorems for second order neutral difference equations*, Appl. Mat. Comput. **62** (1994), 47-60.

[10] B. S. LALLI and B. G. ZHANG, *On existence of positive solutions and bounded oscillations for neutral difference equations*, J. Math. Anal. Appl. **166** (1992), 272-287.

[11] H. J. LI and S. S. CHENG, *Asymptotically monotone solutions of nonlinear difference equation*, Tamkang J. Math. **24** (1993), 269-282.

[12] Z. SZAFRAŃSKI and B. SZMANDA, *Oscillatory behavior of difference equations of second order*, J. Math. Anal. Appl. **150** (1990), 414-424.

[13] Z. SZAFRAŃSKI and B. SZMANDA, *A note on the oscillation of some difference equations*, Fasc. Math. **21** (1990), 57-63.

[14] B. SZMANDA, *Note on the behaviour of solutions of a second order nonlinear difference equation*, Atti Acad. Naz. Lincei Rend. **69** (1980), 120-125.

[15] B. SZMANDA, *Characterization of oscillation of second order nonlinear difference equations*, Bull. Polish. Acad. Sci. Math. 34 (1986), 133-141.

[16] B. SZMANDA, *Oscillatory behaviour of certain difference equations*, Fasc. Math. **21** (1990), 65-78.

[17] E. THANDAPANI, *Asymptotic and oscillatory behavior of solutions of nonlinear second order difference equations*, Indian J. Pure Appl. Math. 24 (1993), 365-372.

[18] E. THANDAPANI, *Asymptotic and oscillatory behaviour of solutions of a second order nonlinear neutral delay difference equation*, Riv. Mat. Univ. Parma **1** (1992), 105-113.

[19] E. THANDAPANI, I. GYÖRI and B. S. LALLI, *An application of discrete inequality to second order nonlinear oscillation*, J. Math. Anal. Appl. **186** (1994), 200-208.

Sommario

Viene studiato il comportamento asintotico e oscillatorio delle soluzioni di alcune classi di equazioni non lineari alle differenze.

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